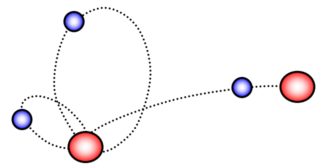
# Forces via exchange of particles



*Classical picture:*

Particle *A* sets filed *F* permitting the whole of space…
Particle *B* interacts with this field…

*Quantum Field Theory picture:*

Particle *A* emits force quanta, particles *C*…
Particle *B* absorbs these quanta and, thus, experiences the force…
The exchange goes back and forth…

Field quanta can carry energy and momentum, which entails energy non-conservation: however, this is allowed for short time $\Delta t \sim 1/\Delta E$. The distance that the quanta can travel is, therefore, $\Delta x \sim c\Delta t \sim 1/\Delta E$: as long as the quantum carries very little energy, it can reach very far away. However, if the field quanta have finite mass, they hardly can go much farther than $\Delta x \sim 1/m$.

## Potential for the force arising from an exchange with particles of mass m:

$$V = \frac{g_A g_B}{4\pi} \frac{e^{-mr}}{r}$$

We will see how this form can be guessed when we discuss the relativistic quantum mechanics.

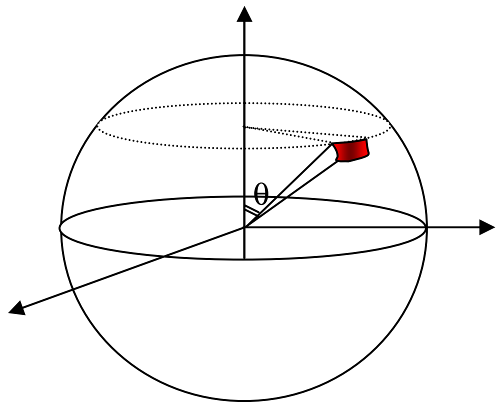
Here $g_A$ represents a charge of particle A generating the field, $g_A$—a charge of a particle B interacting with this field.

For an exchange with zero-mass particles (like photons), the potential represents a classical Coulomb's law.

For an exchange with non-zero mass particles, this field behaves like Coulomb's law field at short distances *(r<<1/m)*, but dies out exponentially at distances larger than *1/m*—the behavior that we guessed from the uncertainty principle.

## Matrix Element:

$$M(\vec{q}) = \int V(r) e^{-i\vec{q}\vec{r}} dV = \int V(r) e^{-iqr\cos\theta} dV$$




$$dV = (r\sin\theta) d\varphi \cdot r d\theta \cdot dr = -r^2 d(\cos\theta) d\varphi dr$$

$$m(q) = -\int V(r) e^{-iqr\cos\theta} r^2 d(\cos\theta) d\varphi dr$$

$$= -\int \frac{g_A g_B}{4\pi} \frac{e^{-mr}}{r} e^{-iqr\cos\theta} r^2 d(\cos\theta) d\varphi dr$$

$$= -\frac{g_A g_B}{4\pi} \int e^{-mr} e^{-iqr\cos\theta} r d(\cos\theta) d\varphi dr$$

$$= -\frac{g_A g_B}{4\pi} (2\pi) \int e^{-mr} e^{-iqr\cos\theta} d(\cos\theta) r dr$$

$$= -\frac{g_A g_B}{2} \int e^{-mr} \left. \frac{e^{-iqr\cos\theta}}{-iqr} \right|_{\theta=0}^{\theta=\pi} r dr$$

$$= \frac{g_A g_B}{2iq} \int e^{-mr} \left( e^{iqr} - e^{-iqr} \right) dr$$

$$= \frac{g_A g_B}{2iq} \left. \left( \frac{e^{-mr+iqr}}{-m+iq} - \frac{e^{-mr-iqr}}{-m-iq} \right) \right|_0^\infty$$

$$= \frac{g_A g_B}{2iq} \left( \frac{-1}{-m+iq} - \frac{-1}{-m-iq} \right)$$

$$= \frac{g_A g_B}{2iq} \left( \frac{m+iq}{m^2-(iq)^2} - \frac{m-iq}{m^2-(iq)^2} \right)$$

