Summary

1) Vector calculus (basis, component rep.) — scalar product $\vec{a}\cdot\vec{b}$, cross product $\vec{a}\times\vec{b}$ $\Big\}$ higher products

2) Differential calculus $\frac{d}{dt}\vec{r}(t)$

$\vec{\nabla}\varphi(\vec{r}) \qquad d\varphi = \sum_i \frac{\partial\varphi}{\partial x_i}dx_i$

3) Matrices : $C = A\,B$, $\det A$, A^{-1}

rotation matrix D

4) Coordinate transformations : polar, spherical

integration : $dV = r^2 \sin\vartheta \, d\vartheta \, d\varphi \, dr$

5) Mechanics of the mass point

$\vec{F} = \dot{\vec{p}}$ (inertial system)

differential equations

$\ddot{x} + \lambda x = 0 \qquad$ ansatz $x(t) = e^{\alpha t}, \alpha \in \mathbb{C}$

→ harmonic oscillator

→ kepler problem, 1D motion

6) Theorems

Energy conservation, conservative force $\vec{F} = -\vec{\nabla}V$

Angular momentum $\frac{d}{dt}\vec{L} = \vec{M}$, $\vec{L} = \text{const}$ (central force)

7) Many particle systems

$m_i \ddot{\vec{r}}_i = \vec{F}_i \qquad$ conservation laws

$N = 2 \qquad \mu\,\ddot{\vec{r}} = \vec{F}_{12}$

rigid body $\quad r_{ij} = \text{const} \qquad T = \frac{1}{2}\vec{\omega}\,\underline{J}\,\vec{\omega}$ — tensor

$\vec{L} = \underline{J}\,\vec{\omega}$

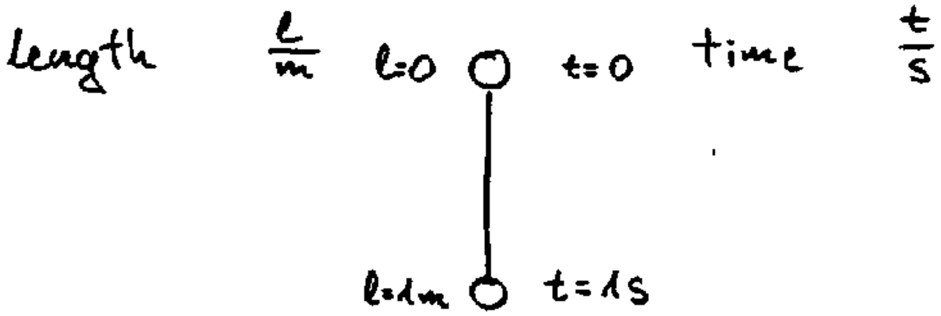
8) Lagrange mechanics

constraints, generalized coordinates

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \qquad L = T - V$$

Physical quantities

Experiment: ball falls down



3 specifications:

dimension	unit of measure	coefficient of measure
length	meter (m)	1
time	second (s)	1
mass	kilogramm (kg)	75
temperature	degrees celsius (°C)	22

→ ......

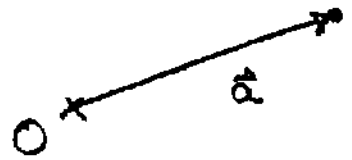
→ quantities that are fixed by dimension, unit of measure and one coeficient of measure are <u>scalars</u>.

→ quantities that aditionally need the specification of a direction are vectors.

→ quantities that need the specification of n directions are tensors (n=0 : scalar, n=1 vector)

Example of a vector: position vector $\vec{a}$

need to define a coordinate system with origin O



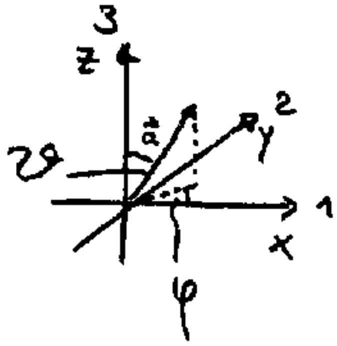
typically $\vec{a}$: vector in the Euclidian space E_3

length of $\vec{a}$: $a = |\vec{a}|$

direction of $\vec{a}$: $\hat{a} = \dfrac{\vec{a}}{|\vec{a}|}$

Coordinate system:

cartesian coordinate system
(right handed)



direction given by two angles: ϑ, φ

<u>Elementary mathematical operations</u>

Notes a) Vectors with the same length and direction are identical

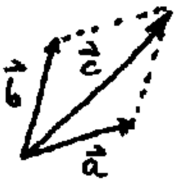
b) $\forall\ \vec{a}\ \exists\ -\vec{a}$ $\qquad \vec{a} \uparrow\downarrow -\vec{a}$

"antiparallel vector" $\qquad a = |\vec{a}| = |-\vec{a}|$

c) unitary vector $\vec{e}$, $|\vec{e}| = 1$

example $|\hat{a}| = 1$

1) Addition of vectors

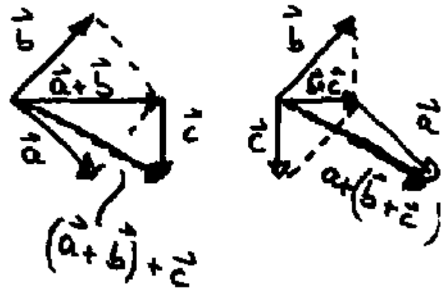


$$\vec{c} = \vec{a} + \vec{b}$$

a) Commutativity

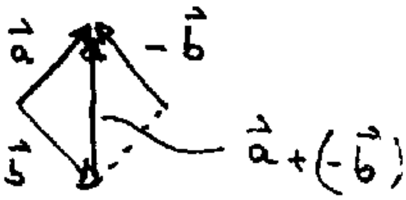
(1) $\vec{a} + \vec{b} = \vec{b} + \vec{a}$

b) Associativity



(2) $(\vec{a} + \vec{b}) + \vec{c} = \vec{a} + (\vec{b} + \vec{c})$

c) vector subtraction



(3) $\vec{a} - \vec{b} = \vec{a} + (-\vec{b})$

zero (null) vector

$$\vec{a} - \vec{a} = \vec{a} + (-\vec{a}) = 0$$

(only vector without "length" and direction)

(4) $\vec{a} + 0 = \vec{a}$

(1-4) : position vectors build a commutative group

2) Multiplication by a (real) number

$\alpha \in \mathbb{R}$, $\vec{a}$ an arbitrary vector

Definition $\alpha\vec{a}$ is a vector with the following proper

i) $$\alpha\vec{a} \begin{cases} \uparrow\uparrow \vec{a} & \text{if } \alpha > 0 \\ \downarrow\uparrow \vec{a} & \text{if } \alpha < 0 \end{cases}$$

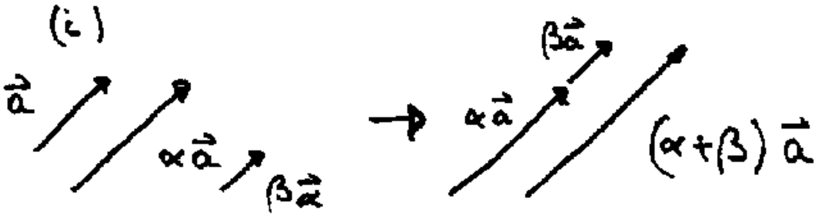
ii) $$|\alpha\vec{a}| = |\alpha| \; a$$

special cases
$$1\,\vec{a} = \vec{a}$$
$$0\,\vec{a} = 0$$
$$-1\,\vec{a} = -\vec{a}$$

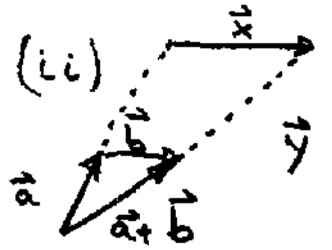
Calculation rules: $\alpha, \beta \in \mathbb{R}$, $\vec{a}, \vec{b} \ldots$ vectors

a) Distributivity
$$(\alpha+\beta)\,\vec{a} = \alpha\vec{a} + \beta\vec{a} \quad \text{(i)}$$
$$\alpha\,(\vec{a}+\vec{b}) = \alpha\vec{a} + \alpha\vec{b} \quad \text{(ii)}$$

(i)



(ii)



read off
$$\alpha\vec{a} + \vec{x} = \vec{y} \quad \text{(I)}$$
$$\vec{y} = \bar{\alpha}\,(\vec{a}+\vec{b}) \quad \text{(II)}$$
$$\vec{x} = \alpha'\,\vec{b} \quad \text{(III)}$$
(so far unknown constants $\bar{\alpha}, \alpha'$)

proof: 1. Intercept Theorem (lengths)
$$\frac{|\vec{y}|}{|\vec{a}+\vec{b}|} = \frac{|\alpha\vec{a}|}{|\vec{a}|}$$
$$\bar{\alpha} = \alpha$$

2. Intercept Theorem
$$\frac{|\vec{x}|}{|\vec{b}|} = \frac{|\alpha\vec{a}|}{|\vec{a}|}$$
$$\alpha' = \alpha$$
insert into definitions (I - III)

b) Associativity

$$\alpha(\beta\vec{a}) = (\alpha\beta)\vec{a} =: \alpha\beta\vec{a}$$

proof : use $|\alpha\beta| = |\alpha||\beta|$

c) construction of unit vector

$$\forall\ \vec{a} \in V\ \exists\ \hat{a} : \frac{1}{a}\vec{a} = \frac{1}{|\vec{a}|}\vec{a} = \hat{a}$$

1) $\hat{a} \uparrow\uparrow \vec{a}$

2) $|\hat{a}| = \left|\frac{1}{a}\vec{a}\right| = \left|\frac{1}{a}\right||\vec{a}| = \frac{1}{a}a = 1$

12.10.2016

Definition of a linear vector space V over the body of real numbers $\mathbb{R}$

1) Define addition : $\vec{a}, \vec{b} \in V \quad \vec{a}+\vec{b} = \vec{c}$, $c \in V$

a) associativity $(\vec{a}+\vec{b})+\vec{c} = \vec{a} + (\vec{b}+\vec{c})$

b) zero, null $0 \in V$: $\vec{a} + 0 = \vec{a} \quad \forall \vec{a}$

c) inverse element $\vec{a} \in V$, exists $-\vec{a} \in V$:

$$\vec{a} + (-\vec{a}) = 0$$

2) Define multiplication of a vector with elements $\alpha, \beta \in \mathbb{R}$

a) distributivity
$$(\alpha+\beta)\vec{a} = \alpha\vec{a} + \beta\vec{a}$$
$$\alpha(\vec{a}+\vec{b}) = \alpha\vec{a} + \alpha\vec{b}$$

b) associativity $\alpha(\beta\vec{a}) = (\alpha\beta)\vec{a}$

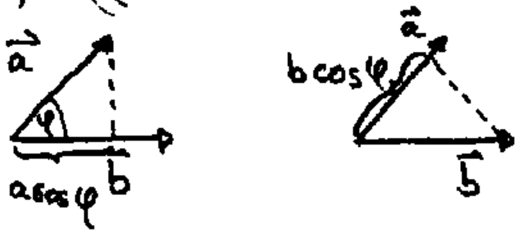
c) identity element $1 \cdot \vec{a} = \vec{a} \quad \forall \vec{a} \in V$

Scalar product (inner, dot product) of two vectors $\vec{a}, \vec{b}$

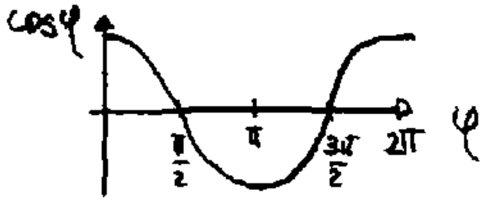
defined by : $\vec{a} \cdot \vec{b} = a\, b \cos\varphi$, φ : angle between $\vec{a}$ and $\vec{b}$

mathematically $\vec{a} \cdot \vec{b} \rightarrow \alpha \in \mathbb{R}$

graphically (for vectors in E_3)



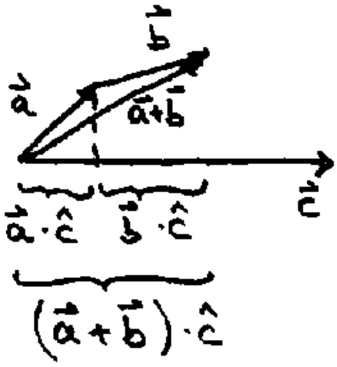
$\vec{a} \cdot \vec{b} = 0$ if
1) $a = 0$ and/or $b = 0$
2) $\varphi = \frac{\pi}{2}$ $(90°)$



Properties:

1) commutativity $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

2) distributivity $(\vec{a} + \vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}$



read off from graph

$(\vec{a} + \vec{b}) \cdot \hat{c} = \vec{a} \cdot \hat{c} + \vec{b} \cdot \hat{c} \quad | \cdot c$

$(\vec{a} + \vec{b}) \cdot \vec{c} = \vec{a} \cdot \vec{c} + \vec{b} \cdot \vec{c}$

3) bilinearity $(\alpha\, \vec{a}) \cdot \vec{b} = \vec{a} \cdot (\alpha\, \vec{b}) = \alpha (\vec{a} \cdot \vec{b})$

$\alpha > 0$ $(\alpha\, \vec{a}) \cdot \vec{b} = (\alpha a)\, b \cos\varphi = \alpha\, (a b \cos\varphi) = \alpha\, \vec{a} \cdot \vec{b}$

$= a\, (\alpha b \cos\varphi) = \vec{a} \cdot (\alpha \vec{b})$

$\alpha < 0$: use that $\sphericalangle(-\vec{a},\vec{b}) = \pi - \varphi$

$$(\alpha\ \vec{a})\cdot\vec{b} = |\alpha|\, a\, b\ \cos(\pi - \varphi) = -|\alpha|\, a\, b\ \cos\varphi$$
$$= \alpha\ a\, b\ \cos\varphi = \alpha\ (a\, b \cos\varphi) = \alpha(\vec{a}\cdot\vec{b})$$
$$= a\ (\alpha\, b\cos\varphi) = \vec{a}\cdot(\alpha\vec{b})$$

4) magnitude (norm) of a vector

$$\vec{a}\cdot\vec{a} = a\, a\ \cos 0 = a^2 \geq 0$$

$\cos 0 = 1$

use: $\sqrt{\vec{a}\cdot\vec{a}} = a$ to calculate magnitude

$$\vec{a}\cdot\vec{a} = 0 \quad \Leftrightarrow \quad \vec{a} = 0$$
$$\hat{a}\cdot\hat{a} = 1 \quad \text{unitary vector}$$

5) Schwarz's inequality : $|\vec{a}\cdot\vec{b}| \leq a\, b$

clear from $|\cos\varphi| \leq 1$

use (1-4) for a proof :

if $\vec{a} = 0$ or $\vec{b} = 0$: trivial

now consider two nonzero vectors

$$0 \leq (\vec{a} + \alpha\vec{b})\cdot(\vec{a} + \alpha\vec{b}) = a^2 + \alpha^2 b^2 + \alpha\,\vec{b}\cdot\vec{a} + \alpha\,\vec{a}\cdot\vec{b}$$
$$= a^2 + \alpha^2 b^2 + 2\alpha\ \vec{a}\cdot\vec{b}$$

choose $\alpha = -\dfrac{\vec{a}\cdot\vec{b}}{b^2}$

$$0 \leq a^2 + \frac{(\vec{a}\cdot\vec{b})^2 b^2}{b^4} - \frac{2(\vec{a}\cdot\vec{b})^2}{b^2} \quad \Big|\cdot b^2$$

$$0 \leq a^2 b^2 - (\vec{a}\cdot\vec{b})^2$$

6) triangle inequality

$$|a-b| \leq |\vec{a}+\vec{b}| \leq a+b$$

use Schwarz's inequality :

$$-ab \leq \vec{a}\cdot\vec{b} \leq a\cdot b$$

$$a^2+b^2-2ab \leq a^2+b^2+2\vec{a}\cdot\vec{b} \leq a^2+b^2+2ab$$

$$(a-b)^2 \leq (\vec{a}+\vec{b})^2 \leq (\vec{a}+\vec{b})^2$$

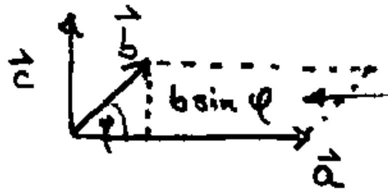
$$|a-b| \leq |\vec{a}+\vec{b}| \leq |a+b| = a+b \quad ✓$$

Vector (outer, cross) product

$$\vec{a},\vec{b},\vec{c} \in E_3 \quad , \quad \vec{a}\times\vec{b} = \vec{c}$$

Properties

a) magnitude $\quad c = ab\sin\varphi \quad , \quad \sphericalangle(\vec{a},\vec{b}) = \varphi$



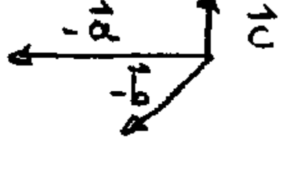
area of parallelogram spanned by $\vec{a}$ and $\vec{b}$

b) direction : $\vec{c}$ is oriented perpendicular to the plane defined by $\vec{a}$ and $\vec{b}$

$(\vec{a},\vec{b},\vec{c})$ build a right-handed coordinate system

Note : $\vec{a},\vec{b}$: ordinary vector ; polar vector
$\vec{c}$: axial vector (does not change direction under space inversion

$$(-\vec{a})\times(-\vec{b}) = \vec{c}$$



Properties of the cross product:

a) anticommutative $\vec{a} \times \vec{b} = - \vec{b} \times \vec{a}$

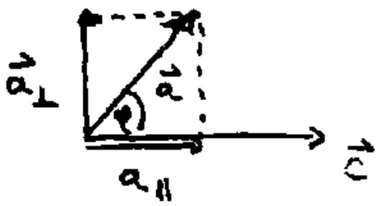
b) $\vec{a} \times \vec{b} = 0$ if 1) $a = 0$ and/or $b = 0$
2) $\vec{b} = \alpha \vec{a}$
(clear from $c = ab \sin \varphi$)

c) distributivity $(\vec{a} + \vec{b}) \times \vec{c} = \vec{a} \times \vec{c} + \vec{b} \times \vec{c}$

For the proof we need two steps

1) decompose $\vec{a}$ and $\vec{b}$ into components parallel and perpendicular to $\vec{c}$

$$\vec{a} = \vec{a}_{\parallel} + \vec{a}_{\perp}$$



$$\vec{a}_{\perp} \times \vec{c} = \vec{a} \times \vec{c}$$

holds because 1) right hand rule (direction)
2) $|\vec{a}_\perp \times \vec{c}| = a_\perp c \sin \frac{\pi}{2} = a_\perp c$
$= (a \sin \varphi) c = a c \sin \varphi$
$= |\vec{a} \times \vec{c}|$

$\rightarrow$ Only perpendicular components contribute

without loss of generality

$\vec{a} \perp \vec{c}$, $\vec{b} \perp \vec{c}$

2) Special case of $\vec{a} \perp \vec{c}$ and $\vec{b} \perp \vec{c}$:

$\vec{a} \times \vec{c}$: a vector arizing from $\vec{a}$ by rotation around $\vec{c}$ of $\frac{\pi}{2}$ and with length $a c$

