Relativistic Kinematics

1905 **Albert Einstein** derives the special relativity theory from a single postulate: speed of light is constant in all inertial reference frames.

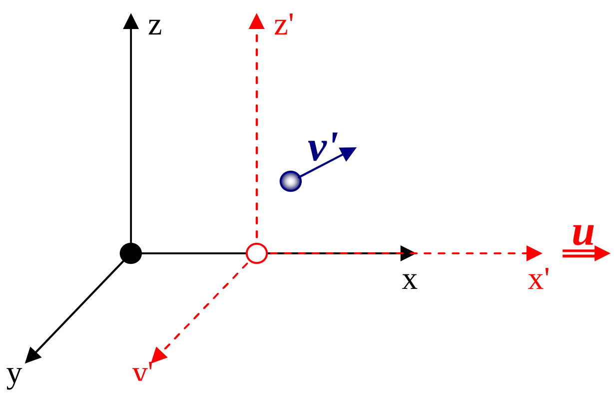
A few important consequences that will be used in the course (remember c=1):

1. Location of a particle is described by a 4-component coordinate vector: $x^{\mu}=(t, x, y, z)$
2. Particle kinematics—by 4-component momentum vector: $p^{\mu}=(E, p_x, p_y, p_z)$
3. Another example: four-vector of electromagnetic field $A^{\mu}=(\varphi, \vec{A})$
4. Energy and 3-component momentum of a particle moving with velocity v are given by:

$$E=\gamma_v m \quad \text{and} \quad \vec{p}=\gamma_v m\vec{v}, \quad \text{where} \quad \gamma_v=\frac{1}{\sqrt{1-v^2}}$$

Important to remember: when v~1 (γ>>1), ***E~p***

5. **Components of all 4-vectors are transformed the same way from one coordinate system to another**



S' system moves with respect to S-system in positive direction x with velocity u; their origins coinciding at t=0

Coordinate system S’	Coordinate system S'
t'	$t=\gamma_u(t'+ux')$
x'	$x=\gamma_u(x'+ut')$
y'	$y=y'$
z'	$z=z'$
E'	$E=\gamma_u(E'+up'_x)$
p'_x	$p_x=\gamma_u(p'_x+uE')$
p'_y	$p_y=p'_y$
p'_z	$p_z=p'_z$

Linear translations naturally result in

$E=E_1+E_2+E_3+\ldots$ $\qquad$ $E'=E'_1+E'_2+E'_3+\ldots$

$\mathbf{p}=\mathbf{p}_1+\mathbf{p}_2+\mathbf{p}_3+\ldots$ $\qquad$ $\mathbf{p}'=\mathbf{p}'_1+\mathbf{p}'_2+\mathbf{p}'_3+\ldots$

6. **Product of any two four-vectors is invariant**, i.e. independent of a coordinate system in which it is calculated (note signs in definition of squaring 4-vectors):

$$ab=a^0b^0-\vec{a}\cdot\vec{b}$$

E.g.: momentum four-vector squared gives particle’s mass:

$$\boldsymbol{E^2-p^2 = E^2-p_x^{\,2}-p_y^{\,2}-p_z^{\,2} = m^2}$$

7. If some process takes time Δt' in one coordinate system S' (without any change in spatial coordinates), it will appear taking longer time in the other (particles live longer in lab frame than at the rest frame):

$$\boldsymbol{\Delta t=\gamma_u \Delta t'}$$

Invariant Mass: Creating New Particles

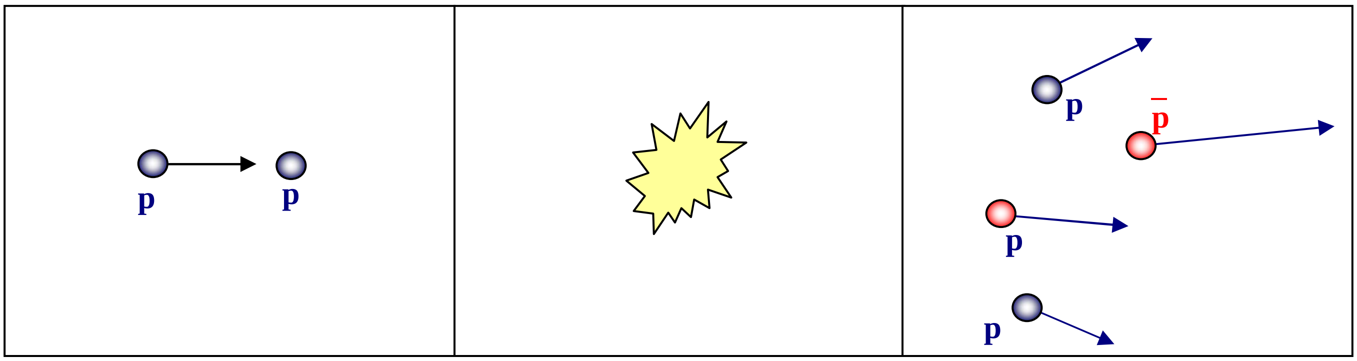
Important concepts:

> **energy/momentum conservation** (before = after)
> **invariant mass**: … = (before in lab frame) = (before in cm frame) = (after in lab frame) = (after in cm frame) = ...