$$= \frac{g_A g_B}{2iq} \left( \frac{2iq}{m^2+q^2} \right)$$

$$= \frac{g_A g_B}{m^2+q^2}$$

# Cross-sections:

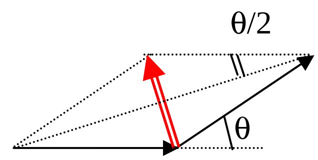
## Massless quanta

Consider elastic *ee* scattering in center-of-mass frame (the cross-section equations obtained earlier are valid for both the non-relativistic and relativistic cases when used in the center-of-mass frame). Ignore spins.

$$m = 0, \qquad M(q) = \frac{e^2}{q^2}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{4\pi^2}|M|^2 \frac{p_f^2}{v_i v_f} = \frac{1}{4\pi^2}|M|^2 \frac{p^2}{v^2} = \frac{1}{4\pi^2}\frac{e^4}{q^4}\frac{p^2}{v^2}$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{4\pi^2}\frac{e^4}{(2p\sin(\theta/2))^4}\frac{p^2}{v^2} = \frac{e^4}{4\pi^2}\frac{1}{16p^2v^2\sin^4(\theta/2)}$$




$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{4p^2v^2\sin^4(\theta/2)}$$

This expression is known as Coulomb's scattering or Rutherford's formulas. Notice that the cross-section is inversely proportional to the square of electron's momentum.

Here I used the following:

Initial and final momenta are the same in magnitude, *p*. The same is true for velocities.
Transferred momentum $q=2p\sin(\theta/2)$.

## Very massive quanta ($m_X$>>$q$)

Consider elastic scattering of massless neutrinos of high energy $E_{lab}$ on a nucleon (proton or neutron) of mass $m_N$ at rest and assume that the scattering occurs due to an exchange with a very heavy particle with mass $m_X$ ($m_X >> q$). Ignore spins.

$$m_X, \qquad M(q) = \frac{g^2}{m_X^2 + q^2} \approx \frac{g^2}{m_X^2} = G$$

$$\frac{d\sigma}{d\Omega} = \frac{1}{4\pi^2}G^2\frac{p_f^2}{v_i v_f} = \frac{G^2}{4\pi^2}p_{cm}^2$$

$$\frac{d\sigma}{d\Omega} = \frac{G^2}{8\pi^2}m_N E_{lab}$$

This is a famous linear dependence of neutrino cross-section on its energy in the lab frame.

Here I used the following:

For massless neutrinos, $v_i$ and $v_f$ equal to speed of light, or 1.
In the center-of-mass frame, the neutrino's final momentum is the same as initial momentum, $p_{cm}$.

Since the energy $E_{lab}$ is high, both neutrino and nucleon are relativistic, and we can neglect nucleon mass in comparison to $p_{cm}$ or $E_{lab}$. In the center-of-mass frame, neutrino's and nucleon's momenta are same in magnitude and also equal to their energies.

Relationship between the lab and center-of-mass variables can be derived from the invariant masses.

In the lab frame: 

$m_{inv}^2=(E_{lab} + m_N)^2 - (p_{lab} + 0)^2 = E_{lab}^2 + m_N^2 + 2E_{lab} m_N - p_{lab}^2 = E_{lab}^2 - p_{lab}^2 + m_N^2 + 2E_{lab} m_N \approx 2E_{lab} m_N$

In the CM frame: 