$\vec{b} \times \vec{c}$: same, but with length $b c$

$(\vec{a} + \vec{b}) \times \vec{c}$: same, but with length $|a + b| c$

from geometry it follows for the (just) rotated vectors

$$\frac{1}{c}(\vec{a}\times\vec{c}) + \frac{1}{c}(\vec{b}\times\vec{c}) = \frac{1}{c}\left[(\vec{a}+\vec{b})\times\vec{c}\right]$$

and with multiplication by c, the full proof.

d) bilinear for real numbers

$$(\alpha\,\vec{a})\times\vec{b} = \vec{a}\times(\alpha\vec{b}) = \alpha\,(\vec{a}\times\vec{b})$$

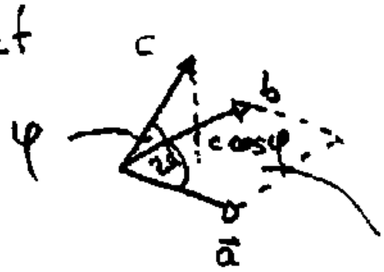
$\alpha > 0$ clear from the definition (magnitude, direction)
$\alpha < 0$: use the right hand rule to prove the direction properties

e) not associative

$$\vec{a}\times(\vec{b}\times\vec{c}) \neq (\vec{a}\times\vec{b})\times\vec{c}$$

Higher Vector products

1) scalar triple product



$$V = (\vec{a}\times\vec{b})\cdot\vec{c}$$
$$= F\cdot c\cos\varphi$$
$$= a\,b\,c\,\sin\vartheta\,\cos\varphi$$

(Volume of parallelepiped)

$F = |\vec{a}\times\vec{b}|$
$= a\,b\,\sin\vartheta$

cyclic interchanges

$$V = (\vec{a}\times\vec{b})\cdot\vec{c} = (\vec{b}\times\vec{c})\cdot\vec{a} = (\vec{c}\times\vec{a})\cdot\vec{b}$$

2) double vector product

$$\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b}) \quad \text{(expansion rule)}$$

3) Jacobi identity

$$\vec{a}\times(\vec{b}\times\vec{c}) + \vec{b}\times(\vec{c}\times\vec{a}) + \vec{c}\times(\vec{a}\times\vec{b}) = 0$$

Basis vectors and component representation

Note: a) separate length and direction by

$$\vec{a} = a\,\hat{a}$$

b) consider two collinear vectors $\vec{a}, \vec{b}$

$$\vec{a} = a\,\hat{a} = a\,\hat{b} = \frac{a}{b}\,b\,\hat{b} = \frac{a}{b}\,\vec{b}$$

$$\Leftrightarrow \; b\,\vec{a} - a\cdot\vec{b} = 0$$

$\vec{a}$ and $\vec{b}$ are linear dependend if $\exists\, \alpha, \beta$:

$$\alpha\,\vec{a} + \beta\,\vec{b} = 0$$

Definition: $\{\vec{a}_1 \dots \vec{a}_n\}$ Linearly independent if:

$$\sum_{j=1}^{n} \alpha_j \vec{a}_j = 0 \quad \Rightarrow \{\alpha_j\} = \{0, \dots, 0\}$$

Definition: Dimension of a vector space: maximal number of linearly independent vectors

Theorem: In a d-dimensional vector space each ensemble of d linear independent vectors build a basis, i.e. any other vector $\vec{b} \in V$ can be expressed as a linear combination of these d vectors.

best choice for basis: unitary vectors, pairwise orthogonal:

$$\vec{e}_i \cdot \vec{e}_j = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases} \quad \text{Kronecker-Delta}$$

$$\forall \vec{a} \in V: \quad \vec{a} = \sum_{j=1}^{d} a_j \vec{e}_j$$

(every vector can be expressed as linear combination of unitary vectors)

$\rightarrow a_i$: components of $\vec{a}$, projection on basis:

$$\vec{e}_i \cdot \vec{a} = \sum_{j}^{d} a_j \, \vec{e}_i \cdot \vec{e}_j = \sum_{j=1}^{d} a_j \, \delta_{ij} = a_i$$

column vector　　　　row vector

$$\vec{a} = \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{pmatrix} \qquad \vec{a} = (a_1, a_2, \dots a_d)$$

Example: Basis of E_3 $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$

$$\vec{a} = a_1 \vec{e}_1 + a_2 \vec{e}_2 + a_3 \vec{e}_3$$

$$a_i = \vec{a} \cdot \vec{e}_i = a \cdot \cos \vartheta_i \qquad \text{(directional cosine)}$$

magnitude : $a = \sqrt{\vec{a}\cdot\vec{a}} = \sqrt{\sum_i a_i \vec{e}_i \cdot \sum_j a_j \vec{e}_j}$

$$= \sqrt{\sum_{ij} a_i a_j \vec{e}_i \cdot \vec{e}_j} = \sqrt{\sum_j a_j^2}$$

$d = 3$

$$a = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

$$\Rightarrow 1 = \sqrt{\frac{a_1^2}{a^2} + \frac{a_2^2}{a^2} + \frac{a_3^2}{a^2}} = \sqrt{\cos^2\vartheta_1 + \cos^2\vartheta_2 + \cos^2\vartheta_3}$$

$$\cos\vartheta_1 = \sqrt{1 - \cos^2\vartheta_2 - \cos^2\vartheta_3}$$

Component representation

a) special vectors $\quad \vec{0} = (0,0,0)$ null vector

$$\left.\begin{aligned} \vec{e}_1 &= (1,0,0) \\ \vec{e}_2 &= (0,1,0) \\ \vec{e}_3 &= (0,0,1) \end{aligned}\right\} \text{Basis vectors}$$

b) Addition $\quad \vec{c} = \vec{a} + \vec{b} = \sum_i a_i \vec{e}_i + \sum_i b_i \vec{e}_i$

$$= \sum_i (a_i + b_i)\vec{e}_i = \sum_i c_i \vec{e}_i$$

$$\Rightarrow c_i = a_i + b_i$$

$$\vec{c} = (a_1 + b_1,\ a_2 + b_2,\ a_3 + b_3)$$

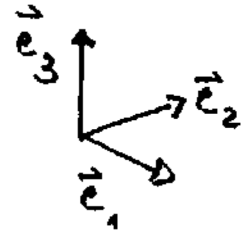
c) Multiplication with real numbers

$$\vec{b} = \alpha\vec{a} = \alpha\sum_{i=1}^{d} a_i \vec{e}_i = \sum_i \alpha a_i \vec{e}_i$$

$$b_i = \alpha a_i \quad , \quad \vec{b} = (\alpha a_1, \alpha a_2, \alpha a_3)$$

d) Scalar product $\quad \vec{a}\cdot\vec{b} = \sum_{ij} a_i b_i \underbrace{\vec{e}_i \cdot \vec{e}_j}_{\delta_{ij}} = \sum_i a_i b_i$

e) Vector product : right handed coordinate system
orthogonal basis vectors



$$\vec{e}_1 \times \vec{e}_2 = \vec{e}_3$$
$$\vec{e}_2 \times \vec{e}_3 = \vec{e}_1$$
$$\vec{e}_3 \times \vec{e}_1 = \vec{e}_2$$

$$\vec{e}_i \cdot (\vec{e}_j \times \vec{e}_k) = \begin{cases} 1 & \text{if } (i,j,k) \text{ cyclic permutation of } (1,2,3) \\ -1 & \text{if } (i,j,k) \text{ antcyclic permutation of } (1,2,3) \\ 0 & \text{otherwise} \end{cases}$$

$\epsilon_{ijk} = \vec{e}_i \cdot (\vec{e}_j \times \vec{e}_k)$ fully antisymmetric tensor of third rank

$\vec{e}_i \times \vec{e}_j = \sum_{k=1}^{3} \epsilon_{ijk} \vec{e}_k$ (to be checked from relations above)

consider:

$$\vec{e}_\ell \cdot (\vec{e}_i \times \vec{e}_j) = \vec{e}_\ell \cdot \sum_k \epsilon_{ijk} \vec{e}_k = \sum_k \epsilon_{ijk} \underbrace{\vec{e}_\ell \cdot \vec{e}_k}_{\delta_{\ell k}} = \epsilon_{ij\ell}$$
$$= -\epsilon_{i\ell j} = \epsilon_{\ell ij}$$

(antisymmetric)

$\epsilon_{iik} = \epsilon_{iki} = \epsilon_{kii} = 0 \quad \forall \; i,k$

convolution

$$\sum_j \epsilon_{ikj} \epsilon_{j\ell m} = \delta_{i\ell} \delta_{km} - \delta_{im} \delta_{k\ell}$$

$$\vec{c} = \vec{a} \times \vec{b} = \sum_{ij} a_i b_j \; \vec{e}_i \times \vec{e}_j = \sum_{ijk} a_i b_j \, \epsilon_{ijk} \vec{e}_k$$

$$c_k = \sum_{ij=1}^{3} a_i b_j \epsilon_{ijk} \qquad \vec{a} \times \vec{b} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

f) Scalar Triple Product

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \sum_{ijk} a_i b_j c_k \, \vec{e}_i \cdot (\vec{e}_j \times \vec{e}_k)$$

$$= \sum_{ijk} a_i b_j c_k \, \epsilon_{ijk}$$

g) Double Vector Product

$$\left[\vec{a}\times(\vec{b}\times\vec{c})\right]_k = \sum_{ij} a_i \underbrace{\left[\vec{b}\times\vec{c}\right]_j}_{\left[\vec{b}\times\vec{c}\right]_j = \sum_{lm} b_l c_m \epsilon_{lmj}} \epsilon_{ijk}$$

$$= \sum_{ijlm} a_i b_l c_m \, \epsilon_{lmj} \epsilon_{ijk}$$

$$= \sum_{ilm} a_i b_l c_m \sum_j \epsilon_{lmj} \underbrace{\epsilon_{ijk}}_{+\epsilon_{jki}}$$

$$= \sum_{ilm} a_i b_l c_m \left(\delta_{lk}\delta_{mi} - \delta_{li}\delta_{mk}\right)$$

$$= \sum_i a_i b_k c_i - \sum_i a_i b_i c_k$$

$$= b_k (\vec{a}\cdot\vec{c}) - c_k (\vec{a}\cdot\vec{b})$$

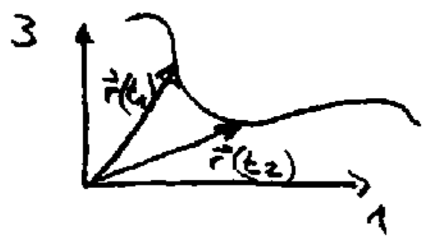
$$\vec{a}\times(\vec{b}\times\vec{c}) = \vec{b}\,(\vec{a}\cdot\vec{c}) - \vec{c}\,(\vec{a}\cdot\vec{b})$$

Vector valued functions : $\vec{r}(t)$

$t \in N : t \longmapsto \vec{r}(t) \in M$

$N \subset \mathbb{R}$ $\quad M \subset \mathbb{R}^d$

example : space curve

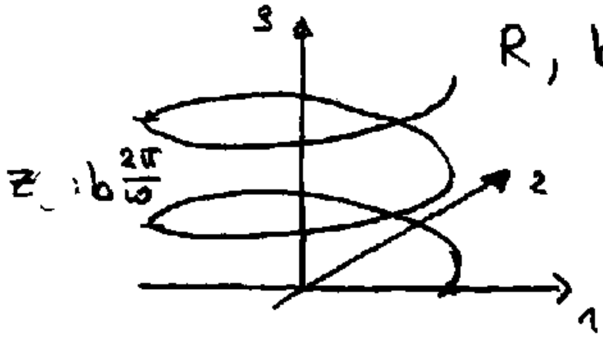


$$\vec{r}(t) = \sum_{i=1}^{3} r_i(t)\, \vec{e}_i$$

$\{\hat{e}_i\}$ time independend orthogonal basis

Example : Helical line

$$\vec{r}(t) = \big(R\cos(\omega t),\ R\sin(\omega t),\ b\,t\big)$$



R, b, ω : constants

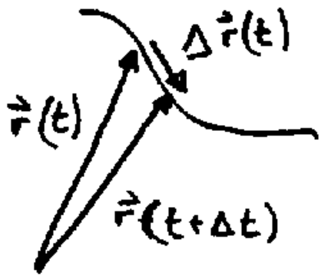
z_0 pitch of skrew

Continuity of path lines (compare continuity of functions $\mathbb{R} \to \mathbb{R}$)

$\vec{r}(t)$ is contious at t_0 if for each $\epsilon > 0$ exists a $\delta(\epsilon, t_0)$ such that for $|t - t_0| < \epsilon$ is always valid $|\vec{r}(t) - \vec{r}(t_0)| < \epsilon$

$\Rightarrow$ $\vec{r}(t)$ is continous if all component functions are continous in the ordinary sense.

Differentiation of Vector-valued functions



$$\Delta \vec{r}(t) = \vec{r}(t+\Delta t) - \vec{r}(t)$$

derivative (of position vector)

$$\dot{\vec{r}}(t) = \frac{d\vec{r}(t)}{dt} = \lim_{\Delta t\to 0} \frac{\Delta\vec{r}(t)}{\Delta t}$$

$$= \lim_{\Delta t\to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$$

$$= \vec{v}(t) \quad \text{(velocity)}$$

higher derivatives

$$\frac{d^n\vec{r}(t)}{dt^n} = \sum_{i=1}^{3} \frac{d^n r_i(t)}{dt^n}\,\vec{e}_i \qquad \text{(component notation)}$$

$$\ddot{\vec{r}}(t) = \dot{\vec{v}}(t) = \vec{a}(t) \qquad \text{(acceleration)}$$

Calculation rules (follow immediately from component representation)

1) $\frac{d}{dt}\left[\vec{a}(t) + \vec{b}(t)\right] = \frac{d\vec{a}(t)}{dt} + \frac{d\vec{b}(t)}{dt} = \dot{\vec{a}}(t) + \dot{\vec{b}}(t)$

2) $\frac{d}{dt}\left[f(t)\,\vec{a}(t)\right] = \frac{df(t)}{dt}\,\vec{a}(t) + f(t)\,\frac{d\vec{a}(t)}{dt}$

3) $\frac{d}{dt}\left[\vec{a}(t)\cdot\vec{b}(t)\right] = \dot{\vec{a}}(t)\cdot\vec{b}(t) + \vec{a}(t)\cdot\dot{\vec{b}}(t)$

4) $\frac{d}{dt}\left[\vec{a}(t)\times\vec{b}(t)\right] = \dot{\vec{a}}(t)\times\vec{b}(t) + \vec{a}(t)\times\dot{\vec{b}}(t)$

Note: From 3 follows: (unitary vector) $0 = \frac{d}{dt}\left[\hat{a}(t)\cdot\hat{a}(t)\right] = 2\,\dot{\hat{a}}(t)\cdot\hat{a}(t)$

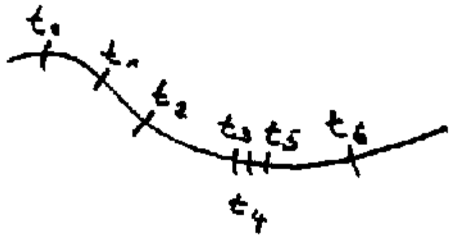
$$\hat{a}(t) \perp \frac{d\hat{a}(t)}{dt}$$

Arc Length

Definition : smooth space curve, if there exists at least one continously differentiable curve $\vec{r}(t)$ for which we have no point with

$$\frac{d\vec{r}(t)}{dt} = 0$$

→ use "arc length" as parametrization



$$L_N(t_a, t_b) = \sum_{n=0}^{N-1} |\vec{r}(t_{n+1}) - \vec{r}(t_n)|$$

$$= \sum_{n=0}^{N-1} \frac{|\vec{r}(t_{n+1}) - \vec{r}(t_n)|}{\Delta t} \Delta t$$

with $\lim\limits_{\Delta t \to 0} \dfrac{\vec{r}(t_{n+1}) - \vec{r}(t_n)}{\Delta t} = \left.\dfrac{d\vec{r}}{dt}\right|_{t=t_n}$

sum becomes Riemann Integral

$$s(t_b) = \int_{t_a}^{t_b} \left|\frac{d\vec{r}(t')}{dt}\right| dt' = \int_{t_a}^{t_b} dt' \, |\vec{v}(t')|$$

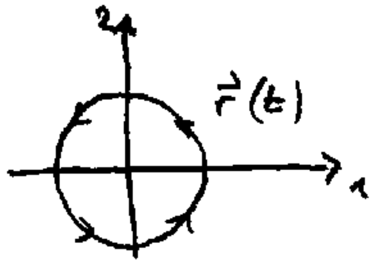
consider :

$$\frac{ds}{dt} = \left|\frac{d\vec{r}}{dt}\right| > 0$$

$s(t)$ monotonously increasing, can be inverted $t(s)$

$$\vec{r}(t) = \vec{r}(t(s)) = \vec{r}(s)$$

Example : Circular motion



$$\vec{r}(t) = R\,(\cos\omega t, \sin\omega t, 0)$$

$$\frac{d\vec{r}}{dt} = R\omega\,(-\sin(\omega t), \cos\omega t, 0)$$

$$\left|\frac{d\vec{r}}{dt}\right| = R\omega \quad , \quad s(t) = \int_0^t R\omega\, dt = R\omega t$$

$$\Rightarrow \quad t(s) = \frac{s}{R\omega}$$

$$\vec{r}(s) = R\left(\cos\left(\frac{s}{R}\right), \sin\left(\frac{s}{R}\right), 0\right)$$

20.10.2016

Fields

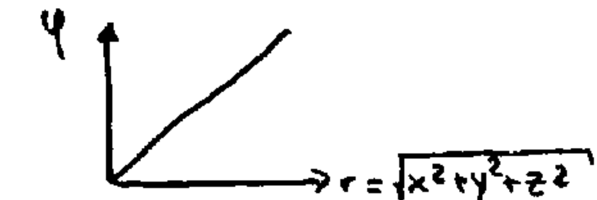
Examples : gravitational field (static)

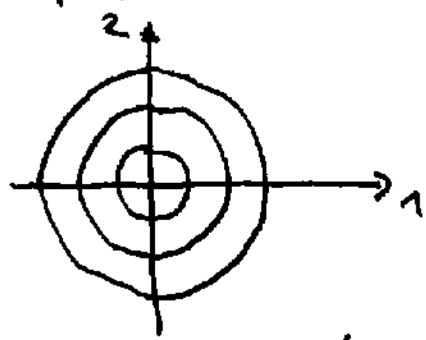
electic field
magnetic field $\Big\}$ charged particle (can be dynamic)