Helpful tip:

- write an for energy/momentum conservation in 4-vector notations
- move terms left/right as needed
- square left/right parts of an equation (pick the frame that simplifies calculations)

A beam proton of energy E collides with a target proton at rest. What is the minimum energy E_{min} to allow for creation of anti-proton and proton pair? For reasons to be discussed later, anti-protons or protons cannot be created alone. This was the process through which anti-protons were discovered at Berkley in 1956.



To create proton-antiproton pair one needs to add energy E that is at least as much as 2m=2×0.940GeV=1.9 GeV. However, it will not work since the system of beam proton plus target proton will have a non-zero momentum

$$\vec{p}_{\text{beam}} + \vec{p}_{\text{target}} = \vec{p}_{\text{beam}} + 0 = \vec{p}_{\text{beam}}$$

that have to be conserved. Therefore the four particles will have to be moving, which will require additional energy.

Let's define the center-of-mass frame where the total momentum sum (of all four particles in this case) is zero. The minimum energy in this CM frame is just the mass of 4 particles, or E_{cm}=4m (2m comes from the beam and target protons and other 2m—from a new born pair).

- The energy-momentum conservation in four-vector notations: $p_{initial} = p_{final}$
- By squaring both sides, we get invariant masses-squared of the system at the beginning and at the end
- The final system invariant mass-squared, calculated in the center-of-mass frame, is:
  $m_{inv}^2 = E^2 - \vec{p}^2 = (4m)^2 - 0^2 = (4m)^2$
- The initial invariant mass-squared in the lab coordinate system:
  $m_{inv}^2 = E_{lab}^2 - \vec{p}_{lab}^2 = (E_{beam} + m)^2 - (\vec{p}_{beam} + 0)^2 = E_{beam}^2 - \vec{p}_{beam}^2 + 2mE_{beam} + m^2 = 2mE_{beam} + 2m^2$
- Both must be the same, which gives $E_{beam} = 7m = 6.6$ GeV.

However, one can do the experiment at even lower energy, if the target is a nucleus instead of bare proton. In this case, protons, being bound to stay within a small spatial region of the nucleus, will necessarily have some momentum as governed by the uncertainty principle, ΔxΔp ≥ ħ. This non-zero momentum of the order of 0.2 GeV/c, when directed toward the beam particle, helps reduce the kinematical threshold by a substantial amount. Note that the proton moving with momentum 0.2 GeV will have energy E~m, since m>>0.2 GeV.

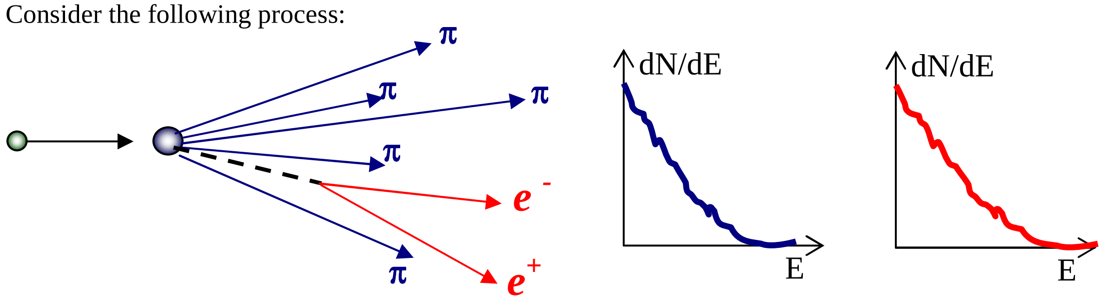
$$m_{inv}^2 = E_{lab}^2 - \vec{p}_{lab}^2 = (E_{beam} + m) - (p_{beam} - 0.2)^2 = E_{beam}^2 - p_{beam}^2 + 2mE_{beam} + m^2 + 0.4\,p_{beam} - 0.04 \approx 2m^2 + 2mE_{beam}\left(1 + \frac{0.2}{m}\right)$$

from where:

$$E_{beam} = \frac{7m}{\left(1 + \dfrac{0.2}{m}\right)} = 5.4 \text{ GeV.}$$

Invariant Mass: Particle Decays

Consider the following process:



Looking at distributions of energy for electrons, can one deduce that electron-positron pairs come from decays of an intermediate particle (as shown) or created in the same manner as numerous pions?