$m_{inv}^2=(p_{cm} + p_{cm})^2 - (p_{cm} - p_{cm})^2 = 4p_{cm}^2$

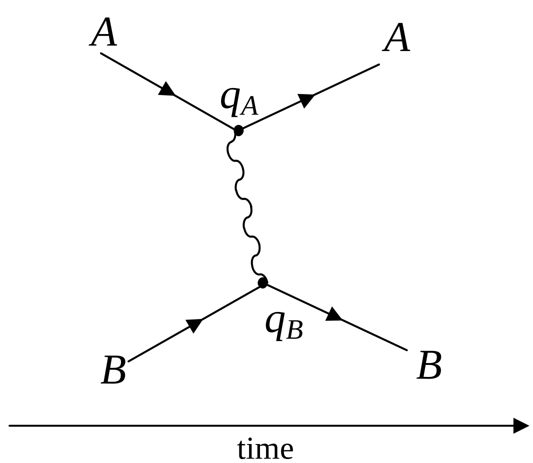
From where:
$p_{cm}^2 = (E_{lab}\, m_N)/2$

# Feynman Diagrams

Scattering probability (see previous lecture) for two particles A and B, A + B → A + B, via an exchange with a force particle C of mass M is:

$$P \sim |M|^2 = q_A^2 \cdot \left(\frac{1}{q^2 + M^2}\right)^2 \cdot q_B^2 = q_A^2 \cdot (\text{Propagator of particle C})^2 \cdot q_B^2$$

**Feynman** proposed a pictorial scheme to represent particle exchange processes—the scheme that proved to be an indispensable tool in visualization of (sometimes very complicated) quantum field theory equations for matrix elements… Here are the rules for drawing such Feynman diagrams:



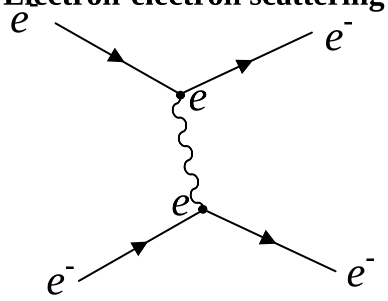

- Time line: from left to right (some prefer, upward; in this course it will be from left to right).
- Fermions are represented by solid lines with arrows (forward in time for fermions and backward for anti-fermions).
- Bosons are represented by various lines: typically, sin-wave line for photons, spiral lines for gluons, dashed lines for Z, $W^{\pm}$, H.
- A coupling constant (charge) is assigned to each vertex; this charge describes the strength of the force (e.g., a negative elementary charge *-e* for an electron emitting a photon). The larger this constant is, the larger the matrix element is, and, therefore, the larger corresponding cross-sections are going to be (and shorter lifetime for particle decays).
- In each vertex, one must conserve:
  - Energy
  - All components of momentum
  - All components of angular momentum
  - All appropriate charges: electric, color, etc.
- Lines that start and end at vertices represent particles that are not present at the beginning nor at the end, these particles are called *virtual*. For these particles, the relationship $E^2\text{-}p^2\text{=}m^2$ does NOT hold true (more on this further below)[1]. Proper propagators are assigned to such virtual particles; e.g., spin-0 particles have a non-relativistic propagator that we obtained in the previous lecture: $1/(q^2+M^2)$.

There is a corresponding set of rules to restore the quantum field equations from the pictorial Feynman diagrams. One-to-one correspondence between the diagrams (with a well defined set of rules) and the quantum field theory equations was proven by **Dyson**.

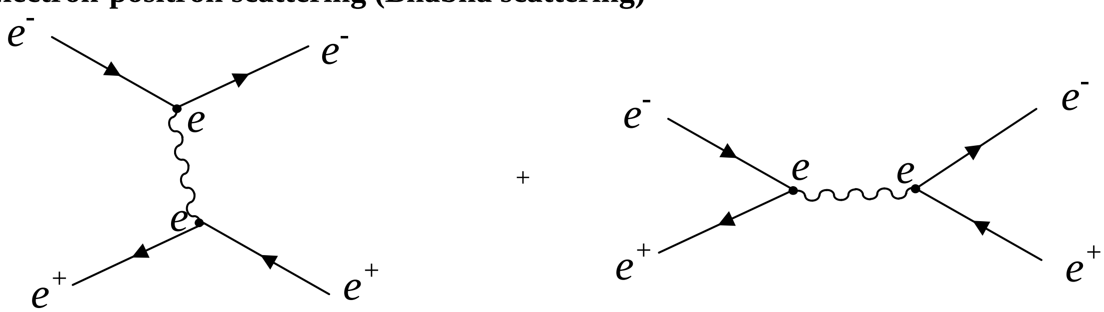
[1] *Often, one can think of these particles popping up in existence for only short time $\Delta t \sim 1/E$, where E is an amount of energy they would need to carry to remain real forever. See the introductory remarks at the beginning of this note.*