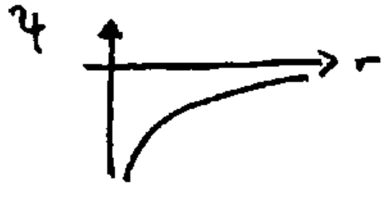
1) Scalar field $A(\vec{r})$ is a scalar

$$M \subset \mathbb{R}^3 \longrightarrow N \subset \mathbb{R}$$
$$\vec{r} \rightarrow A$$

examples a) $\varphi(\vec{r}) = \alpha r = \alpha |\vec{r}|$







b) $\psi(\vec{r}) = -\frac{\beta}{r}$

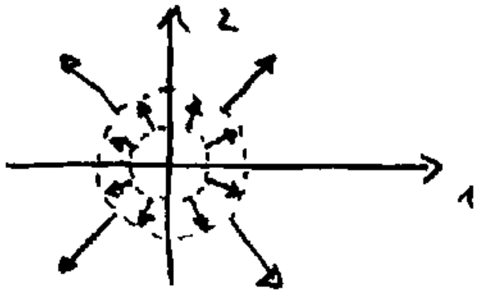
contour lines

2) Vector fields $\vec{A}(\vec{r})$

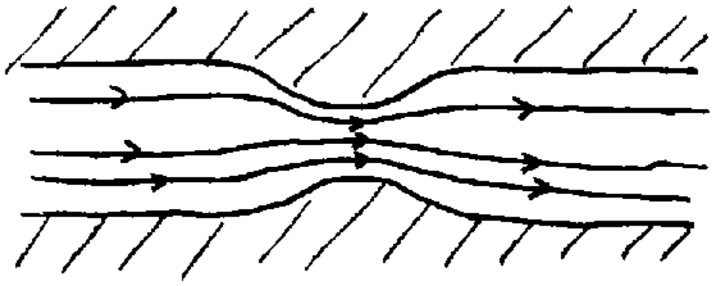
$$M \subset \mathbb{R}^3 \longrightarrow N \subset \mathbb{R}^3$$
$$\vec{r} \longrightarrow \vec{A}$$

examples:

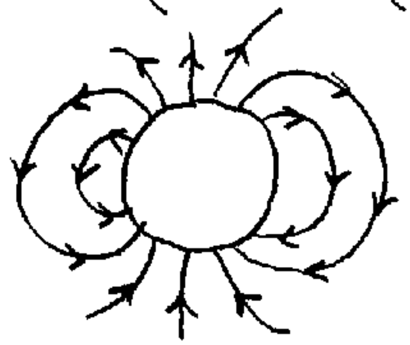
a) $\vec{A}(\vec{r}) = \alpha \vec{r}$
$\alpha > 0$



b) velocity field of a liquid:



c) magnetic field of the earth



Continuity:

1) scalar field is continous $\underbrace{\text{if}}^{\text{at } \vec{r}_0}$ for each ~~ϵ~~ exists $\delta > 0$ such that for all $\vec{r}$ with $|\vec{r} - \vec{r}_0| < \delta$ holds $|\varphi(\vec{r}) - \varphi(\vec{r}_0)| < \epsilon$

2) vector field is continous at $\vec{r}_0$ if all components are continous at $\vec{r}_0$: $\vec{A}(\vec{r}) = \sum_i A_i \vec{e}_i$

Partial derivatives $\varphi(x_1, x_2, x_3)$

derivative along path (holding other variables fixed)

$$\left.\frac{\partial \varphi}{\partial x_1}\right|_{x_2,x_3} = \lim_{\Delta x_1 \to 0} \frac{\varphi(x_1+\Delta x_1, x_2, x_3) - \varphi(x_1, x_2, x_3)}{\Delta x_1}$$

$$= \partial_{x_1} \varphi = \partial_1 \varphi$$

$$\left.\frac{\partial \varphi}{\partial x_2}\right|_{x_1,x_3} = \lim_{\Delta x_2 \to 0} \frac{\varphi(x_1, x_2+\Delta x_2, x_3) - \varphi(x_1, x_2, x_3)}{\Delta x_2}$$

Examples: $\varphi(\vec{r}) = r$

$$\varphi(x_1, x_2, x_3) = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

$$\frac{\partial \varphi}{\partial x_j} = \frac{\partial}{\partial x_j}\sqrt{x_1^2 + x_2^2 + x_3^2} = \frac{2x_j}{2r} = \frac{x_j}{r}$$

$$\vec{a}(\vec{r}) = \alpha \vec{r}$$

$$\frac{\partial \vec{a}}{\partial x_j} = \alpha \frac{\partial}{\partial x_j}\left(x_1\vec{e}_1 + x_2\vec{e}_2 + x_3\vec{e}_3\right) = \alpha\, \vec{e}_j$$

Calculation rules

$$\partial_i (\varphi_1 + \varphi_2) = \partial_i \varphi_1 + \partial_i \varphi_2$$
$$\partial_i (\vec{a} \cdot \vec{b}) = (\partial_i \vec{a}) \cdot \vec{b} + \vec{a} \cdot (\partial_i \vec{b})$$
$$\partial_i (\vec{a} \times \vec{b}) = (\partial_i \vec{a}) \times \vec{b} + \vec{a} \times (\partial_i \vec{b})$$

Multiple partial derivatives

$$\frac{\partial^2 \varphi}{\partial x_i^2} = \frac{\partial}{\partial x_i} \frac{\partial \varphi}{\partial x_i} \quad , \quad \frac{\partial^n \varphi}{\partial x_i^n} = \frac{\partial}{\partial x_i}\left(\frac{\partial^{n-1}\varphi}{\partial x_i^{n-1}}\right)$$

mixed partial derivatives

continous partial derivatives to second order

$$\frac{\partial^2 \varphi}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial \varphi}{\partial x_j}\right) = \frac{\partial}{\partial x_j}\left(\frac{\partial \varphi}{\partial x_i}\right) = \frac{\partial^2 \varphi}{\partial x_j \partial x_i}$$

Total differential of a function

Note: chain rule for ordinary function $f[x(t)]$

$$\frac{df}{dt} = \frac{df}{dx}\frac{dx}{dt}$$

Similar for different parameters t_i $\quad \varphi(x_1(t_1), x_2(t_2), x_3(t_3))$

$$\frac{d\varphi}{dt_1} = \frac{d\varphi}{\partial x_1}\frac{dx_1}{dt_1}$$

Usually: all components depend on the <u>same</u> parameter t

$$\varphi(\vec{r}(t)) = \varphi(x_1(t), x_2(t), x_3(t))$$

difference quotient

$$D = \frac{\varphi(x_1(t+\Delta t), x_2(t+\Delta t), x_3(t+\Delta t)) - \varphi(x_1(t), x_2(t), x_3(t))}{\Delta t}$$

$$= \frac{1}{\Delta t}\Big[\varphi(x_1(t+\Delta t), x_2(t+\Delta t), x_3(t+\Delta t)) - \varphi(x_1(t), x_2(t+\Delta t), x_3(t+\Delta t))$$

$$+ \varphi(x_1(t), x_2(t+\Delta t), x_3(t+\Delta t)) - \varphi(x_1(t), x_2(t), x_3(t+\Delta t))$$

$$+ \varphi(x_1(t), x_2(t), x_3(t+\Delta t)) - \varphi(x_1(t), x_2(t), x_3(t))\Big]$$

$$D = \frac{1}{\Delta x_1}\left[\varphi(x_1+\Delta x_1, x_2+\Delta x_2, x_3+\Delta x_3) - \varphi(x_1, x_2+\Delta x_2, x_3+\Delta x_3)\right] \times \frac{\Delta x_1}{\Delta t}$$

$$+ \frac{1}{\Delta x_2}\left[\varphi(x_1, x_2+\Delta x_2, x_3+\Delta x_3) - \varphi(x_1, x_2, x_3+\Delta x_3)\right]\frac{\Delta x_2}{\Delta t}$$

$$+ \frac{1}{\Delta x_3}\left[\varphi(x_1, x_2, x_3+\Delta x_3) - \varphi(x_1, x_2, x_3)\right]\frac{\Delta x_3}{\Delta t}$$

take limit

$$\lim_{\Delta t \to 0} D = \frac{\partial \varphi}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial \varphi}{\partial x_2}\frac{dx_2}{dt} + \frac{\partial \varphi}{\partial x_3}\frac{dx_3}{dt}$$

$$= \frac{d\varphi}{dt} \quad \text{(total derivative)}$$

$$d\varphi = \sum_{i=1}^{3} \frac{\partial \varphi}{\partial x_i} dx_i \quad \text{(total differential of } \varphi\text{)}$$

Gradient, Divergence and Curl (Rotation)

$$\vec{\nabla}\varphi(\vec{r}) = \left(\frac{\partial \varphi}{\partial x_1}, \frac{\partial \varphi}{\partial x_2}, \frac{\partial \varphi}{\partial x_3}\right) = \sum_{j=1}^{3} \frac{\partial \varphi}{\partial x_j}\vec{e}_j$$

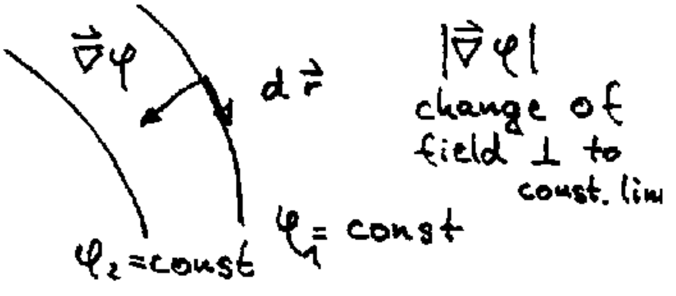
$$\vec{\nabla} = \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right) \quad \text{Nabla operator}$$

rewrite total differential:

$$d\varphi = \vec{\nabla}\varphi \cdot d\vec{r} \qquad d\vec{r} = \sum_i dx_i \vec{e}_i$$

$d\varphi = 0$ along lines of constant field

$\Rightarrow \vec{\nabla}\varphi \perp d\vec{r}$



$$\vec{\nabla}(\varphi_1+\varphi_2) = \vec{\nabla}\varphi_1 + \vec{\nabla}\varphi_2$$

$$\vec{\nabla}(\varphi_1\varphi_2) = (\vec{\nabla}\varphi_1)\varphi_2 + \varphi_1(\vec{\nabla}\varphi_2) = \varphi_2\vec{\nabla}\varphi_1 + \varphi_1\vec{\nabla}\varphi_2$$

Examples

$$\vec{\nabla}(\vec{a}\cdot\vec{r}) = \vec{\nabla}\sum_j a_j x_j = \sum_{ij} a_j \vec{e}_i \underbrace{\frac{\partial x_i}{\partial x_j}}_{\delta_{ij}} = \sum_i a_i \vec{e}_i = \vec{a}$$

$$\vec{\nabla} r = \sum_i \vec{e}_i \frac{\partial r}{\partial x_i} = \sum_i \vec{e}_i \frac{x_i}{r} = \frac{1}{r}\sum_i x_i \vec{e}_i = \hat{r}$$

$$\vec{\nabla}\cdot\vec{a}(\vec{r}) = \sum_j \frac{\partial a_i}{\partial x_i} = \operatorname{div}\vec{a}$$

divergence
(source field)

Example

$$\vec{\nabla}\cdot\vec{r} = \sum_i \frac{\partial x_i}{\partial x_i} = 3$$

$$\vec{\nabla}\cdot(\vec{a}+\vec{b}) = \vec{\nabla}\cdot\vec{a} + \vec{\nabla}\cdot\vec{b}$$

$$\vec{\nabla}\cdot(\alpha\vec{a}) = \alpha\,\vec{\nabla}\cdot\vec{a} \qquad \alpha: \text{constant}$$

$$\vec{\nabla}\cdot(\varphi\vec{a}) = \varphi\,\vec{\nabla}\cdot\vec{a} + \vec{a}\cdot\vec{\nabla}\varphi$$

Divergence of gradient field

$$\vec{\nabla}\cdot\vec{\nabla}\varphi = \operatorname{div}(\operatorname{grad}\varphi) = \sum_{i=1}^{3}\frac{\partial^2\varphi}{\partial x_i^2} = \Delta\varphi$$

(Laplace operator)

$$\operatorname{rot}\vec{a} = \vec{\nabla}\times\vec{a} = \sum_{ijk}\epsilon_{ijk}\,\partial_i\,a_j(\vec{r})\,\vec{e}_k$$

(curl or rotation
of field $\vec{a}$)

Calculation rules

$$\vec{\nabla}\times(\vec{a}+\vec{b}) = \vec{\nabla}\times\vec{a} + \vec{\nabla}\times\vec{b}$$
$$\vec{\nabla}\times(\alpha\,\vec{a}) = \alpha\,\vec{\nabla}\times\vec{a} \qquad \alpha : \text{const}$$
$$\vec{\nabla}\times(\varphi\,\vec{a}) = \varphi\,\vec{\nabla}\times\vec{a} + (\vec{\nabla}\varphi)\times\vec{a}$$
$$= \varphi\,\vec{\nabla}\times\vec{a} - \vec{a}\times\vec{\nabla}\varphi$$

$$\vec{\nabla}\times(\vec{\nabla}\varphi) = \sum_{ijk} \epsilon_{ijk}\,\partial_i(\partial_j\varphi)\,\vec{e}_k$$
$$= \sum_{ijk} \tfrac{1}{2}(\epsilon_{ijk}-\epsilon_{jik})\,(\partial_i\partial_j\varphi)\,\vec{e}_k$$

$\partial_i\partial_j\varphi = \partial_j\partial_i\varphi$

$$= \tfrac{1}{2}\sum_{ijk} \epsilon_{ijk}(\partial_i\partial_j\varphi)\vec{e}_k - \tfrac{1}{2}\sum_{ijk}\epsilon_{jik}(\partial_j\partial_i\varphi)\vec{e}_k$$
$$= 0 \qquad \text{(gradient fields are curl-free)}$$

$$\vec{\nabla}\cdot(\vec{\nabla}\times\vec{a}) = \sum_i \partial_i \sum_{jk}\epsilon_{jki}\,\partial_j a_k = \sum_{ijk}\epsilon_{jki}(\partial_i\partial_j a_k)$$
$$= \sum_{ijk}\tfrac{1}{2}\Big(\epsilon_{jki} - \epsilon_{\substack{i\\ \downarrow\\ j}\,k\,\substack{j\\ \downarrow\\ i}}\Big)\,\underbrace{(\partial_i\partial_j a_k)}_{=\partial_j\partial_i a_k}$$
$$= \sum_{ijk}\tfrac{1}{2}(\epsilon_{jki}-\epsilon_{jki})\,\partial_i\partial_j a_k = 0$$

(curl fields are source free)

$$\vec{\nabla}\times[f(r)\,\vec{r}\,] = \sum_{ijk}\epsilon_{ijk}\,\partial_i[f(r)\,x_j]\,\vec{e}_k$$
$$= \sum_{ijk}\epsilon_{ijk}\left[f'(r)\frac{x_i x_j}{r} + f(r)\,\delta_{ij}\right]\vec{e}_k$$

$\epsilon_{ijk} = -\epsilon_{jik}$

$$= 0 \qquad \text{(central forces are curl free)}$$

Matrices and Determinants

Matrix : rectangular array of numbers

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \vdots \\ \vdots & & & \vdots \\ a_{m1} & & \cdots & a_{mn} \end{pmatrix} = (a_{ij})$$

Definition A and B are identical if $a_{ij} = b_{ij} \quad \forall\, i,j$

Special matrices :

1) quadratic matrix $n = m$
2) row vector $m = 1$
   column vector $n = 1$
3) zero matrix $a_{ij} = 0$
4) symmetric matrix $a_{ij} = a_{ji}$
5) diagonal matrix $a_{ij} = a_i \delta_{ij}$
   unitary matrix $a_{ij} = \delta_{ij}$, $A = \mathbb{1}$
6) transposed matrix A^T : $a^T_{ij} = a_{ji}$

Definition : rank of a matrix
maximal number of linear ly independent row / column vectors

Calculation rules

a) Addition $C = A + B$ $\quad c_{ij} = a_{ij} + b_{ij}$
b) multiplication with real numbers $B = \lambda A$, $b_{ij} = \lambda a_{ij}$

Matrix multiplication

A : $(m \times n)$ matrix
B : $(n \times r)$ matrix

$C = A\,B$ $(m \times r)$ matrix

$$c_{ij} = \sum_k a_{ik} b_{kj}$$

$$\begin{pmatrix} c_{11} & c_{12} & \cdots & c_{1r} \\ \vdots & & & \\ c_{m1} & \cdots & & c_{mr} \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & & & \\ a_{m1} & \cdots & & a_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & \cdots & b_{1r} \\ b_{21} & & \vdots \\ \vdots & & \vdots \\ b_{n1} & \cdots & b_{nr} \end{pmatrix}$$

special case $m = 1$, $r = 1$

$$(a_1, a_2 \ldots a_n) \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix} = \sum_{i=1}^{n} a_i b_i = \vec{a}^T \vec{b}$$

(scalar product)

Note: multiplication is not commutative

$$A\,B \neq B\,A$$

clear for $m \neq r$, not def
for $m = r$: product could be $(m \times m)$ or $(n \times n)$

inverse matrix :

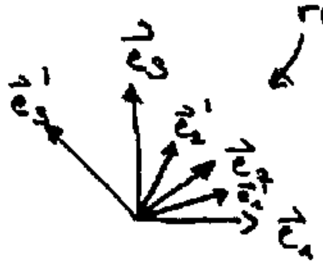
$$A^{-1} A = A\,A^{-1} = \mathbb{1} = \delta_{ij}$$

Transformation of coordinates :

two coordinate systems

Σ $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$
Σ' $\{\vec{e}_1{}', \vec{e}_2{}', \vec{e}_3{}'\}$

Σ and Σ' related by rotation



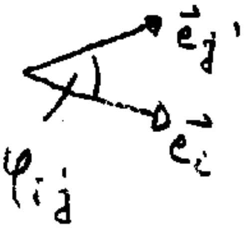
position vector identical in Σ and Σ'

$$\vec{r} = \sum_{i=1}^{3} x_i \vec{e}_i = \sum_{i=1}^{3} x_i' \vec{e}_i' \qquad | \cdot \vec{e}_j{}'$$

want to calculate components x_i'

$$x_i' = \sum_{j=1}^{3} x_j \; \vec{e}_j \cdot \vec{e}_i{}' \qquad\qquad \vec{e}_i{}' \cdot \vec{e}_j{}' = \delta_{ij}$$

$$= \sum_j x_j \; \cos \varphi_{ij}$$



$$= \sum_j d_{ij} \; x_j$$

$$\vec{r}_{\Sigma'} = D \; \vec{r}_{\Sigma}$$

D describes rotation

$$\vec{e}_i{}' = \sum_j d_{ij} \vec{e}_j$$

d_{ij} : components of $\vec{e}_i{}'$ in Σ

$$\delta_{ij} = \vec{e}_i{}' \cdot \vec{e}_j{}' = \sum_{km} d_{ik} d_{jm} \; \vec{e}_k \cdot \vec{e}_m$$

$$= \sum_k d_{ik} d_{jk}$$

(rows of D are orthonormalized)

Introduce inverse matrix D^{-1} with

$D^{-1} D = D D^{-1} = \mathbb{1}$, apply D^{-1} :

$$D^{-1} \vec{r}_{\Sigma'} = D^{-1} D \, \vec{r}_{\Sigma} = \mathbb{1} \, \vec{r}_{\Sigma} = \vec{r}_{\Sigma}$$

D^{-1} describes back rotation

Calculate components in Σ: multiply with $\vec{e}_j$

$$x_i = \sum_{j=1}^{3} x_j' \underbrace{\vec{e}_j' \cdot \vec{e}_i}_{\vec{e}_i \cdot \vec{e}_j' = \cos\varphi_{ji} = d_{ji}} = \sum_{j=1}^{3} d_{ji}\, x_j'$$

$$= \sum_{j=1}^{3} d_{ij}^{T}\, x_j' \qquad \text{(need transpose matrix)}$$

comparison yields $D^{-1} = D^{T}$ (orthogonal matrix)