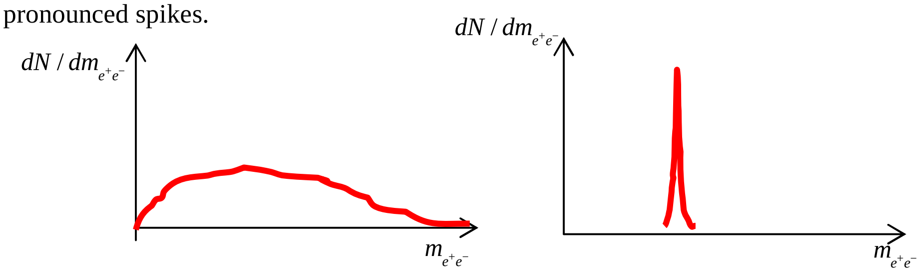
- If there is an intermediate particle X, it may be born with variety (spread) of energies.
- After decaying, energies of electrons will be further spread from event to event, depending on the direction of the decays…
- So the final distribution of electron energies may look not very distinct from similarly spread of energies of other particles emerging directly from the collisions…

However:

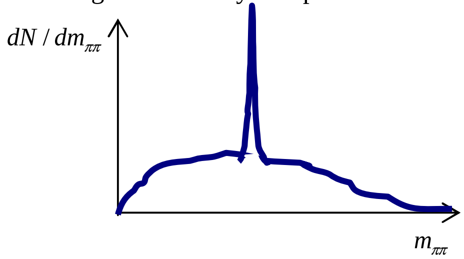
- The energy-momentum conservation in four-vector notations: $p_X = p_{e^+} + p_{e^-}$
- By squaring both sides, we get invariant masses-squared of the system at the beginning, m_X^2, and at the end, $m_{e^+e^-}^2 = \left(E_{e^+} + E_{e^-}\right)^2 - \left(\vec{p}_{e^+} + \vec{p}_{e^-}\right)^2$

So, X-particle's mass is just the invariant mass of its decay products, in this case electron-positron pair.

Therefore, an experimentalist should expect to see a narrow spike in the distribution of $dN/dm_{e^+e^-}$. Its width would be defined by the errors in measurements and/or the natural width of the intermediate particles related to its finite lifetime (ΔEΔt≥ħ). If electrons and positrons were born independently, there would be some broad spectrum without any pronounced spikes.

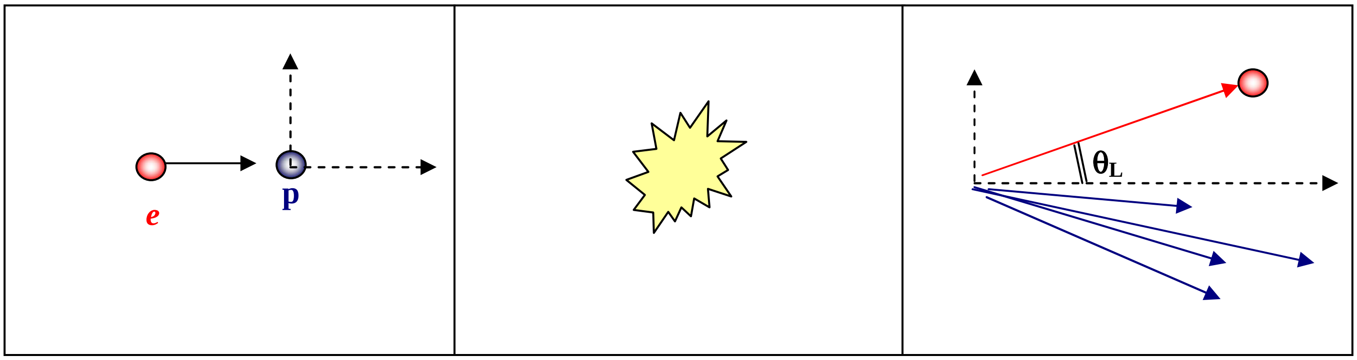


What if we are looking for ππ-decay of a particle of mass M born in presence of many other π-particles?



Angle Transformations

Let's consider a process of an electron of mass m and very large energy (say, 20 GeV) scattering off a stationary proton (mass M~1 GeV). Transform the scattering angle in the center-of-mass frame (theoretical calculations are almost invariably done in such a system) to the observed scattering angle θ_L to the lab frame.



First, we need to find the velocity *u* of the center-of-mass frame:

$$u = \mathbf{p_{TOT}/E_{TOT}} = p_L / (E_L + M) \approx 1, \text{ since } p_L \approx E_L >> M, \text{ and}$$
$$\gamma_u \approx \mathrm{sqrt}(E_L/2M)$$

Therefore, if in the CM system the scattered electron has momentum q_{cm} (and scattering angle θ_{cm}), the same vector in the center-of-mass frame will have the following components:

$$q_{xL} = \gamma_u (q_{cm}\cos\theta_{cm} + uE_{cm}) \quad \text{and} \quad q_{yL} = q_{cm}\sin\theta_{cm},$$

from where:

$$\tan\theta_L = q_{yL} / q_{xL} = (1/\gamma_u)\cdot\sin\theta_{cm} / (\cos\theta_{cm} + 1)$$

From the last equation one can derive an **important observation** for decay products:

- decays in rest frame give particles uniformly distributed in 4π → the most probable polar angle is $\pi/2$
- the most probable angle in lab frame is $1/\gamma$ (and will be small for large boosts!)

Colliding billiard balls in the center of mass frame are scattered uniformly in 4π. Looking at the same process in lab frame, where one ball is at rest before collision, one should expect to see both ball scattered at ~$1/\gamma$ angle, which would get smaller and smaller as energy increases…