# Cross section

Consider typical fixed-target scattering experiment:

There is a target slab of some material of thickness dx [cm] and very large area and with the density of scattering centers n [cm^{-3}]. Each scattering center has a tiny projected area, or cross section, σ [cm^2].

There is a beam of incoming particles, thought of as point-like, with flux J [s^{-1}] and a cross section A [cm^2] smaller than the target area. The beam particles are scattered only when they hit the scattering centers. Probability of such scattering is the ratio of all scattering center areas to the overall area available for a beam particle:

$$dp = (nAdx)\sigma / A \ = \sigma n dx$$
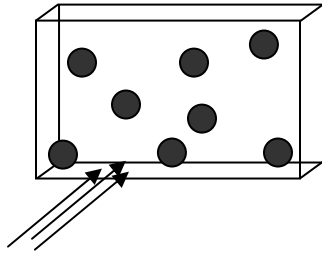
If N is the number of particles fallen on the target, then the number of particles coming through dx thickness without scattering is $N+dN$, where

$$dN = -dp\,N$$

Therefore, $dN/N = -\sigma n dx$. And for a target of finite thickness Δx, we get:

$$N(x) = N_0 \cdot e^{-\sigma n \Delta x}$$

Rate of scattered particles (number of particles scattered per second) is $J_{scattered} = J\,dp = J \cdot \sigma n dx$. From this expression, we can derive experimental definition of cross-section σ:

$$\sigma = \frac{J_{scattered}}{J\,n\,dx},$$

This is an experimental definition for a cross section for interactions of beam particles with the target particles.

If the beam is much wider than the scattering target, then the expression for rate can be re-written in the following form:

$$J_{scattered} = \sigma \cdot J(\text{thru area A}) \cdot n \cdot dx = \sigma \cdot j = \sigma \cdot j \cdot nAdx = \sigma \cdot j \cdot N,$$

where j is the flux per unit of area [cm^{-2}s^{-1}] and N is the number of scattering centers on the way of the beam. This will become very handy to relate the experimentally measured cross section to theory. In theory N=1 (one scattering center) and j will represent an incoming flat wave.

Units of cross section are cm^2; however, a much smaller units of barn (b), mb, nb, pb, etc. are more commonly used, 1 barn being 10^{-24} cm^2. One barn is a typical size of heavy element nuclei. The proton-proton cross section is ~40 mb. This approximately corresponds to a radius of proton of ~1 fm.

Note of caution: One should be cautious about interpreting the cross section as a real geometrical cross section.

- If scattering centers are "semi-transparent", the measured cross section will be much smaller than the actual size of scattering particles. The case for neutrino is very representative in this case: neutrino-proton cross section for typical solar neutrinos of 1 MeV is around 10^{-41} cm^2, despite of a very larger proton "size".

- Cross section of scattering of a point-like electron in columb's potential of another point-like electron is infinite since no matter how large impact parameter is, there is always some force between the two particles that will result in some scattering (albeit very small for large impact parameters, but non-zero nevertheless).

- Cross sections change with the energy of an incoming particle.

- It is more proper to interpret the cross section as a measure of a range and strength of the force between two particles. Note that the amount of scattering depends not only a strength of a force acting between two particles, but also on their energies (it is harder to scatter fast moving particles). Thus, one may rightfully expect that cross sections may also depend on the center of mass energy.

Cross section jargon

Differential: cross section for scattering within a restricted range of some observable, e.g. angle range dθ around some direction θ:

$$\frac{d\sigma}{d\theta} = \frac{dJ_{scattered}/d\theta}{J\,n\,dx}$$

Elastic: we count only the cases when neither beam particle or target particle disintegrated

Inelastic: just opposite, i.e., when either beam particle or target particle or both disintegrate

Total: sum of the elastic and inelastic

Inclusive (e.g., $p + p \rightarrow \pi^+ + anything$): any process with at least one π^+ in the final state (all π^+ would typically be counted in this case)

Exclusive (e.g., $p + p \rightarrow p + p + \pi^0$): exclusively for some fully defined final state (no extras).

Cross section for production of new particles
Definition of the scattering cross section can be easily generalized to describe a probability of production of new particles

$$\sigma = \frac{J(\text{new particle})}{J(\text{projectile particle})\,n(\text{target particles})\,dx}$$

Particle lifetime, width, branching ratios…

Let the particles P decay into some set of particles, set A.
The rate of such transition will be proportional to the number of the original particles present:

$$\frac{dN}{dt}(P \to set\ A) = -\Gamma_A N(t)$$

If the particles P can decay into some other set of particles, set B, we can similarly write:

$$\frac{dN}{dt}(P \to set\ B) = -\Gamma_B N(t)$$

The overall rate of disappearance of particles P will be, therefore:

$$\frac{dN}{dt} = \frac{dN}{dt}(P \to set\ A) + \frac{dN}{dt}(P \to set\ B) + ... = -(\Gamma_A + \Gamma_B + ...)N(t) = -\Gamma N(t),$$

form where one immediately obtains

$$N(t) = N(0)e^{-\Gamma t} = N(0)e^{-t/\tau}.$$

Here $\tau = \dfrac{1}{\Gamma}$ is called *particle's lifetime*,

$\Gamma = \Gamma_A + \Gamma_B + ...$ and is called the *full width* of particle P,

and Γ_f is a *partial width* of particle P decaying into a particular final set (a.k.a. channel) f.