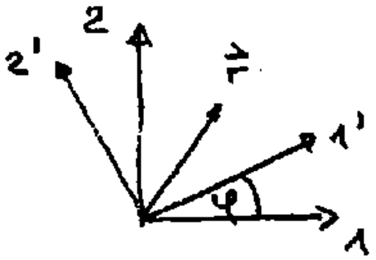
use $\mathbb{1} = D^{-1} D = D^{T} D$

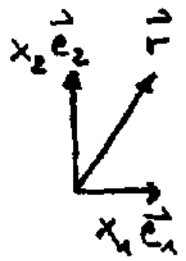
$$\delta_{ij} = \sum_{k=1}^{3} d_{ik}^{T}\, d_{kj} = \sum_{k=1}^{3} d_{ki}\, d_{kj}$$

(colums are orthonormalized)

27.10.2016

Example: rotation around z-axis





geometry

$$x_1\vec{e}_1 = x_1\cos\varphi\,\vec{e}_1{}' - x_1\sin\varphi\,\vec{e}_2{}'$$
$$x_2\vec{e}_2 = x_2\sin\varphi\,\vec{e}_1{}' + x_2\cos\varphi\,\vec{e}_2{}'$$

calculate $\vec{r} = x_1\vec{e}_1 + x_2\vec{e}_2 = (x_1\cos\varphi + x_2\sin\varphi)\,\vec{e}_1{}' + (-x_1\sin\varphi + x_2\cos\varphi)\,\vec{e}_2{}'$

$$\Rightarrow \quad x_1' = x_1\cos\varphi + x_2\sin\varphi$$
$$x_2' = -x_1\sin\varphi + x_2\cos\varphi$$

read off angles between basis vectors

$$d_{11} = \vec{e}_1{}' \cdot \vec{e}_1 = \cos\varphi$$
$$d_{12} = \vec{e}_1{}' \cdot \vec{e}_2 = \cos\left(\tfrac{\pi}{2} - \varphi\right) = \sin\varphi$$

$$d_{21} = \vec{e}_2{}' \cdot \vec{e}_1 = \cos\left(\frac{\pi}{2} + \varphi\right) = -\sin\varphi \qquad d_{31} = d_{13} = 0$$
$$d_{22} = \vec{e}_2{}' \cdot \vec{e}_2 = \cos\varphi \qquad d_{23} = d_{32} = 0$$
$$d_{33} = 1$$

$$\Rightarrow \quad D = \begin{pmatrix} \cos\varphi & \sin\varphi & 0 \\ -\sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\vec{r}_{\Sigma'} = D \; \vec{r}_\Sigma$ yields above equations and $x_3' = x_3$

Determinants

$$A = (a_{ij}) = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \vdots \\ \vdots & & & \\ a_{n1} & \cdots & \cdots & a_{nn} \end{pmatrix} \qquad (n \times n) \text{ matrix}$$

Definition :

$$\det A = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & & & \vdots \\ \vdots & & & \\ a_{n1} & \cdots & & a_{nn} \end{vmatrix} = \sum_P \underbrace{\text{sign}\, P}_{\text{sign of permutation}} \; a_{1P(1)} a_{2P(2)} \cdots a_{nP(n)}$$

permutation of $1 \ldots n$: $[P(1), P(2), \ldots, P(n)] = P(1, 2, \ldots n)$

sequence of numbers

$n = 1$ $\quad \det A = |a_{11}| = a_{11}$

$n = 2$ $\quad (1, 2)$: two permutations $\quad [1, 2]$, $\text{sign}(P) = 1$
$[2, 1]$, $\text{sign}(P) = -1$

$$\det A = 1 \cdot a_{11} a_{22} - 1 \; a_{12} a_{21}$$
$$= a_{11} a_{22} - a_{12} a_{21}$$

$n = 3$

$3! = 3 \cdot 2 \cdot 1 = 6$ terms

$$\begin{aligned}
\det A = {} & a_{11}\,(a_{22}a_{33} - a_{23}a_{32}) \\
& + a_{12}\,(-a_{21}a_{33} + a_{23}a_{31}) \\
& + a_{13}\,(a_{21}a_{32} - a_{22}a_{31}) \\
= {} & a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \\
& + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix} \\
= {} & \sum_{ijk} \epsilon_{ijk}\, a_{1i}\, a_{2j}\, a_{3k}
\end{aligned}$$

Expansion with respect to row i:

$$\det A = \sum_{j=1}^{n} a_{ij} \underbrace{(-1)^{i+j} \underbrace{\det A_{ij}}_{\substack{\text{sub determinant of} \\ (n-1)\times(n-1) \text{ matrix} \\ \text{with row } i\text{, column } j \text{ eliminated}}}}_{\text{algebraic complement}}$$

Want: row with zeros!

Calculation rules

1)
$$\begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ \alpha a_{i1} & \cdots & \alpha a_{in} \\ \vdots & & \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = \alpha \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{i1} & \cdots & a_{in} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

multiplication of <u>one</u> row with number

$$\Rightarrow \det(\alpha A) = \alpha^n \det A$$

2) addition of two rows

$$\begin{vmatrix} a_{11}+b_{11} & \cdots & a_{1n}+b_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} b_{11} & \cdots & b_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}$$

3) permutation of 2 rows changes sign

4) two identical rows $\det A = 0$

5) $\det A^T = \det A$ (expansion with respect to colums)

6) Matrix product : $\det(AB) = \det A \det B$

7) diagonal matrix $\det\begin{pmatrix} a_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn} \end{pmatrix} = \prod_{i=1}^{n} a_{ii}$

$\Rightarrow \det \mathbb{1} = 1$

Applications :

1) inverse matrix $A A^{-1} = A^{-1} A = \mathbb{1}$

$\Rightarrow \det A A^{-1} = \det A \det(A^{-1}) = 1$

A^{-1} exists only when $\det A \neq 0$

calculate elements of A^{-1} by

$$(a^{-1}{}_{ij}) = \frac{(-1)^{i+j} \det A_{ji}}{\det A}$$ (order of indices!)

2) cross product

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$

similar rotation

$$\vec{\nabla} \times \vec{a} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ \partial_1 & \partial_2 & \partial_3 \\ a_1 & a_2 & a_3 \end{vmatrix}$$

3) scalar triple product

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

special case

$$\vec{e}_1 \cdot (\vec{e}_2 \times \vec{e}_3) = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

4) rotation matrix D : orthogonal matrix $D^T = D^{-1}$

$$D^{-1} D = D^T D = \mathbb{1}$$

$$\Rightarrow \det \mathbb{1} = \det(D^{-1} D) = \det(D^T D) = \det D^T \det D$$

$$1 = \det D^T \det D = \det D \det D$$

$$\Rightarrow \det(D)^2 = 1 \qquad \det D = \pm 1$$

$O(n)$: set of orthonormal $(n \times n)$ matrices

$SO(n)$: set of orthonormal $(n \times n)$ matrices with $\det D = +1$

Consider : two rotation matrices

$$D = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix} \qquad D' = \begin{pmatrix} d_{11} & d_{12} & d_{13} \\ -d_{21} & -d_{22} & -d_{23} \\ d_{31} & d_{32} & d_{33} \end{pmatrix}$$

assume

$$\det D = +1 \qquad \Rightarrow \det D' = -1 \det D = -1$$

$$\vec{e}_i{}' = \sum_{j=1}^{3} d_{ij} \vec{e}_j$$

transformation with D : $\{\vec{e}_1{}', \vec{e}_2{}', \vec{e}_3{}'\}$

transformation with D' : $\{\vec{e}_1{}', -\vec{e}_2{}', \vec{e}_3{}'\}$

right handed coordinate system:

$$1 = \vec{e}_1{}' \cdot (\vec{e}_2{}' \times \vec{e}_3) = \sum_{lmn} d_{1l}\, d_{2m}\, d_{3n}\, \underbrace{\vec{e}_l \cdot (\vec{e}_m \times \vec{e}_n)}$$

assume right handed $\rightarrow \epsilon_{lmn}$

$$= \det D$$

$\det D = +1$: right handed $\rightarrow$ right handed
$\det D = -1$: right handed $\rightarrow$ left handed

Linear system of equations

$$\begin{aligned} a_{11} x_1 + a_{12} x_2 + \dots a_{1n} x_n &= b_1 \\ \vdots \quad &= \vdots \\ a_{n1} x_1 + a_{n2} x_2 + \dots a_{nn} x_n &= b_n \end{aligned}$$

$$A \vec{x} = \vec{b}$$

$$\vec{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix} \quad \vec{b} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix}$$

homogeneous system $\vec{b} = 0$
inhomogeneous system $\vec{b} \neq 0$

Find solutions of the equations:

$$x_k = \frac{\det A_k}{\det A}$$

((Cramer's rule)

$$A_k = \begin{array}{c} \text{k th column} \\ \downarrow \\ \begin{pmatrix} a_{11} \cdots & b_1 \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1} \cdots & b_n \cdots & a_{nn} \end{pmatrix} \end{array}$$

→ require $\det A \neq 0$ for a unique solution

special case: homogeneous system $\vec{b} = 0$

$\Rightarrow \det A_k = 0 \quad \forall k$

$x_k = \det A_k \cdot \det A$

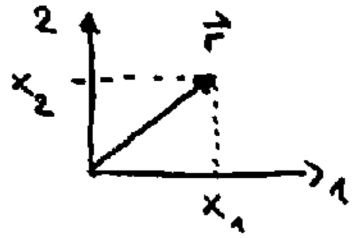
1) $x_k = 0 \quad \forall k$

2) $\det A = 0$ not all rows/columns linear independent

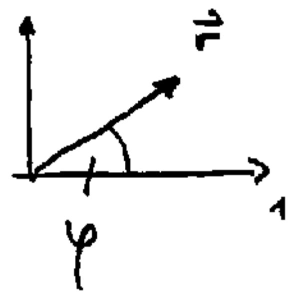
$\mathrm{rank}(A) < n$

Coordinate Systems

so far: Cartesian coordinates



polar coordinates



$(r, \varphi) \longrightarrow (x_1, x_2)$

$$x_1 = r\cos\varphi = x_1(r,\varphi)$$
$$x_2 = r\sin\varphi = x_2(r,\varphi)$$

not uniqely defined

$(0,\varphi) \to (0,0)$
$\varphi \in \mathbb{R}$

$(x_1, x_2) \longrightarrow (r, \varphi)$

$$r = \sqrt{x_1^2 + x_2^2} = r(x_1, x_2)$$
$$\varphi = \arctan\left(\frac{x_2}{x_1}\right) = \varphi(x_1, x_2)$$

$\frac{\sin\varphi}{\cos\varphi} = \tan\varphi = \frac{x_2}{x_1}$

(for $r \neq$ uniquely reversible)

General coordinate transformation (d-dimensional space)

$$(y_1, \dots y_d) \to (x_1, \dots x_d)$$

$$x_i(y_1, \dots y_d) \qquad i = 1 \dots d$$

requirements:

1) each point specificiable by coordinates y_i
2) almost always locally reversible

allowed to be violated in regions of dimensionality $d' < d$

to each point exists neighborhood in which mapping is unique

Check for the local reversibility?

Consider point $\vec{y}$ which is mapped to $\vec{x}$:
$\vec{y} + d\vec{y}$ is mapped to $\vec{x} + d\vec{x}$

total differential of $x_i(y_1, \dots, y_d)$:

$$dx_i = \sum_{j=1}^{d} \frac{\partial x_i}{\partial y_j} dy_j \qquad i = 1 \dots d$$

$$d\vec{x} = \begin{pmatrix} dx_1 \\ \vdots \\ dx_d \end{pmatrix} = \begin{pmatrix} \frac{\partial x_1}{\partial y_1} & & \frac{\partial x_1}{\partial y_d} \\ \vdots & & \vdots \\ \frac{\partial x_d}{\partial y_1} & \dots & \frac{\partial x_d}{\partial y_d} \end{pmatrix} \begin{pmatrix} dy_1 \\ \vdots \\ dy_d \end{pmatrix} = F^{(xy)} \, d\vec{y}$$

reversibility:

$$F^{(xy)^{-1}} d\vec{x} = d\vec{y} \qquad \Rightarrow \text{condition: } \det F^{(xy)} \neq 0$$

(Jacobian determinant)

The transformation of variables $x_i = x_i(y_1 \dots y_d)$, $i = 1 \dots d$ (with continously differentiable functions x_i) is in the proximity of point P bijective if and only if

$$\left. \frac{\partial (x_1 \dots x_d)}{\partial (y_1 \dots y_d)} \right|_P \neq 0 \qquad \det F^{(xy)} \neq 0$$

Example : plane polar coordinates

$$x_1 = r \cos \varphi \qquad \frac{\partial x_1}{\partial r} = \cos \varphi \qquad \frac{\partial x_1}{\partial \varphi} = -r \sin \varphi$$

$$x_2 = r \sin \varphi \qquad \frac{\partial x_2}{\partial r} = \sin \varphi \qquad \frac{\partial x_2}{\partial \varphi} = r \cos \varphi$$

$$\det F^{(x,y)} = \begin{vmatrix} \cos \varphi & -r \sin \varphi \\ \sin \varphi & r \cos \varphi \end{vmatrix} = r \cos^2 \varphi + r \sin^2 \varphi = r \neq 0$$

(except on single point $r=0$)

Consider : two transformations

$$x_i = x_i(y_1, \dots y_d) \qquad i = 1 \dots d$$
$$y_i = y_i(z_1, \dots, z_d)$$

Jacobian determinants : $\dfrac{\partial (x_1 \dots x_d)}{\partial (y_1 \dots y_d)} = F^{(xy)}$

$$\frac{\partial (y_1, \dots, y_d)}{\partial (z_1, \dots, z_d)} = F^{(yz)}$$

Now: calculate Jacobian of the full transformation:

$$F^{(xz)} = \frac{\partial (x_1, \dots x_d)}{\partial (z_1, \dots, z_d)}$$

with $x_i = x_i(y_1(z_1, \dots, z_d), \dots, y_d(z_1, \dots, z_d))$

we use the chain rule :

$$\frac{\partial x_i}{\partial z_j} = \sum_{k=1}^{d} \frac{\partial x_i}{\partial y_k} \frac{\partial y_k}{\partial z_j} \qquad (F^{xz})_{ij} = \sum_k (F^{xy})_{ik} (F^{yz})_{kj}$$

$$F^{(xz)} = F^{(xy)} F^{(yz)} \quad \Rightarrow \quad \det F^{(xz)} = \det F^{(xy)} \cdot \det F^{(yz)}$$

Special case : Transformation and its inverse :

$$x_i (z_1, \dots, z_d) = z_i \qquad i = 1..d$$

$$\frac{\partial x_i}{\partial z_j} = \delta_{ij}$$

$$\mathbb{1} = F^{(xz)} = F^{(xy)} \underbrace{F^{(yz)}}_{F^{(yx)}} = F^{(xy)} F^{(yx)}$$

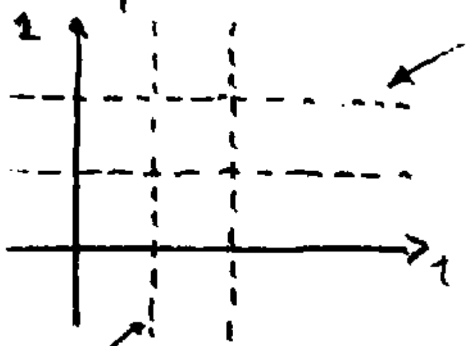
$$F^{-1}(xy) = F^{(yx)}$$ and for the det : $\det F^{(yx)} = \frac{1}{\det F^{(xy)}}$

Coordinate lines :

Consider transformation $\quad x_i = x_i (y_1, \dots, y_d)$

space curve with all coordinates $(y_2, \dots, y_d)$ kept constant : y_1 -coordinate line
(similar y_i coordinate line : all y_j kept constant $j \neq i$)

Examples : 1) Cartesian coordinates



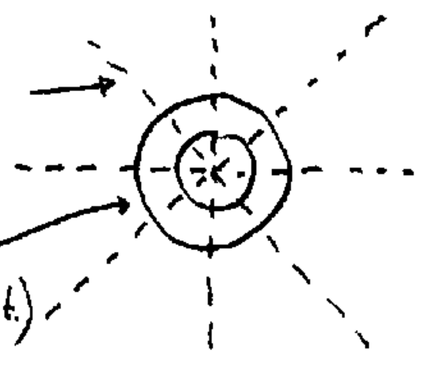
x_1 line
x_2 kept constant

x_2 line (x_1 kept constant)

2) Polar coordinates

r line
(φ const.)

φ line
(r kept const.)



Volume element in curvilinear coordinates

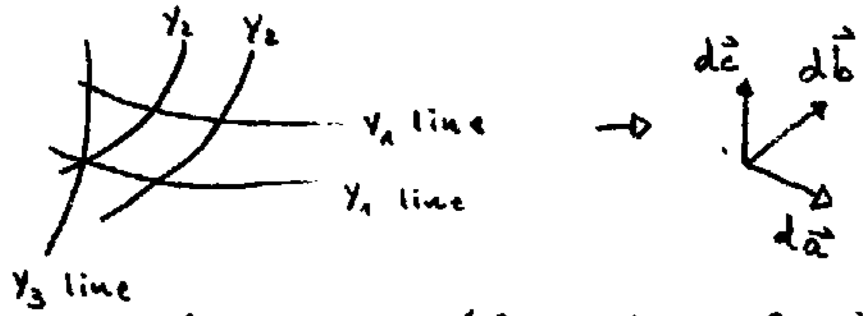
1) Integration in cartesian coordinates $V = \int dV$

$dV = dx_1\, dx_2\, dx_3$

Volume element in cartesian coordinates

2) curvilinear coordinates

coordinate lines



(locally parallel to coordinate lines)

$$d\vec{a} = \frac{\partial \vec{r}}{\partial y_1} dy_1 = \left(\frac{\partial x_1}{\partial y_1}, \frac{\partial x_2}{\partial y_1}, \frac{\partial x_3}{\partial y_1} \right) dy_1$$

similar $\quad d\vec{b} = \frac{\partial \vec{r}}{\partial y_2} dy_2 \quad , \quad d\vec{c} = \frac{\partial \vec{r}}{\partial y_3} dy_3$

$$dV = d\vec{a} \cdot (d\vec{b} \times d\vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial x_1}{\partial y_1} & \frac{\partial x_2}{\partial y_1} & \frac{\partial x_3}{\partial y_1} \\ \vdots & & \vdots \\ \frac{\partial x_1}{\partial y_3} & \cdots & \end{vmatrix} dy_1\, dy_2\, dy_3 = \underbrace{\det F^{(xy)^T}}_{\det F^{(xy)}} dy_1 dy_2 dy_3$$

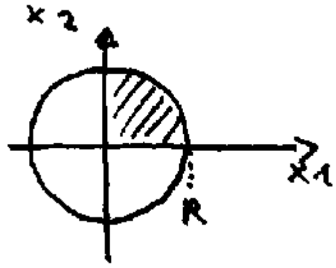
$$= \frac{\partial (x_1, x_2, x_3)}{\partial (y_1, y_2, y_3)} dy_1\, dy_2\, dy_3$$

$$V = \int dV = \int dx_1\, dx_2\, dx_3 = \int \det F^{(xy)}\, dy_1\, dy_2\, dy_3$$

Example: Integration in curvilinear coordinates

Calculation of the area of a cicle:

1) cartesian coordinates



$$A_0 = 4\int_0^R dx_1 \int_0^{\sqrt{R^2-x_1^2}} dx_2$$

$$= 4\int_0^R dx_1 \sqrt{R^2-x_1^2}$$

$$= 4\left[\frac{x_1}{2}\sqrt{R^2-x_1^2} + \frac{R^2}{2}\arcsin\frac{x_1}{R}\right]_0^R$$

$$= 4\,\frac{R^2}{2}\arcsin\frac{R}{R} = \pi R^2$$

2) polar coordinates

$$dx_1 dx_2 = \frac{\partial(x_1,x_2)}{\partial(r,\varphi)}\,dr\,d\varphi$$

$$= r\,dr\,d\varphi$$

$$A_0 = \int_0^R r\,dr\int_0^{2\pi} d\varphi = \left[\frac{r^2}{2}\right]_0^R \left[\varphi\right]_0^{2\pi} = \frac{R^2}{2}\,2\pi = \pi R^2$$

Basis vectors in curvilinear coordinates

→ so far: cartesian coordinates

$$\{\vec{e}_1, \vec{e}_2, \vec{e}_3\} \qquad \vec{e}_1 = \begin{pmatrix}1\\0\\0\end{pmatrix},\ \vec{e}_2 = \begin{pmatrix}0\\1\\0\end{pmatrix},\ \vec{e}_3 = \begin{pmatrix}0\\0\\1\end{pmatrix}$$

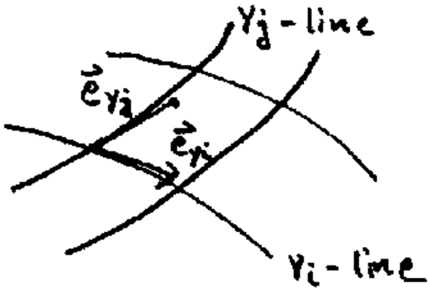
$$\vec{r} = \sum_{i=1}^{3} x_i \vec{e}_i \quad , \quad d\vec{r} = \sum_{i=1}^{3} dx_i\,\vec{e}_i$$

consider $\vec{r} = \vec{r}(x_1,x_2,x_3)$ such that the total differential reads

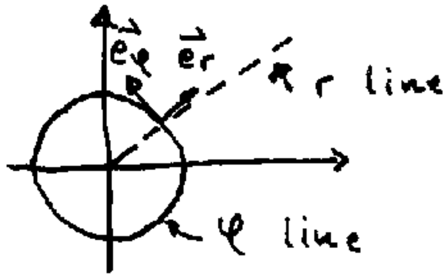
$$d\vec{r} = \sum_{i=1}^{3}\frac{\partial\vec{r}}{\partial x_i}\,dx_i \qquad \text{comparison: } \vec{e}_i = \frac{\partial\vec{r}}{\partial x_i}$$

tangent unit vector along i-th coordinate line general basis vectors (need to normalize)

$$\vec{e}_i = b_{y_i}^{-1} \frac{\partial \vec{r}}{\partial y_i} \quad , \quad b_{y_i} = \left| \frac{\partial \vec{r}}{\partial y_i} \right|$$



Example: polar coordinates



$$\left. \begin{array}{l} x_1 = r\cos\varphi \\ x_2 = r\sin\varphi \end{array} \right\} \vec{r}(\vec{r},\varphi) = \begin{pmatrix} r\cos\varphi \\ r\sin\varphi \end{pmatrix}$$

$$\frac{\partial \vec{r}}{\partial r} = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix} \qquad b_r = 1 \quad , \quad \vec{e}_r = \begin{pmatrix} \cos\varphi \\ \sin\varphi \end{pmatrix}$$

$$\frac{\partial \vec{r}}{\partial \varphi} = \begin{pmatrix} -r\sin\varphi \\ r\cos\varphi \end{pmatrix} \qquad b_\varphi = r \qquad \vec{e}_\varphi = \frac{1}{r}\begin{pmatrix} -r\sin\varphi \\ r\cos\varphi \end{pmatrix} = \begin{pmatrix} -\sin\varphi \\ \cos\varphi \end{pmatrix}$$

$\vec{e}_r \cdot \vec{e}_\varphi = 0$ curvilinear orthogonal

$$\delta_{ij} = \vec{e}_{y_i} \cdot e_{y_j} \quad \forall\ i,j$$

differential of the position vector:

$$d\vec{r} = \sum_{i=1}^{3} \frac{\partial \vec{r}}{\partial y_i} dy_i = \sum_{i=1}^{3} b_{y_i} dy_i \vec{e}_{y_i}$$

$$d\vec{r} = 1\, dr\, \vec{e}_r + r\, d\varphi\, \vec{e}_\varphi$$

$$= dr\, \vec{e}_r + r\, d\varphi\, \vec{e}_\varphi$$

Differential operators in curvilinear coordinates

a) Gradient

$$\vec{\nabla}\varphi = \left(\frac{\partial \varphi}{\partial x_1}, \frac{\partial \varphi}{\partial x_2}, \frac{\partial \varphi}{\partial x_3}\right)$$

$$\vec{\nabla} = \sum_i \vec{e}_i \frac{\partial}{\partial x_i}$$

now: project gradient field onto y_i coordinate line

$$\vec{\nabla}_{y_i}\varphi = \vec{e}_{y_i}\cdot\vec{\nabla}\varphi = b_{y_i}^{-1}\frac{\partial \vec{r}}{\partial y_i}\cdot\vec{\nabla}\varphi = b_{y_i}^{-1}\Big($$

$$\vec{e}_{y_i} = b_{y_i}^{-1}\frac{\partial \vec{r}}{\partial y_i}$$

$$= b_{y_i}^{-1}\left(\frac{\partial x_1}{\partial y_i}\frac{\partial \varphi}{\partial x_1} + \frac{\partial x_2}{\partial y_i}\frac{\partial \varphi}{\partial x_2} + \frac{\partial x_3}{\partial y_i}\frac{\partial \varphi}{\partial x_3}\right)$$

chain rule

$$= b_{y_i}^{-1}\frac{\partial \varphi}{\partial y_i}$$

$$\vec{\nabla} = \left(b_{y_1}^{-1}\frac{\partial}{\partial y_1}, b_{y_2}^{-1}\frac{\partial}{\partial y_2}, b_{y_3}^{-1}\frac{\partial}{\partial y_3}\right) = \sum_{j=1}^{3}\vec{e}_{y_i} b_{y_i}^{-1}\frac{\partial}{\partial y_i}$$

b) Divergence

given $\vec{a} = \sum_{i=1}^{3} a_{y_i}\vec{e}_i$ differentiable vector field

$$\vec{\nabla}\cdot\vec{a} = \frac{1}{b_{y_1}b_{y_2}b_{y_3}}\left[\frac{\partial}{\partial y_1}\left(b_{y_2}b_{y_3}a_{y_1}\right) + \frac{\partial}{\partial y_2}\left(b_{y_3}b_{y_1}a_{y_1}\right) + \frac{\partial}{\partial y_3}\left(b_{y_1}b_{y_2}a_{y_3}\right)\right]$$

Derivation: 1) use $\vec{\nabla} = \sum_{j=1}^{3}\vec{e}_{y_i} b_{y_i}^{-1}\frac{\partial}{\partial y_i}$

$$\vec{\nabla}\cdot\vec{a} = \sum_{i,j}\left(\vec{e}_{y_i} b_{y_i}^{-1}\frac{\partial}{\partial y_i}\right)\cdot\left(a_{y_i}\vec{e}_{y_i}\right) = \sum_i \frac{1}{b_{y_i}}\frac{\partial a_{y_i}}{\partial y_i} + \sum_{i} \frac{a_{y_i}}{b_{y_i}}\vec{e}_{y_i}\cdot\frac{\partial \vec{e}_{y_i}}{\partial y_i}$$

product rule

Use (Schwarz' theorem) $\frac{\partial^2 \vec{r}}{\partial y_i \partial y_j} = \frac{\partial^2 \vec{r}}{\partial y_j \partial y_i}$

$$\frac{\partial \vec{r}}{\partial y_i} = \vec{e}_{y_i} b_{y_i}$$

$$\Rightarrow \quad \frac{\partial}{\partial y_i}\left(b_{y_j} \vec{e}_{y_j} \right) = \frac{\partial}{\partial y_j}\left(b_{y_i} \vec{e}_{y_i} \right)$$

Product rule

$$b_{y_j} \frac{\partial \vec{e}_{y_j}}{\partial y_i} + \vec{e}_{y_j} \frac{\partial b_{y_j}}{\partial y_i} = b_{y_i} \frac{\partial \vec{e}_{y_i}}{\partial y_j} + \vec{e}_{y_i} \frac{\partial b_{y_i}}{\partial y_j}$$

multiply with $\vec{e}_{y_i}$:

$$b_{y_j} \vec{e}_{y_i} \cdot \frac{\partial \vec{e}_{y_j}}{\partial y_i} + \delta_{ij} \frac{\partial b_{y_j}}{\partial y_i} = b_{y_i} \vec{e}_{y_i} \cdot \frac{\partial \vec{e}_{y_i}}{\partial y_j} + \frac{\partial b_{y_i}}{\partial y_j}$$

Derivative of unitary vector $\perp$ to that vector

$$\vec{e}_{y_i} \cdot \frac{\partial \vec{e}_{y_i}}{\partial y_j} = \frac{1}{2} \frac{\partial}{\partial y_j}\left(\vec{e}_{y_i}^{\,2} \right) = 0$$

$$\Rightarrow \quad b_{y_j} \vec{e}_{y_i} \cdot \frac{\partial \vec{e}_{y_j}}{\partial y_i} = \frac{\partial b_{y_i}}{\partial y_j} - \delta_{ij} \frac{\partial b_{y_j}}{\partial y_i} = \begin{cases} 0 & i = j \\ \frac{\partial b_{y_i}}{\partial y_j} & i \neq j \end{cases}$$

$$\vec{\nabla} \cdot \vec{a} = \sum_i b_{y_i}^{-1} \frac{\partial a_{y_i}}{\partial y_i} + \sum_{\substack{i,j=1 \\ i \neq j}}^{3} b_{y_i}^{-1} b_{y_j}^{-1} a_{y_j} \frac{\partial b_{y_i}}{\partial y_j}$$

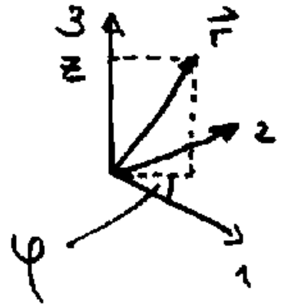
c) Rotation

$$\vec{\nabla} \times \vec{a} = \begin{vmatrix} b_{y_1} \vec{e}_{y_1} & b_{y_2} \vec{e}_{y_2} & b_{y_3} \vec{e}_{y_3} \\ \frac{\partial}{\partial y_1} & \frac{\partial}{\partial y_2} & \frac{\partial}{\partial y_3} \\ b_{y_1} a_{y_1} & b_{y_2} a_{y_2} & b_{y_3} a_{y_3} \end{vmatrix}$$

d) Laplace operator

$$\Delta \vec{a} = \frac{1}{b_{y_1} b_{y_2} b_{y_3}} \left[\frac{\partial}{\partial y_1}\left(\frac{b_{y_2} b_{y_3}}{b_{y_1}} \frac{\partial}{\partial y_1} \right) + \frac{\partial}{\partial y_2}\left(\frac{b_{y_1} b_{y_3}}{b_{y_2}} \frac{\partial}{\partial y_2} \right) + \frac{\partial}{\partial y_3}\left(\frac{b_{y_1} b_{y_2}}{b_{y_3}} \frac{\partial}{\partial y_3} \right)\right.$$

Examples: 1) cylindrical coordinates



transformation formulae

$$\left.\begin{aligned} x_1 &= \rho \cos \varphi \\ x_2 &= \rho \sin \varphi \\ x_3 &= z \end{aligned}\right\}$$ suitable if there is rotation symmetry around axis

Jacobian determinant (need all partial derivatives)

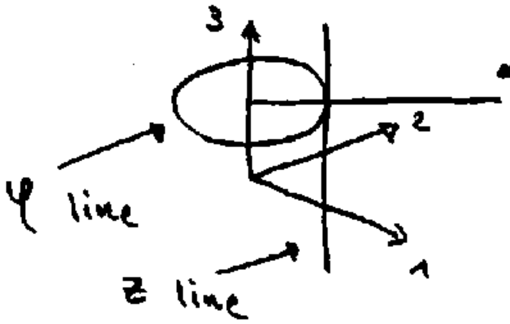
$$\left|\frac{\partial(x_1,x_2,x_3)}{\partial(\rho,\varphi,z)}\right| = \begin{vmatrix} \cos\varphi & -\rho\sin\varphi & 0 \\ \sin\varphi & \rho\cos\varphi & 0 \\ 0 & 0 & 1 \end{vmatrix} = \begin{vmatrix} \cos\varphi & -\rho\sin\varphi \\ \sin\varphi & \rho\cos\varphi \end{vmatrix}$$

$$= \rho$$ uniquely reversible except for $\rho = 0$

Volume element:

$$dV = \rho \, d\rho \, d\varphi \, dz$$ (using Jacobian determinant)

<u>coordinate lines</u>:



<u>scale factors</u>

$$\vec{r} = (\rho\cos\varphi, \rho\sin\varphi, z)$$

$$\frac{\partial\vec{r}}{\partial\rho} = (\cos\varphi, \sin\varphi, 0)$$

$$b_\rho = \left|\frac{\partial\vec{r}}{\partial\rho}\right| = 1$$

$$\frac{\partial\vec{r}}{\partial\varphi} = (-\rho\sin\varphi, \rho\cos\varphi, 0)$$

$$b_\varphi = \left|\frac{\partial\vec{r}}{\partial\varphi}\right| = \rho$$

$$\frac{\partial\vec{r}}{\partial z} = (0, 0, 1)$$

$$b_z = \left|\frac{\partial r}{\partial z}\right| = 1$$

<u>unit vectors</u>

$$\vec{e}_\rho = (\cos\varphi, \sin\varphi, 0)$$
$$\vec{e}_\varphi = (-\sin\varphi, \cos\varphi, 0)$$
$$\vec{e}_z = (0, 0, 1)$$

$\{\vec{e}_\varrho, \vec{e}_\varphi, \vec{e}_z\}$ (curvilinear, right-handed orthogonal basis)

differential of position vector

$$d\vec{r} = \sum_j b_{y_j}\, dy_j\, \vec{e}_{y_j} = d\varrho\, \vec{e}_\varrho + \varrho\, d\varphi\, \vec{e}_\varphi + dz\, \vec{e}_z$$

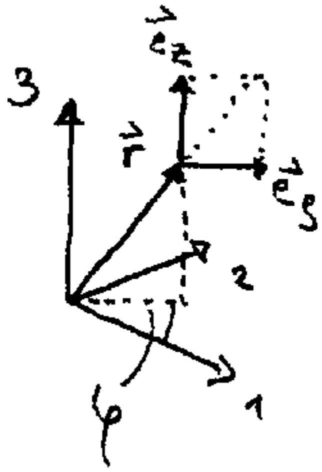
Differential operators

$$\vec{\nabla} = \vec{e}_\varrho \frac{\partial}{\partial \varrho} + \vec{e}_\varphi \frac{1}{\varrho}\frac{\partial}{\partial \varphi} + \vec{e}_z \frac{\partial}{\partial z}$$

$$\vec{\nabla}\cdot\vec{a} = \frac{1}{\varrho}\left[\frac{\partial}{\partial \varrho}(\varrho\, a_\varrho) + \frac{\partial}{\partial \varphi} a_\varphi + \frac{\partial}{\partial z}(\varrho\, a_z)\right]$$
$$= \frac{\partial a_\varrho}{\partial \varrho} + \frac{a_\varrho}{\varrho} + \frac{1}{\varrho}\frac{\partial a_\varphi}{\partial \varphi} + \frac{\partial a_z}{\partial z}$$

Position vector

$$\vec{r}(t) = \varrho(t)\, \vec{e}_\varrho(t) + z(t)\, \vec{e}_z$$



Velocity and acceleration

$$\vec{v}(t) = \dot{\vec{r}}(t) = \dot{\varrho}(t)\, \vec{e}_\varrho + \varrho\, \dot{\vec{e}}_\varrho + \dot{z}\, \vec{e}_z$$

total differential:

$$\vec{v} = \frac{d\vec{r}}{dt} = \frac{d\varrho}{dt}\vec{e}_\varrho + \varrho\frac{d\varphi}{dt}\vec{e}_\varphi + \frac{dz}{dt}\vec{e}_z$$

comparison yields: $\dot{\vec{e}}_\varrho = \dot{\varphi}\, \vec{e}_\varphi$ (from $\vec{e}_{y_i}\cdot\vec{e}_{y_j} = \delta_{ij}$

calculation of $\dot{\vec{e}}_\varphi = -\dot{\varphi}\, \vec{e}_\varrho$ yields

$$\vec{a}(t) = \dot{\vec{v}}(t) = \ddot{\varrho}\, \vec{e}_\varrho + \dot{\varrho}\, \dot{\vec{e}}_\varrho + \varrho\ddot{\varphi}\, \vec{e}_\varphi + \dot{\varrho}\dot{\varphi}\, \vec{e}_\varphi + \varrho\dot{\varphi}\, \dot{\vec{e}}_\varphi + \ddot{z}$$
$$= (\ddot{\varrho} - \varrho\dot{\varphi}^2)\, \vec{e}_\varrho + (\varrho\ddot{\varphi} + 2\dot{\varrho}\dot{\varphi})\, \vec{e}_\varphi + \ddot{z}\, \vec{e}_z$$

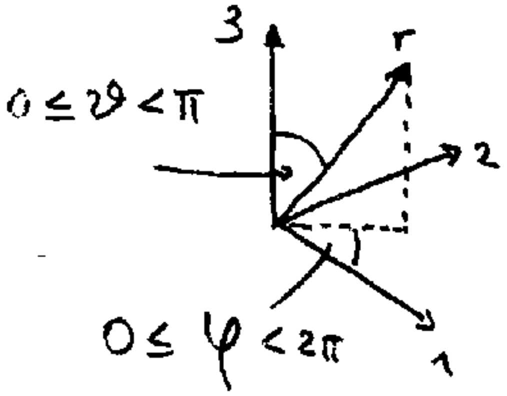
2) Spherical coordinates

$$x_1 = r \sin\vartheta \cos\varphi$$
$$x_2 = r \sin\vartheta \sin\varphi$$
$$x_3 = r \cos\vartheta$$

r : magnitude of position vector

ϑ : angle between $\vec{r}$ and $\vec{e}_3$, (polar angle)

φ : angle from projection of $\vec{r}$ onto $x_1 - x_2$ plane (azimuthal angle)