# Examples

### Electron-electron scattering (Møller scattering)



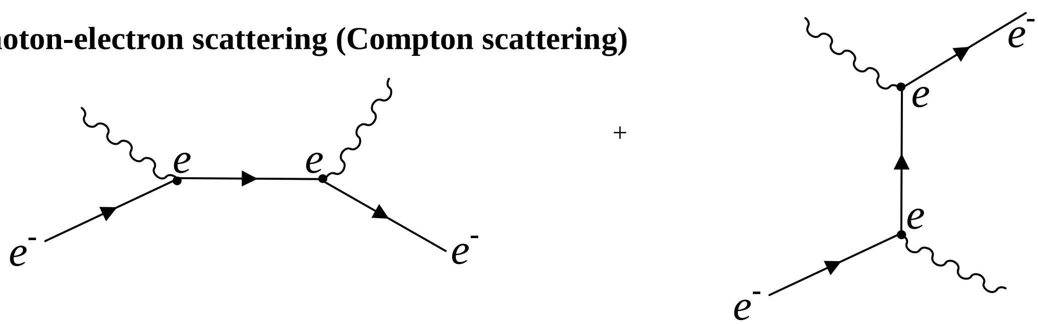

### Electron-positron scattering (Bhabha scattering)



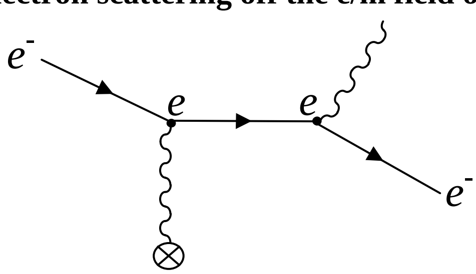

Note that there are two process: Moller-like scattering and the other one via intermediate annihilation of the electron-positron pair. Since we have no means to distinguish experimentally what has actually happened, the amplitudes of these two processes should be added together to represent the overall amplitude. When matrix-element-squared is calculated there will be three terms: amplitude-squared for each of the two sub-processes and one term corresponding to the interference between them.

The first diagram represents so-called t-channel process, since photon's four-vector-momentum squared $q^2$ equals to the Lorentz-invariant variable $t^2=(p_{out}-p_{in})^2$, with p's being electron's (or positron's) four-vector momentum. The second diagram is called s-channel process. Here photon's four-vector-momentum squared $q^2$ equals to the Lorentz-invariant variable $s^2=(p_{electron} + p_{positron})^2$.

### Photon-electron scattering (Compton scattering)




### Electron scattering off the e/m field of nucleus (Bremsstrahlung = breaking radiation)



# Virtual (off-shell) particles

Note that the virtual particle's four-momentum-vector-squared does not necessarily equal to its mass-squared:

$$q^2 = E^2 - \vec{p}^2 \neq m^2$$

This is allowed since the virtual particles exist only for short time and energy does not have to be conserved. As an example, consider the electron-electron scattering diagram in the center of mass-frames. The photon's $q^2$ for this process is not equal to zero and negative:

$$q^2 = (p_{out} - p_{in})^2 = (E_{out} - E_{in})^2 - (\vec{p}_{out} - \vec{p}_{in})^2 = 0 - \left(2|\vec{p}|\sin(\theta/2)\right)^2 = -4|\vec{p}|^2 \sin^2(\theta/2) \neq 0 \ .$$

The virtual photon's $q^2$ of the second annihilation diagram is also not equal to zero and positive this time:

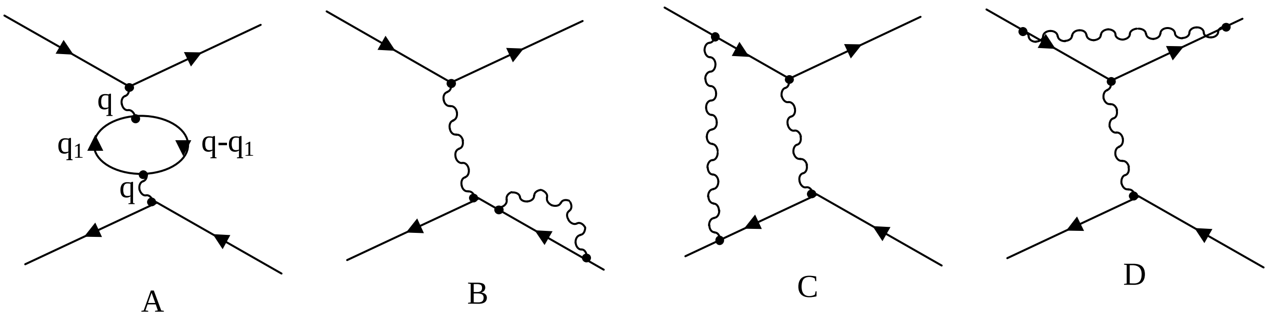
$$q^2 = (p_{electron} + p_{positron})^2 = (E_{electron} + E_{positron})^2 - (\vec{p}_{electron} + \vec{p}_{positron})^2 = (2E_{electron})^2 - 0 = (2E_{electron})^2 \neq 0$$

The virtual particles are said to be "off their mass shells". The real particle's $E$ and $\vec{p}$ must be within the 3-dimentional surface (shell) in the four-dimensional space (E, $p_x$, $p_y$, $p_z$), the surface being defined by the equation $E^2 - p_x^2 - p_y^2 - p_z^2 = m^2$

# Radiative corrections

## Problem No. 1 (resolved)

Note that one can add extra virtual lines to any of the diagrams of the previous page without changing the process from the point of view of an observer. Here are just a few additional diagrams for the Bhabha scattering:



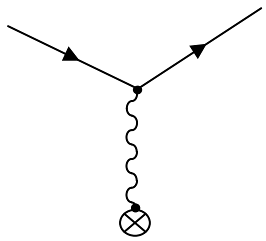

All these diagrams still represent $e^+e^- \rightarrow e^+e^-$ and, therefore, their contributions should be added in calculations of the Bhabha scattering amplitude. Note that these sub-processes contain two extra vertices and their contributions to the matrix element should be of the $\alpha^2$-order, i.e., should contribute at the level of (*some numerical factor)*$\times$(*1/*137) in comparison to the two 1$^{st}$ order in $\alpha$ diagrams from the previous page. The diagrams from the previous page are often called tree-level or leading-order (LO) or 1$^{st}$ order in $\alpha$ (for matrix element), while the diagrams shown above are NLO corrections, or 2$^{nd}$ order in $\alpha$. If one adds more loops, the power of $\alpha$ in the corresponding matrix element becomes larger and larger and their contributions are called n$^{th}$ order corrections.