Probability of decaying into a particular channel f $P_f = \dfrac{\dfrac{dN}{dt}(P \to set\ A)}{\dfrac{dN}{dt}(P \to all)} = \dfrac{\Gamma_f}{\Gamma}$ is called *branching ratio*.

Note 1: There are only a few particles in nature that have lifetimes long enough that we can trace them experimentally from the point to birth to a point of decay. Not counting stable particles (p, e, γ, ν), the relatively long lived particles are: n (~15 min), μ (~10^{-6} s) , $\pi^{\pm}/K^{\pm}/K_L$ (~10^{-8} s)

Note 2: If a particle has lots of possible decay channels, its lifetime would tend to be shorter…

From the energy-time uncertainty principle ($\Delta E \Delta t \sim 1$):
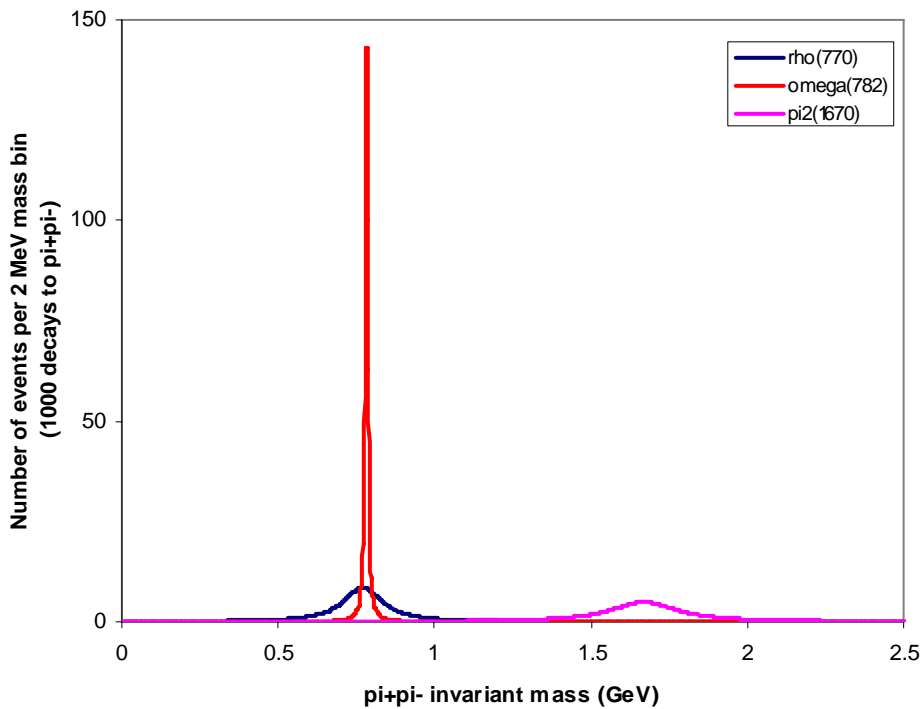
- A particle of mass M having a finite lifetime can be born in a collision of some other two particles with their center of mass energy E_{cm} different from M by as much as $\Delta E \sim 1/\tau \sim \Gamma$. Therefore, one may expect that the cross section for its production would have a bump of width $\pm\Gamma$ around $E=M$. The actual shape is given by the Breit-Wigner formula (it is actually a classical resonance formula):

$$\sigma(E_{cm}) \sim \frac{1}{\left(E_{cm} - M\right)^2 + \dfrac{\Gamma^2}{4}}$$

- Similarly, if one searches for such a particle by building an invariant mass of its decay products among other particles produced in the collision, the invariant mass distribution does not appear as a $\delta(m_{inv}\text{-}M)$ function; rather it looks like a resonance curve:

$$\frac{dN}{dm_{inv}} \sim \frac{1}{\left(m_{inv} - M\right)^2 + \dfrac{\Gamma^2}{4}}$$

Important: the width of the observed resonance is defined by particle's lifetime (or its total particle width), i.e. it does not depend on which partial decay channel is actually used for reconstructing its invariant mass.

$\rho(770)$:	M=771 MeV	Γ=149 MeV
$\omega(782)$:	M=781 MeV	Γ=8.44 MeV
$\pi_2(1670)$	M=1670 MeV	Γ=259 MeV