Jacobian determinant:

$$\left|\frac{\partial(x_1,x_2,x_3)}{\partial(r,\vartheta,\varphi)}\right| = \begin{vmatrix} \sin\vartheta\cos\varphi & r\cos\vartheta\cos\varphi & -r\sin\vartheta\sin\varphi \\ \sin\vartheta\sin\varphi & r\cos\vartheta\sin\varphi & r\sin\vartheta\cos\varphi \\ \cos\vartheta & -r\sin\vartheta & 0 \end{vmatrix}$$

$$= r^2 \sin\vartheta$$

uniquely reversible except for $r = 0$, $\vartheta = 0$

Volume element

$$dV = r^2 \sin\vartheta \, dr \, d\vartheta \, d\varphi$$

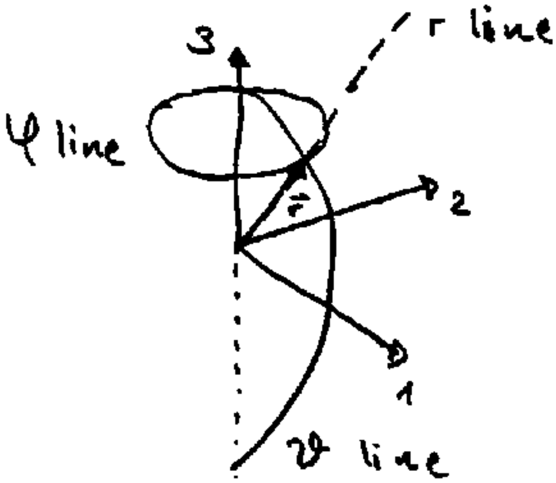
Example: Calculation of the volume of a sphere

$$V_s = \int_0^R dr\, r^2 \int_0^\pi d\vartheta \sin\vartheta \int_0^{2\pi} d\varphi = 2\pi \int_0^R dr\, r^2 \int_{-1}^{1} da$$

$a = \cos\vartheta$

$$= 2\pi \frac{R^3}{3} \, 2 = \frac{4\pi}{3} R^3$$

Coordinate lines



Scale factors

$$\vec{r} = r(\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)$$

$$\frac{\partial\vec{r}}{\partial r} = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)$$

$$b_r = 1$$

$$\frac{\partial\vec{r}}{\partial\vartheta} = r(\cos\vartheta\cos\varphi, \cos\vartheta\sin\varphi, -\sin\vartheta)$$

$$b_\vartheta = r$$

$$\vec{e}_r = (\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta) \qquad \frac{\partial\vec{r}}{\partial\varphi} = r(-\sin\vartheta\sin\varphi, \sin\vartheta\cos\varphi\ 0)$$

$$\vec{e}_\vartheta = (\cos\vartheta\cos\varphi, \cos\vartheta\sin\varphi, -\sin\vartheta) \qquad b_\varphi = r\sin\vartheta$$

$$\vec{e}_\varphi = (-\sin\varphi, \cos\varphi, 0)$$

$\{\vec{e}_r, \vec{e}_\vartheta, \vec{e}_\varphi\}$ (curvilinear orthogonal)

total differential

$$d\vec{r} = d\vec{r}\ \vec{e}_r + r\,d\vartheta\,\vec{e}_\vartheta + r\sin\vartheta\,d\varphi\,\vec{e}_\varphi$$

Differential operator

$$\vec{\nabla} = \vec{e}_r\frac{\partial}{\partial r} + \vec{e}_\vartheta\frac{1}{r}\frac{\partial}{\partial\vartheta} + \vec{e}_\varphi\frac{1}{r\sin\vartheta}\frac{\partial}{\partial\varphi}$$

Position vector: $\vec{r}(t) = r(t)\,\vec{e}_r$

$$\left.\begin{aligned}
\dot{\vec{e}}_r &= \dot\vartheta\,\vec{e}_\vartheta + \sin\vartheta\,\dot\varphi\,\vec{e}_\varphi \\
\dot{\vec{e}}_\vartheta &= \dot\varphi\cos\vartheta\,\vec{e}_\varphi - \dot\vartheta\,\vec{e}_r \\
\dot{\vec{e}}_\varphi &= -\dot\varphi\cos\vartheta\,\vec{e}_\vartheta - \dot\varphi\sin\vartheta\,\vec{e}_r
\end{aligned}\right\}$$

Derivatives of basis vectors: useful for $\vec{v} = \dot{\vec{r}}$, $\vec{a} = \ddot{\vec{r}}$

Mechanics of the free mass point

without restraining condition

negligible extension in all directions (in comparison to the length scales of the movement)

Kinematics: describe movement
(no question for the cause of the movement)

Task: calculate $\vec{r}(t)$ for given $\vec{a}(t) = \ddot{\vec{r}}(t)$ and initial conditions $\vec{v}(t_0) = \dot{\vec{r}}(t_0)$ and $\vec{r}(t_0)$

two integrations: $\vec{a} = \frac{d\vec{v}}{dt}$ $\vec{v} = \frac{d\vec{r}}{dt}$

$$\vec{v}(t) = \vec{v}(t_0) + \int_{t_0}^{t} dt' \, \vec{a}(t')$$

$$\vec{r}(t) = \vec{r}(t_0) + \int_{t_0}^{t} dt' \, \vec{v}(t') = \vec{r}(t_0) + \int_{t_0}^{t} dt' \left[\vec{v}_0(t_0) + \int_{t_0}^{t'} dt'' \vec{a}(t'') \right]$$

$$= \vec{r}(t_0) + \vec{v}(t_0)(t - t_0) + \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \, \vec{a}(t'')$$

Cartesian coordinates (time-independent basis)

$$\vec{r}(t) = \sum_i x_i(t) \vec{e}_i$$

$$\vec{v}(t) = \frac{d\vec{r}}{dt} = \sum_i \dot{x}_i(t) \vec{e}_i$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = \sum_i \ddot{x}_i(t) \vec{e}_i$$

Cylindrical coordinates : $\vec{r}(t) = \rho \, \vec{e}_\rho + z \, \vec{e}_z$

$$\vec{v}(t) = \dot{\rho} \, \vec{e}_\rho + \rho \dot{\varphi} \, \vec{e}_\varphi + \dot{z} \, \vec{e}_z$$

$$\vec{a}(t) = (\ddot{\rho} - \rho \dot{\varphi}^2) \vec{e}_\rho + (\rho \ddot{\varphi} + 2 \dot{\rho} \dot{\varphi}) \vec{e}_\varphi + \ddot{z} \, \vec{e}_z$$

Spherical coordinates: $\vec{r}(t) = r\,\vec{e}_r$

use total differential: $d\vec{r} = dr\,\vec{e}_r + r\,d\vartheta\,\vec{e}_\vartheta + r\sin\vartheta\,d\varphi\,\vec{e}_\varphi$

$\Rightarrow \vec{v}(t) = \frac{d\vec{r}}{dt} = \dot{r}\,\vec{e}_r + r\dot{\vartheta}\,\vec{e}_\vartheta + r\sin\vartheta\,\dot{\varphi}\,\vec{e}_\varphi$

comparison with $\dot{\vec{r}}(t) = \dot{r}\,\vec{e}_r + r\,\dot{\vec{e}}_r$ yields

$$\dot{\vec{e}}_r = \dot{\vartheta}\,\vec{e}_\vartheta + \sin\vartheta\,\dot{\varphi}\,\vec{e}_\varphi$$

We use that $\dot{\hat{b}} \perp \hat{b}$ for each unitary vector $\hat{b}$:

$$\dot{\vec{e}}_\vartheta = \alpha\,\vec{e}_\varphi + \beta\,\vec{e}_r$$
$$\dot{\vec{e}}_\varphi = \gamma\,\vec{e}_\vartheta + \delta\,\vec{e}_r \quad (\rightarrow \text{need to calculate } \alpha, \beta, \gamma, \delta)$$

use orthogonality: $0 = \vec{e}_\vartheta \cdot \vec{e}_r = \vec{e}_\vartheta \cdot \vec{e}_\varphi = \vec{e}_\varphi \cdot \vec{e}_r$
with $0 = \frac{d}{dt}(0) = \frac{d}{dt}(\vec{e}_\vartheta \cdot \vec{e}_r) = \dot{\vec{e}}_\vartheta \cdot \vec{e}_r + \vec{e}_\vartheta \cdot \dot{\vec{e}}_r$ etc.

$$\beta = \dot{\vec{e}}_\vartheta \cdot \vec{e}_r = -\vec{e}_\vartheta \cdot \dot{\vec{e}}_r = -\dot{\vartheta}$$
$$\alpha = \dot{\vec{e}}_\vartheta \cdot \vec{e}_\varphi = -\vec{e}_\vartheta \cdot \dot{\vec{e}}_\varphi = -\gamma$$
$$\delta = \dot{\vec{e}}_\varphi \cdot \vec{e}_r = -\vec{e}_\varphi \cdot \dot{\vec{e}}_r = -\sin\vartheta\,\dot{\varphi}$$

$\vec{e}_\varphi = (-\sin\varphi, \cos\varphi, 0) \qquad \vec{e}_\varphi \cdot \vec{e}_z = 0, \; \dot{\vec{e}}_\varphi \cdot \vec{e}_z = 0$

$$0 = \gamma\,\vec{e}_\vartheta \cdot \vec{e}_z + \delta\,\vec{e}_r \cdot \vec{e}_z = -\gamma\sin\vartheta + \delta\cos\vartheta$$

$$\Rightarrow 0 = -\gamma\sin\vartheta - \sin\vartheta\,\dot{\varphi}\cos\vartheta$$

$$\dot{\vec{e}}_\vartheta = \dot{\varphi}\cos\vartheta\,\vec{e}_\varphi - \dot{\vartheta}\,\vec{e}_r$$
$$\dot{\vec{e}}_\varphi = -\dot{\varphi}\cos\vartheta\,\vec{e}_\vartheta - \sin\vartheta\,\dot{\varphi}\,\vec{e}_r$$

$$\vec{a}(t) = \frac{d\vec{v}}{dt} = (\ddot{r} - r\dot{\vartheta}^2 - r\sin^2\vartheta\,\dot{\varphi}^2)\,\vec{e}_r$$
$$+ (r\ddot{\vartheta} + 2\dot{r}\dot{\vartheta} - r\sin\vartheta\cos\vartheta\,\dot{\varphi}^2)\,\vec{e}_\vartheta$$
$$(r\sin\vartheta\,\ddot{\varphi} + 2\sin\vartheta\,\dot{r}\dot{\varphi} + 2r\cos\vartheta\,\dot{\varphi}\dot{\vartheta})\,\vec{e}_\varphi$$

10.11.2016

Kinematics

Examples : 1) Uniformly accelerated motion : $\vec{a}(t) = \vec{a}_0$

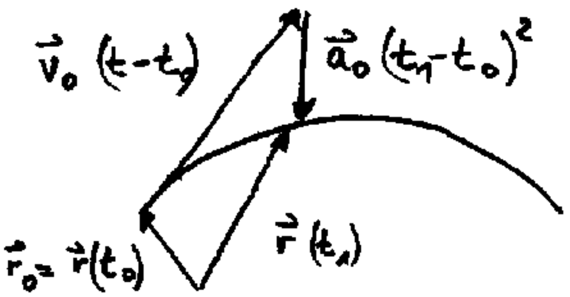
initial conditions : $\vec{r}(t_0) = \vec{r}_0$, $\vec{v}(t_0) = \vec{v}_0$

$$\vec{v}(t) = \vec{v}_0 + \int_{t_0}^{t} \vec{a}_0 \, dt = \vec{v}_0 + \vec{a}_0 (t - t_0)$$

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0 (t - t_0) + \int_{t_0}^{t} dt' \int_{t_0}^{t'} dt'' \vec{a}_0$$

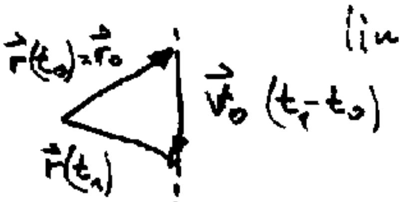
$$= \vec{r}_0 + \vec{v}_0 (t - t_0) + \int_{t_0}^{t} dt' \, \vec{a}_0 (t' - t_0)$$

$$= \vec{r}_0 + \vec{v}_0 (t - t_0) + \vec{a}_0 \, \frac{1}{2} (t - t_0)^2$$

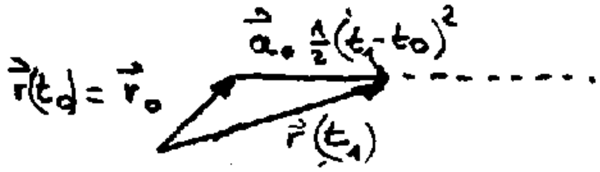


special cases

1) $\vec{a}_0 = 0$ uniform straight line motion



2) uniform accelerated straight line motion
$\vec{v}_0 = 0$, $\vec{a}_0 \neq 0$

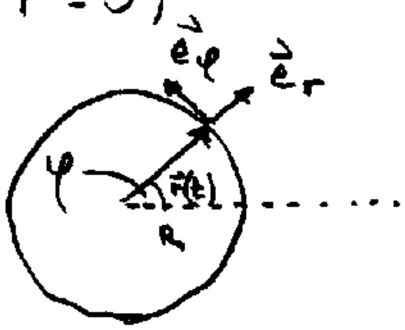


2) Circular motion on circle with (fixed) radius R

$\vec{r}(t) = R \, \vec{e}_r \qquad (r = R, \ \dot{r} = 0, \ \ddot{r} = 0)$

$\vec{v}(t) = R \, \dot{\varphi} \, \vec{e}_\varphi$

$\vec{a}(t) = -R \, \dot{\varphi}^2 \, \vec{e}_r + R \, \ddot{\varphi} \, \vec{e}_\varphi$



define : angular velocity $\omega = \dot{\varphi}$

velocity magnitude $|\vec{v}(t)| = v = R\omega$
centripetal acceleration $a_r = -R\omega^2$
tangential acceleration $a_\varphi = R\dot{\omega}$ $\qquad \dot{\omega} = 0$ (uniform circular motion)

define axial vector $\vec{\omega} = \omega \vec{e}_3$
and rewrite $\vec{v}(t) = \vec{\omega} \times \vec{r}(t) = \omega R \vec{e}_\varphi$

Fundamental laws of dynamics

→ Question / Investigation : cause of motion

Note : definitions a) basis definitions : "position" "time"
b) following definitions $\vec{v} = \frac{d\vec{r}}{dt}$

theorems : a) Axioms (not provable)
b) conclusions (emerge from math. proof)

Newton's Laws of motion

2 basis definitions :

force : effort to change state of motion (or shape of a body) ; indirect definition, vector

mass (inertial mass) : property of each body to resist against changes of motion, scalar "m_{in}"

Axiom 1 (Galilei's law of inerta):

(There exist systems of coordinates in which)
a force-free body (mass point) persists in
the state in the state of rest or in state
of uniform straight-line motion

following definitions 1) force-free body : no external influence

2) inertial system : coordinate system in which axiom holds

3) (linear) momentum : $\vec{p} = m_{in} \vec{v}$
(product of mass and velocity)

Axiom 2 (Law of motion)

rate of change of momentum = force

$$\dot{\vec{p}} = \vec{F} \qquad , \quad \dot{\vec{p}} = \dot{m}\vec{v} + m\dot{\vec{v}}$$

Remarks

a) time-independent mass $\vec{p}(t) = m_{in} \vec{v}(t)$

$$\dot{\vec{p}} = m_{in} \dot{\vec{v}} = m_{in} \vec{a} = \vec{F}$$

(basic dynamical equation)

b) special relativity

$$m_{in} = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \qquad \dot{m}_{in} \neq 0$$

for $v \ll c$ $m_{in} \approx m_0$

c) changes of mass also possible:
rocket, car (fuel), ... $\dot{m}_{in} < 0$

d) Axiom : $\vec{F} = \alpha \dot{\vec{p}}$, choose $\alpha = 1$

e) dynamical equation of motion : $\frac{\vec{F}}{m_{in}} = \vec{a}$
r.h.s. well defined : yields definition of ratio

Axiom 3 : Law of reaction "actio = reactio"

$\vec{F}_{12}$: Force of body 2 on body 1
$\vec{F}_{21}$: Force of body 1 on body 2

$$\vec{F}_{12} = - \vec{F}_{21}$$

Note : allows to measure masses ($\dot{m}_{in} = 0$)

$$\vec{F}_{12} = - \vec{F}_{21}$$

$$m_1 \vec{a}_1 = - m_2 \vec{a}_2$$

absolute value

$$m_1 a_1 = m_2 a_2$$

$$\frac{a_1}{a_2} = \frac{m_2}{m_1} \quad \text{(independent of type of force)}$$

$$m_2 = m_1 \frac{a_1}{a_2} \quad m_1 : \text{fixed mass } 1\,\text{kg}$$

$$[m_{in}] = \text{kg}$$

force is following definition $\vec{F} = m_{in} \vec{a}$

$$[\vec{F}] = \frac{\text{kg m}}{\text{s}^2} = \text{N}$$

Axiom 4 (Superposition principle)

$$\vec{F} = \sum_{i=1}^{n} \vec{F}_i \quad \text{(forces add up like vectors)}$$

Forces : Force field $\vec{F} = \vec{F}(\vec{r}, \dot{\vec{r}}, t)$

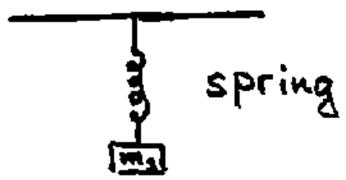
Examples of model forces:

a) Weight, gravitational force

$$\vec{F}_s = m_h \vec{g}$$

m_s : heavy (gravitational) mass
$\vec{g} = (0, 0, -g)$ gravity acceleration
$g \approx 9.81 \frac{m}{s^2}$

measurement of m_s (elongation of spring)



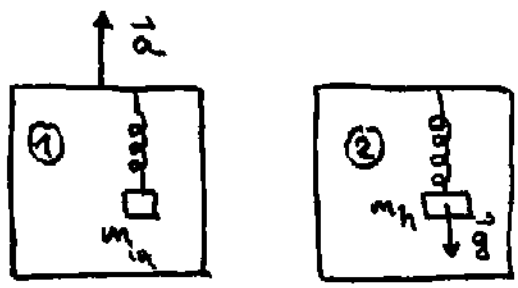
mass point in gravitational field

$$m_{in} a = m_h g \quad \Rightarrow \quad a = \frac{m_h}{m_{in}} g$$

$\frac{m_h}{m_{in}}$: independent of substace / mass point

Einstein's Equivalence principle $m_h = m_{in}$

→ measure methods for m_{in} and m_h are equivalent



measurement of inertial mass in accelerated Lab

measurement of heavy mass in lab subject to gravitational force field

b) Central forces

$$\vec{F}(\vec{r},\dot{\vec{r}},t) = f(\vec{r},\dot{\vec{r}},t)\, r\vec{e}_r \qquad \text{(radial force)}$$

gravitational force $\quad f(r) = -\gamma \frac{mM}{r^3}$

Coulomb force $\quad f(r) = \frac{q_1 q_2}{4\pi\varepsilon_0 r^3} \quad q_i$: charg

harmonic oscillator $\quad f(r) = -k$

c) Lorentz force

$$\vec{F} = q\left[\vec{E}(\vec{r},t) + \vec{v}\times\vec{B}(\vec{r},t)\right]$$

(force on charged particle in electric field $\vec{E}$ and magnetic induction $\vec{B}$, depends on velocity $\vec{v}$)

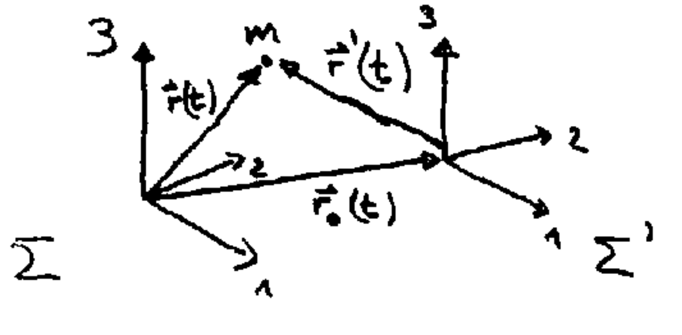
d) Frictional forces

$$\vec{F} = -\alpha(v)\,\vec{v} \qquad \alpha(v) > 0$$

Inertial systems, Galilei transformation

recall: inertial system : force free mass moves on a straight line $\vec{v}$ = con