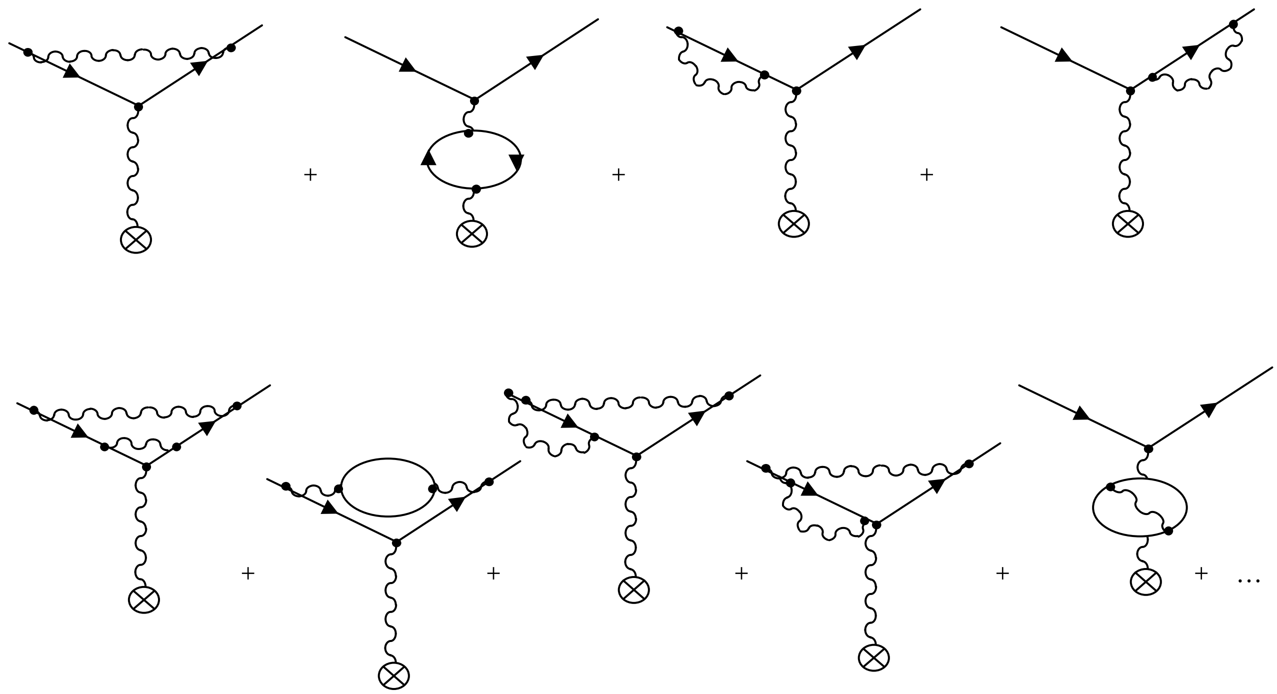
The problem arises from the fact the momenta of virtual particles in the loops are not constrained by the kinematics of the incoming and outgoing particles. Indeed, in the diagram A above, the virtual photon has the energy-momentum q set by the energy-momenta of the incoming and outgoing electrons; however, once it splits into a virtual "bubble" of an electron and a positron, one has a full freedom of assigning energy-momentum to one of them ($q_1$) as long as the other one takes care of the balance. Therefore, all these contributions with different $q_1$'s should be accounted for in calculations, which leads to summation/integration over all possible momenta in the loops and, as it turns out, these integrals are divergent at the limit of very large momenta—the mentioned above "*some numerical factor*" is infinity!

This problem plagued the development of the quantum field theory throughout 1930s and most of 1940s until **Tomanaga**, **Schwinger**, and **Feynman** independently found a way to fix it by 1947. We will discuss it later.

### Problem No. 2 (no solution)

One can make an interesting observation that the number of diagrams of large order grows very fast. Indeed, consider a scattering of an electron in Coulomb's field of very heavy nucleus:





It can be shown that the number of diagrams grows as factorial of the order number: n!

Each of them contributes at the level of $\alpha^n$, and their sum is proportional to $n!\alpha^n$—and this is the problem: for any small $\alpha$, there will be a number n for which $n!\alpha^n$ becomes larger than 1 and our nice perturbation theory may become invalid[2]. For QED it should happen at n>100. This is so far away from what we can possibly compute that the problem is often just ignored... provided that the finite order calcualtions continue to work… This problem is intrinsic to our choice to work in the framework of perturbation theory—and this, unfortunately, is about all we can do for now.

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[2] One can imagine that different signs of individual contributions may affectively prevent an "explosion". For example, series -1+1-1+1-1+1-1+…+n does not explode. However, there is not a proof that something like is actually guaranteed to happen in a perturbative expansion.