Consider two coordinate systems



Σ' moves relatively to Σ with constant velocity $\vec{V}_0$

$$\vec{r}_0(t) = \vec{V}_0(t-t_0) + \vec{r}_0(t_0)$$

Without loss of generality: $t_0 = 0, \ \vec{r}_0(t_0) = 0$

$$\vec{r}(t) = \vec{r}\,'(t) + \vec{r}_0(t) \qquad , \ \vec{r}_0(t) = \vec{v}_0\, t$$
$$\vec{v}(t) = \dot{\vec{r}}(t) = \dot{\vec{r}}\,'(t) + \dot{\vec{r}}_0(t) = \vec{v}\,'(t) + \vec{v}_0$$
$$\vec{a}(t) = \ddot{\vec{r}}(t) = \ddot{\vec{r}}\,'(t) + \ddot{\vec{r}}_0(t) = \vec{a}\,'(t)$$

Newtons law : $\quad m\,\vec{a}(t) = m\,\vec{a}\,'(t)$
$$\vec{F} = \vec{F}'$$

force free particle in Σ $\Leftrightarrow$ force free particle in Σ'
Σ: inertial system $\Leftrightarrow$ Σ': inertial system

Transformation between inertial systems

$$\left.\begin{array}{ll} t = t' & \text{(absolute time)} \\ \vec{r}(t) = \vec{r}\,'(t) + \vec{v}_0(t) & \end{array}\right\} \text{Galilei transformation}$$

special relativity : Lorentz-Transformation

Rotating Coordinate Systems: Pseudo Forces

from $\vec{v}_0 = \vec{v}_0(t)$ follows

$$\vec{v}(t) = \vec{v}\,'(t) + \vec{v}_0(t) + \dot{\vec{v}}_0(t)\, t$$
$$\vec{a}(t) = \vec{a}\,'(t) + 2\,\dot{\vec{v}}_0(t) + \ddot{\vec{v}}_0(t)\, t \neq \vec{a}\,'(t)$$

$\dot{\vec{v}}_0 \neq 0$: Σ' is not an inertial system if Σ is one:

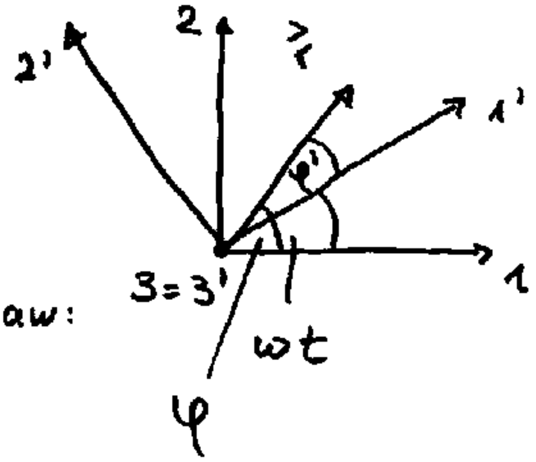
for $\vec{a} = 0$ follows $\frac{\vec{F}}{m} = \vec{a} = 0$, $\vec{F} = 0$ in Σ
but $\frac{\vec{F}'}{m} = \vec{a}\,'(t) = -2\dot{\vec{v}}_0(t) - \ddot{\vec{v}}_0(t)\, t \neq 0$

Example: Σ inertial system

Σ' rotating coordinate system with constant angular velocity ω

cylindrical coordinates : $t=0$, $\Sigma \hat{=} \Sigma'$

$$\varrho = \varrho'$$
$$\varphi = \varphi' + \omega t$$
$$z = z'$$



Forces from Newton's law:

Σ : $\vec{F} = m\ddot{\vec{r}}$

$$F_\varrho = m a_\varrho = m\left(\ddot{\varrho} - \varrho\dot{\varphi}^2\right)$$
$$F_\varphi = m a_\varphi = m\left(\varrho\ddot{\varphi} + 2\dot{\varrho}\dot{\varphi}\right)$$
$$F_z = m a_z = m\ddot{z}$$

Σ' : $\vec{F}' = m\ddot{\vec{r}}'$

$$F'_\varrho = m a'_\varrho = m\left(\ddot{\varrho}' - \varrho'\dot{\varphi}'^2\right) = m\left[\ddot{\varrho} - \varrho\left(\dot{\varphi} - \omega\right)^2\right]$$
$$F'_\varphi = m a'_\varphi = m\left(\varrho'\ddot{\varphi}' + 2\dot{\varrho}'\dot{\varphi}'\right) = m\left[\varrho\ddot{\varphi} + 2\dot{\varrho}\left(\dot{\varphi} - \omega\right)\right]$$
$$F'_z = m a'_z = m\ddot{z}' = m\ddot{z}$$

substitute F_ϱ, F_φ, F_z

$$F'_\varrho = F_\varrho + m\varrho\omega\left(2\dot{\varphi}' + \omega\right)$$
$$F'_\varphi = F_\varphi - 2m\dot{\varrho}\omega$$
$$F'_z = F_z$$

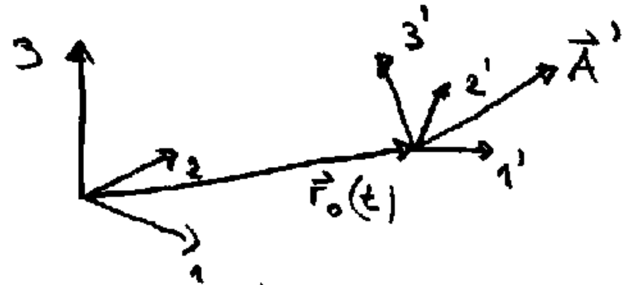
force free body in Σ has pseudo forces in Σ':

$F'_\varrho = m\varrho\omega\left(2\dot{\varphi}' + \omega\right)$ ($\dot{\varphi}' = 0$, centrifugal force)

$F'_\varphi = -2m\dot{\varrho}\omega$ (Coriolis force)

Arbitrarily Accelerated Reference Systems

Consider: inertial system Σ with $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$
arbitrarily moving system Σ' with $\{\vec{e}_1{}', \vec{e}_2{}', \vec{e}_3{}'\}$



vector $\vec{A}'$ in Σ': $\vec{A}' = \sum_{i=1}^{3} a_i' \vec{e}_i{}'$

calculate time derivative in Σ:

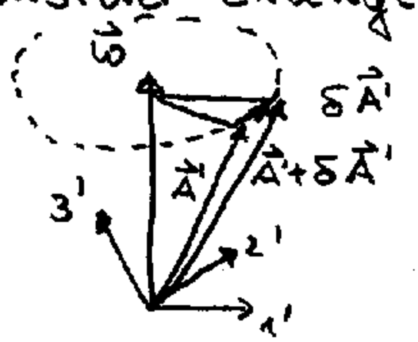
$$\frac{d\vec{A}'}{dt} = \sum_{i=1}^{3} \dot{a}_i' \vec{e}_i{}' + \sum_{i=1}^{3} a_i' \dot{\vec{e}}_i{}'$$

& identification of quantities

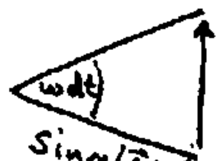
$$\left.\frac{d\vec{A}'}{dt}\right|_{\Sigma} = \sum_{i=1}^{3} \left.\frac{da_i'}{dt}\right|_{\Sigma} \vec{e}_i{}' + \sum_{i=1}^{3} a_i' \left.\frac{d\vec{e}_i{}'}{dt}\right|_{\Sigma}$$

$$= \left.\frac{d\vec{A}'}{dt}\right|_{\Sigma'} + \underbrace{\sum_{i=1}^{3} a_i' \left.\frac{d\vec{e}_i{}'}{dt}\right|_{\Sigma}}_{\frac{\delta\vec{A}'}{dt}}$$

Consider change of $\vec{A}'$ due to rotation



Geometry



$|\delta\vec{A}| = \sin\alpha|\vec{A}|\omega dt$

Note: $\delta\vec{A} \perp \vec{\omega}$
$\delta\vec{A} \perp \vec{A}$
$|\delta\vec{A}| = \sin\alpha|\vec{A}'|\omega dt \;, \; \alpha = \angle(\vec{\omega}, \vec{A})$

$\left.\begin{array}{l} \\ \\ \\ \end{array}\right\} \delta\vec{A} = (\vec{\omega}\times\vec{A}')dt$

$$\delta\vec{A}' = (\vec{\omega}\times\vec{A}')\,dt$$

change due to rotation: $\left.\dfrac{d\vec{A}'}{dt}\right|_{\text{Rotation}} = \vec{\omega}\times\vec{A}'$

in summary:

$$\left.\frac{d\vec{A}'}{dt}\right|_{\Sigma} = \left.\frac{d\vec{A}'}{dt}\right|_{\Sigma'} + \vec{\omega}\times\vec{A}'$$ (holds for arbitrary vector)

Time derivative operator

$$\left.\frac{d}{dt}\right|_{\Sigma} = \left.\frac{d}{dt}\right|_{\Sigma'} + \vec{\omega}\times$$

Use for position vector $\vec{r}\,' = \vec{r} - \vec{r}_0$

$$\left.\frac{d\vec{r}}{dt}\right|_{\Sigma} - \left.\frac{d\vec{r}_0}{dt}\right|_{\Sigma} = \left.\frac{d\vec{r}\,'}{dt}\right|_{\Sigma} = \left.\frac{d\vec{r}\,'}{dt}\right|_{\Sigma'} + \vec{\omega}\times\vec{r}\,'$$

2nd derivative

$$\left.\frac{d^2\vec{r}}{dt^2}\right|_{\Sigma} - \left.\frac{d^2\vec{r}_0}{dt^2}\right|_{\Sigma} = \left.\frac{d}{dt}\right|_{\Sigma}\left[\left.\frac{d\vec{r}\,'}{dt}\right|_{\Sigma'} + \vec{\omega}\times\vec{r}\,'\right]$$

$$= \left.\frac{d^2\vec{r}\,'}{dt^2}\right|_{\Sigma'} + \vec{\omega}\times\left.\frac{d\vec{r}\,'}{dt}\right|_{\Sigma'} + \left.\frac{d}{dt}\right|_{\Sigma}\left[\vec{\omega}\times\vec{r}\,'\right]$$

$$= \left.\frac{d^2\vec{r}\,'}{dt^2}\right|_{\Sigma'} + \vec{\omega}\times\left.\frac{d\vec{r}\,'}{dt}\right|_{\Sigma'} + \left.\frac{d\vec{\omega}}{dt}\right|_{\Sigma}\times\vec{r}\,'$$

$$+ \vec{\omega}\times\underbrace{\left.\frac{d\vec{r}\,'}{dt}\right|_{\Sigma}}_{\left.\frac{d\vec{r}\,'}{dt}\right|_{\Sigma'} + \vec{\omega}\times\vec{r}\,'}$$

$$\left.\ddot{\vec{r}}\,'\right|_{\Sigma'} = \left.\frac{d^2\vec{r}\,'}{dt^2}\right|_{\Sigma'} + 2\,\vec{\omega}\times\left.\frac{d\vec{r}\,'}{dt}\right|_{\Sigma'} + \left.\frac{d\vec{\omega}}{dt}\right|_{\Sigma}\times\vec{r}\,' + \vec{\omega}\times(\vec{\omega}\times\vec{r}\,')$$

mutiply with m and use Newton's law to relate forces

$$m\left.\ddot{\vec{r}}\,'\right|_{\Sigma'} = \vec{F}\,' = \vec{F} - m\ddot{\vec{r}}_0 - 2m\,\vec{\omega}\times\left.\frac{d\vec{r}\,'}{dt}\right|_{\Sigma'} - m\,\vec{\omega}\times(\vec{\omega}\times\vec{r}\,') - m\,\dot{\vec{\omega}}\times\vec{r}\,'$$

additional pseudo forces (inerta forces):

$$\vec{F}' = \underbrace{\vec{F} - m\ddot{\vec{r}}_o}_{\substack{\text{relative acceleration} \\ \text{of coordinate systems}}} - \underbrace{2m\,\vec{\omega} \times \frac{d\vec{r}\,'}{dt}\Big|_{\Sigma'}}_{\substack{\text{Coriolis} \\ \text{force}}} - \underbrace{m\,\vec{\omega}\times(\omega\times\vec{r}\,')}_{\substack{\text{centrifugal} \\ \text{force}}} - m\,\dot{\vec{\omega}}\times\vec{r}\,'$$

Simple Problems of Dynamics

Newton's law $\vec{F} = \dot{\vec{p}} \overset{m=\text{const}}{=} m\,\vec{a} = m\,\ddot{\vec{r}}$

For $\vec{F} = \vec{F}(\vec{r}, \dot{\vec{r}}, t)$: $\overbrace{\text{differential}}^{\text{2nd order}}$ equation for $\vec{r}$

Solution scheme:

1) writing down / setting equations of motion (considering all forces)
2) solution of the differential equation (mathematical methods)
3) physical discussion of solution

For $\vec{F} = \vec{F}(t)$: similar to kinematics $\vec{a}(t) = \frac{\vec{F}(t)}{m}$

solution: already discussed

$$\vec{r}(t) = \vec{r}_0 + \vec{V}_o(t - t_o) + \int_{t_o}^{t} dt' \int_{t_o}^{t'} dt''\,\vec{a}(t)$$

Examples:

1) force-free motion. $\vec{F} = 0$, $\vec{r}(t) = \vec{r}_o + \vec{v}_o(t - t_o)$

2) motion in homogeneous gravitational field

$$\vec{F} = m\vec{g} \quad , \quad m\vec{a} = m\vec{g} \quad \Rightarrow \vec{a} = \vec{g}$$

$$\vec{r}(t) = \vec{r}_0 + \vec{v}_0(t - t_0) + \frac{1}{2}\vec{g}(t - t_0)^2$$

initial conditions: example free fall from height h

$$\vec{r}_0 = (0, 0, h)$$
$$\vec{v}_0 = 0$$

solution: $\vec{r}(t) = \sum_i x_i \vec{e}_i \quad , \quad x_1(t) = x_2(t) = 0$

$$x_3(t) = h - \frac{1}{2} g t^2 \quad , \quad \dot{x}_3(t) = -g t$$

fall-time: condition $x_3(t) = 0$

$$0 = h - \frac{1}{2} g t_F^2$$
$$\Rightarrow t_F = \sqrt{2 \frac{g}{h}}$$

corresponding velocity $\dot{x}_3(t_F) = v_F$

$$v_F = -\sqrt{2hg}$$

For forces that depend on the position or the velocity, more sophisticated mathematical methods required:

$$m\ddot{\vec{r}} = \vec{F}(\dot{\vec{r}}, \vec{r}, t)$$

$m\ddot{\vec{r}} - \vec{F}(\dot{\vec{r}}, \vec{r}, t) = 0$
coupled differential eqn. of 2^{nd} order

differential equation of n-th order:

$$f(x^{(n)}, x^{(n-1)}, \dots, \dot{x}, x, t) = 0$$

with

$$x^{(n)} = \frac{d^n x}{dt^n} \qquad n\text{-th derivative}$$

general solution of differential equation of n-th order:

$$x = x(t, \gamma_1, \gamma_2, \dots, \gamma_n)$$

where $\{\gamma_i\}$ are n independent parameters

special solution (particular solution) for fixed set $\{\gamma_i\}$ which are determined fx. by initial conditions $\{x^{(n)}(t_0)\}$

Example:

$$\ddot{x}_3 + g = 0 \qquad \text{from} \quad m\ddot{\vec{r}} = m\vec{g}$$
$$\vec{g} = (0, 0, -g)$$

general solution

$$x_3(t) = \gamma_1 + \gamma_2 t - \frac{1}{2} g t^2$$

for $t_0 = 0$, $\dot{x}_3(0) = v_0$, $x_3(0) = x_0$

the parameters are : $\gamma_1 = x_0$, $\gamma_2 = v_0$

Definition: Linear differential equation

$$\sum_{j=0}^{n} a_j(t)\, x^{(j)}(t) = \beta(t)$$

derivatives up to n-th order appear linearly

$\beta(t) = 0$ homogeneous
$\beta(t) \neq 0$ inhomogeneous

Superposition principle (linear homogeneous D.E.)

if $x_1(t)$ and $x_2(t)$ solve LDE, also

$\tilde{x}(t) = c_1 x_1(t) + c_2 x_2(t)$ solve it with arbitrary constants c_i

Definition: Linear independency of (solution) functions

$x_1(t), \dots, x_n(t)$ are linear independent if

$$\sum_{j=1}^{n} \alpha_j \, x_n(t) = 0$$

is an identity only for $\alpha_1 = \dots = \alpha_n = 0$

general solution of homogeneous LDE (n-th order):

$$x(t, \gamma_1, \dots, \gamma_n) = \sum_{j=1}^{m} \alpha_j \, x_j(t)$$

where $x_j(t)$ are linearly independent solutions

$m = n$ $\begin{cases} m \geq n & : \text{need minimum } n \text{ parameters for general solution} \\ m \leq n & : \text{cannot have more than } n \ \alpha_j \text{ to parametrize general solution} \end{cases}$

Solution of inhomogeneous LDE

particular solution of inhom. LDE!

$$\bar{x}(t, \gamma_1, \dots, \gamma_n) = \sum_{j=1}^{n} \alpha_j \, x_j(t) + x_0(t)$$

general solution of hom. LDE
(superposition of n special solutions)

Applications : Motion with friction in the homogeneous gravitational field

Friction forces in gases and liquids:

1) Newton's law of friction $\vec{F}_{R_g} = -\beta v \, \vec{v}$
2) Stoke's law of friction $\vec{F}_{R_g} = -\alpha \vec{v}$

Friction between solids
1) sliding friction $\vec{F}_{R_s} = -\mu F_\perp \hat{v}, \ \vec{v} \neq 0$
2) static friction $\vec{F}_{R_s} = -\vec{F}_\parallel \ , \ \vec{v} = 0$
(compensates external force, $F_\parallel < \mu F_\perp$)

Equation of motion:

$$m\ddot{\vec{r}} = m\vec{g} + \vec{F}_{R_g} \iff m\ddot{\vec{r}} - \vec{F}_{R_g}(\dot{\vec{r}}) = m\vec{g}$$

(inhomogeneous DE of 2nd order)

Newton's law : non-linear
Stoke's law : linear DE with inhomogeneity $m\vec{g}$

$$m\ddot{\vec{r}} + \alpha\dot{\vec{r}} = m\vec{g}$$

1) construct solution for homogeneous DE

$$m\ddot{\vec{r}} + \alpha\dot{\vec{r}} = 0 \qquad \text{decouple equations}$$
$$m\ddot{x}_i + \alpha\dot{x}_i = 0 \qquad i = 1,2,3$$

(DE with constant coefficients) 23.11.2016
need : 2 linear independent solutions

Ansatz $x_i(t) = e^{\gamma t}, \ \dot{x}_i(t) = \gamma e^{\gamma t}, \ \ddot{x}_i(t) = \gamma^2 e^{\gamma t}$

insert: $m \gamma^2 e^{\gamma t} + \alpha \gamma e^{\gamma t} = 0$

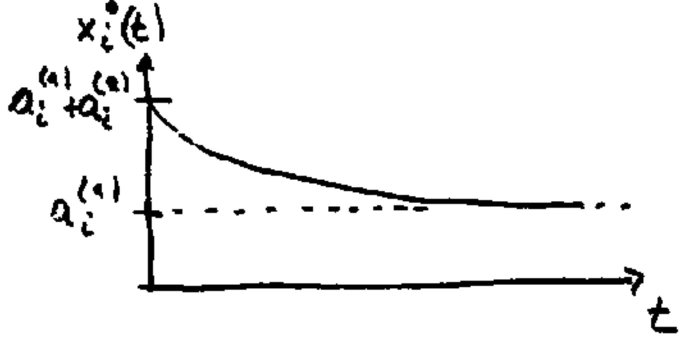
$$\Rightarrow \quad m\gamma^2 + \alpha \gamma = 0$$
$$(m\gamma + \alpha)\gamma = 0 \qquad \gamma_1 = 0 \,,\; \gamma_2 = -\frac{\alpha}{2}$$

two linearly independent solutions

$$x_1(t) = e^{0t} = 1$$
$$x_2(t) = e^{-\frac{\alpha}{2} t}$$

general solution:

$$x_i^0(t) = a_i^{(1)} + a_i^{(2)} e^{-\frac{\alpha}{2} t}$$



motion of particle under influence of friction only (full solution for $i = 1,2$

now consider $i = 3$ $\quad m\ddot{x}_3 + \alpha \dot{x}_3 = -mg$

construct one special solution where friction force and gravitational force are equal:

$$\alpha \dot{x}_3^s = -mg \quad \Rightarrow \quad \dot{x}_3^s = -\frac{mg}{\alpha}$$

(force free motion) $\qquad x_3^s(t) = -\frac{mg}{\alpha} t \,,\; \ddot{x}_3 = 0$

general solution of inhomogeneous equation:

$$x_3(t) = a_3^{(1)} + a_3^{(2)} e^{-\frac{\alpha}{m} t} - \frac{mg}{\alpha} t$$

$$x_{1/2}(t) = a_{1/2}^{(1)} + a_{1/2}^{(2)} e^{-\frac{\alpha}{m} t}$$

Discussion of the solution:

calculate the velocities : $v_{1/2}(t) = -a_{1/2}^{(2)}\,\frac{\alpha}{m}\,e^{-\frac{\alpha}{m}t}$

$$v_3(t) = -a_3^{(2)}\,\frac{\alpha}{m}\,e^{-\frac{\alpha}{m}t} - \frac{m}{\alpha}g$$

choose initial conditions : vertical fall

$$\vec{r}(t_0) = (0,0,h) \qquad t_0 = 0$$
$$\vec{v}(t_0) = (0,0,0)$$

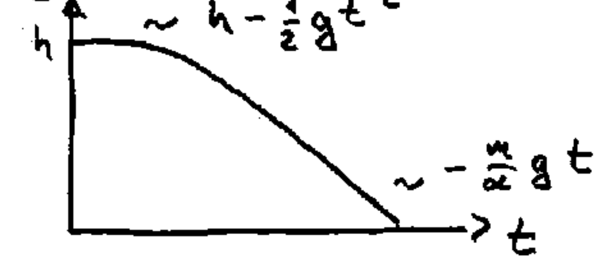
fixing the parameters $a_{1/2}^{(1)} = 0$ and $a_{1/2}^{2} = 0$ yields

$x_1(t) = 0\,,\quad x_2(t) = 0$ no motion in 1 and 2 direction

$$x_3(0) = a_3^{(1)} + a_3^{(2)} = h$$
$$\dot{x}_3(0) = -a_3^{(2)}\,\frac{\alpha}{m} - \frac{m}{\alpha}g = 0 \quad \Rightarrow a_3^{(2)} = -\frac{m^2}{\alpha^2}g$$

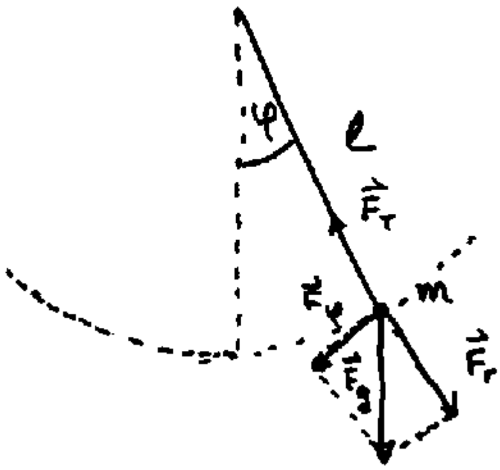
inserting $a_3^{(1)} - \frac{m^2}{\alpha^2}g = h \quad \Rightarrow a_3^{(1)} = h + \frac{m^2}{\alpha^2}g$

$$x_3(t) = h + \frac{m^2}{\alpha^2}g\left(1 - e^{-\frac{\alpha}{m}t}\right) - \frac{m}{\alpha}g\,t$$
$$v_3(t) = \frac{m}{\alpha}g\left(e^{-\frac{\alpha}{m}t} - 1\right)$$



$$\lim_{t\to\infty} v_3(t) = -\frac{m}{\alpha}g$$

Simple Pendulum (mathematical pendulum)



mass point on massless thread

$$\vec{F} = m\ddot{\vec{r}}$$

consider all forces

$$\vec{F}_g = m\vec{g}$$

$$\left.\begin{aligned}\vec{F}_r &= F_r\vec{e}_r\\ \vec{F}_\varphi &= F_\varphi\vec{e}_\varphi\end{aligned}\right\} \vec{F}_g = \vec{F}_r + \vec{F}_\varphi$$

decompose force

calculate components : $F_r = m\,g\cos\varphi$

$F_\varphi = -m\,g\sin\varphi$

equations of motion in polar coordinates

$$m\left[(\ddot{r} - r\dot{\varphi}^2)\vec{e}_r + (r\ddot{\varphi} + 2\dot{r}\varphi)\vec{e}_\varphi\right] = (F_r + F_T)\vec{e}_r + F_\varphi\vec{e}_\varphi$$

F_T : thread tension : $r = \ell$, $\dot{r} = 0$, $\ddot{r} = 0$

radial component $\quad -mr\dot{\varphi}^2 = F_r + F_T$

calculate F_T once dynamics of $\varphi(t)$ is known.

angular component $\quad m\ell\ddot{\varphi} = -mg\sin\varphi$

$$\ell\ddot{\varphi} + g\sin\varphi = 0 \quad \text{nonlinear D.E. (2nd order}$$

small angles $\quad \sin\varphi \approx \varphi$

$$\ddot{\varphi} + \frac{g}{\ell}\varphi = 0 \quad , \text{introduce } \omega^2 = \frac{g}{\ell}$$

$\ddot{\varphi} = -\omega^2 \varphi$ linear D.E. (2nd order)

2 linearly independent solutions

$\varphi_1 = \sin(\omega t)$, $\ddot{\varphi}_1 = -\omega^2 \sin(\omega t)$
$\varphi_2 = \cos(\omega t)$, $\ddot{\varphi}_2 = -\omega^2 \cos(\omega t)$

linearly independent $c_1 \sin(\omega t) + c_2 \cos(\omega t) = 0$
$\Rightarrow c_1 = c_2 = 0$

general solution:

$\varphi(t) = A \sin(\omega t) + B \cos(\omega t)$; A, B fixed by initial cond.

angular frequency : $\omega = \sqrt{\frac{g}{\ell}}$ independent of mass ($m_{in} = m_h$)

oscillation period $T = 2\pi \sqrt{\frac{\ell}{g}} = \frac{2\pi}{\omega}$ (from $\omega T = 2\pi$)

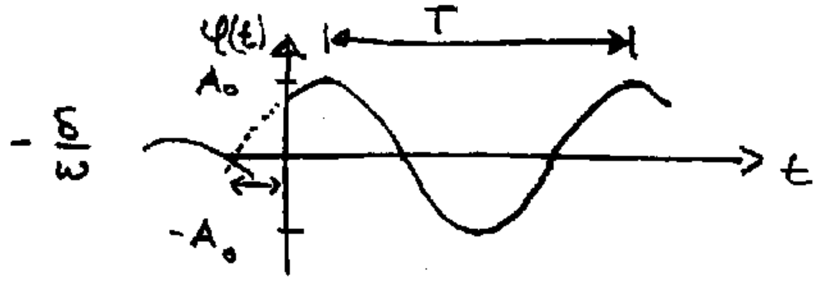
frequency : $\nu = \frac{1}{T} = \frac{1}{2\pi}\sqrt{\frac{g}{\ell}}$

Rewrite solution (with two other parameters):

$$\varphi(t) = \underbrace{\sqrt{A^2+B^2}}_{A_0} \left(\underbrace{\frac{A}{\sqrt{A^2+B^2}}}_{\cos\delta} \sin(\omega t) + \underbrace{\frac{B}{\sqrt{A^2+B^2}}}_{\sin\delta} \cos(\omega t) \right)$$

use $\sin(x+y) = \sin x \cos y + \cos x \sin y$

$\varphi(t) = A_0 \sin(\omega t + \delta)$ A_0: Amplitude
δ : phase shift



Complex Numbers

Imaginary numbers : $\mathbb{I}$ (set of imaginary numbers)

Definition : Unit of imaginary numbers

$$i^2 = -1 \quad , \quad i = \sqrt{-1}$$

Definition : set of imaginary numbers $\{\alpha\}$:

$$\alpha^2 < 0 \quad , \quad \alpha = y \cdot i \qquad y \in \mathbb{R}$$

Definition : Complex numbers $\mathbb{C}$ (set of complex numbers)

$$z = x + iy \qquad x, y \in \mathbb{R}$$

real part $\operatorname{Re} z = x$

imaginary part $\operatorname{Im} z = y$

conjugated complex number $z^* = x - iy$

$z = 0 \Rightarrow \operatorname{Re} z = x = 0, \ \operatorname{Im} z = y = 0$

$\mathbb{C} \to \mathbb{I} : x = 0$

$\mathbb{C} \to \mathbb{R} : y = 0$

Calculation rules $z_1, z_2 \in \mathbb{C}$

Addition $\quad z_1 + z_2 = x_1 + iy_1 + x_2 + iy_2 = (x_1 + x_2) + i(y_1 + y_2)$

Multiplication $\quad z_1 \cdot z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1x_2 + iy_1x_2 + ix_1y_2 - y_1y_2$

$$= (x_1x_2 - y_1y_2) + i\,(y_1x_2 + x_1y_2)$$

special case $\quad z z^* = (x+iy)(x-iy) = x^2 + y^2 = |z|^2$

Division $\quad \dfrac{z_1}{z_2} = \dfrac{x_1 + iy_1}{x_2 + iy_2} = \dfrac{z_1}{z_2} \cdot \dfrac{z_2^*}{z_2^*} = \dfrac{(x_1 + iy_1)(x_2 - iy_2)}{x_2^2 + y_2^2}$

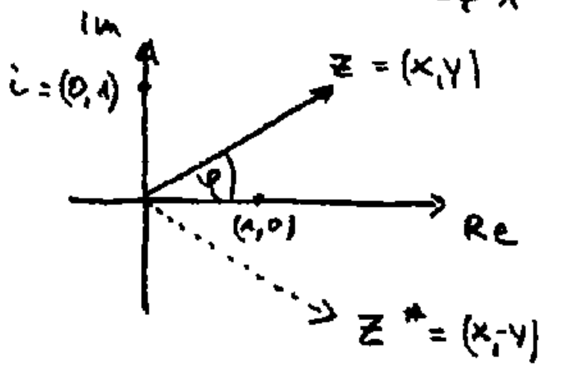
$$= \frac{x_1x_2 + y_1y_2}{x_2^2 + y_2^2} + i\,\frac{y_1x_2 - x_1y_2}{x_2^2 + y_2^2}$$

special case $\quad \dfrac{1}{i} = \dfrac{1}{i}\,\dfrac{-i}{-i} = \dfrac{-i}{-i^2} = \dfrac{-i}{-(-1)} = -i$

Complex plane :

Consider real and imaginary part as components of vecto

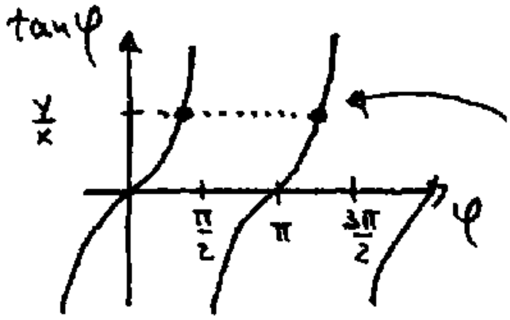
$$z = x + iy = x \underbrace{(1,0)}_{1} + y \underbrace{(0,1)}_{i} = (x, y)$$



polar representation

$$z = |z| \left(\cos\varphi + i \sin\varphi\right)$$

$$|z| = \sqrt{x^2 + y^2}$$



$$\varphi = \arctan\left(\frac{y}{x}\right) = \arg(z)$$

(argument of z)

choose "correct" solution

Exponential form of a complex number

Recall: Series expansions

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$\sin(x) = x - \frac{x^3}{3!} + \dots = \sum_{n=0}^{\infty} \underbrace{(-1)^n}_{\frac{1}{i} i^{2n+1}} \frac{x^{2n+1}}{(2n+1)!}$$

$$= \frac{1}{i} \sum_{n=0}^{\infty} \frac{(ix)^{2n+1}}{(2n+1)!}$$

$$\cos x = 1 - \frac{x^2}{2} + \dots \quad \sum_{n=0}^{\infty} \underbrace{(-1)^n}_{i^{2n}} \frac{x^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} \frac{(ix)^{2n}}{(2n)!}$$

now consider:

$$\cos x + i \sin x = \sum_{n=0}^{\infty} \frac{(ix)^{2n}}{(2n)!} + i \frac{1}{i} \sum_{n=0}^{\infty} \frac{(ix)^{2n+1}}{(2n+1)!}$$

$$= \sum_{m=0}^{\infty} \frac{(ix)^m}{m!} = e^{ix}$$

Euler formula: $e^{i\varphi} = \cos\varphi + i \sin\varphi$

$$\mathrm{Re}\, e^{i\varphi} = \cos\varphi \qquad z = |z|\, e^{i\varphi}$$

$$\mathrm{Im}\, e^{i\varphi} = \sin\varphi \qquad z^* = |z|\, e^{-i\varphi}$$

inverse Euler formula: $\cos\varphi = \frac{1}{2}\left(e^{i\varphi}+e^{-i\varphi}\right)$

$\sin\varphi = \frac{1}{2i}\left(e^{i\varphi}-e^{-i\varphi}\right)$

further calculation rules:

Multiplication $\quad z_1 \cdot z_2 = |z_1|\cdot|z_2| \; e^{i(\varphi_1+\varphi_2)}$

Division $\quad \frac{z_1}{z_2} = \frac{|z_1|}{|z_2|} \; e^{i(\varphi_1-\varphi_2)}$

Power $\quad z^n = |z|^n \; e^{in\varphi}$

Root $\quad \sqrt[n]{z} = \sqrt[n]{|z|} \; e^{i\frac{1}{n}\varphi} \qquad , \sqrt[n]{z} = z^{\frac{1}{n}}$

periodicity in φ: $\quad 1 = e^{i\cdot 0} = e^{2\pi i}$

any complex number $\quad z = |z|\, e^{i\varphi} = |z|\, e^{i(\varphi+2\pi n)}$

Examples:

$$\begin{aligned} \ln(-5) &= \ln\left(5\, e^{i\pi}\right) \\ &= \ln 5 + \ln\left(e^{i(\pi+2\pi n)}\right) \\ &= \ln 5 + i\pi(1+2n) \end{aligned}$$

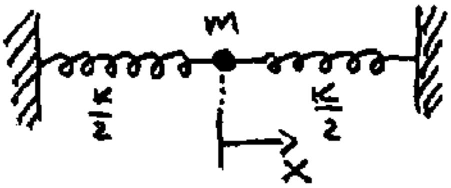
$$z = 1-i = \sqrt{1^2+1^2}\; e^{i\varphi} = \sqrt{2}\; e^{i\frac{7\pi}{4}}$$

$\varphi = \arctan(-1) = \frac{7\pi}{4}$

Linear harmonic oscillator

$$\ddot{x} + \omega_0^2 x = 0 \qquad \text{(L.D.E. 2n order)}$$

1) mass on springs



Hooke's law
$F = -kx$

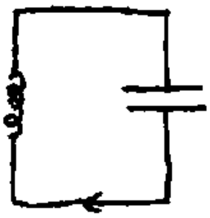
$$m\ddot{x} = -kx$$

$$\ddot{x} + \frac{k}{m}x = 0 \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

2) mathematical pendulum $\quad \ddot{\varphi} + \frac{g}{\ell}\varphi = 0 \quad , \; \omega_0 = \sqrt{\frac{g}{\ell}}$

2) non-mechanical : electrical oscillator

inductance L



C capacity

I (electrical current)

$$L\ddot{I} + \frac{1}{C} I = 0$$

$$\ddot{I} + \frac{1}{LC} I = 0 \quad , \omega_0 = \frac{1}{\sqrt{LC}}$$

Now : solve L.D.E. with exponential ansatz

$$x(t) = e^{\alpha t} \qquad \ddot{x}(t) = \alpha^2 e^{\alpha t}$$

$$\Rightarrow \alpha^2 e^{\alpha t} + \omega_0^2 e^{\alpha t} = 0$$

$$(\alpha^2 + \omega_0^2)\, e^{\alpha t} = 0 \qquad \alpha^2 = -\omega_0^2$$

no solution for $\alpha \in \mathbb{R}$

$$\alpha_\pm = \pm\sqrt{-\omega_0^2} = \pm i\,\omega_0$$

two linearly independent solutions $x_1(t) = e^{i\omega_0 t}$, $x_2(t) = e^{-i\omega_0 t}$

general solution :

$$x(t) = A\, e^{i\omega_0 t} + B e^{-i\omega_0 t}$$

fix A and B with initial conditions and need that $x(t)$ is real quantity : $x = x^*$

$$x(t) = A\, e^{i\omega_0 t} + B e^{-i\omega_0 t} = x^*(t) = A^* e^{-i\omega_0 t} + B^* e^{i\omega_0 t}$$

$$\Rightarrow (A - B^*)\, e^{i\omega_0 t} + (B - A^*)\, e^{-i\omega_0 t} = 0$$

$$A = B^* = a + ib$$

rewrite solution

$$x(t) = (a+ib)\, e^{i\omega_0 t} + (a-ib)\, e^{i\omega_0 t}$$

$$= 2a\tfrac{1}{2}\left(e^{i\omega_0 t} + e^{-i\omega_0 t}\right) - 2b\,\tfrac{1}{2i}\left(e^{i\omega_0 t} - e^{-i\omega_0 t}\right)$$

$$x(t) = 2a\cos(\omega_0 t) - 2b\sin(\omega_0 t)$$

(Solution as discussed before $x(t) = A_0 \sin(\omega_0 t + \delta)$)

Discussion on example:

1) $x(0) = x_0$, $\dot{x}(0) = 0$ (oscillator at rest at $t=0$)

$x_0 = 2a$

$\dot{x}(t) = -2a\omega_0 \sin(\omega t) - 2b\omega_0 \cos(\omega_0 t)$

$0 = -2b\omega_0$

special solution $x(t) = x_0 \cos(\omega_0 t)$

2) $x(0) = 0$, $\dot{x}(0) = v_0$ (oscillator at origin, but with finite velocity at $t=0$)

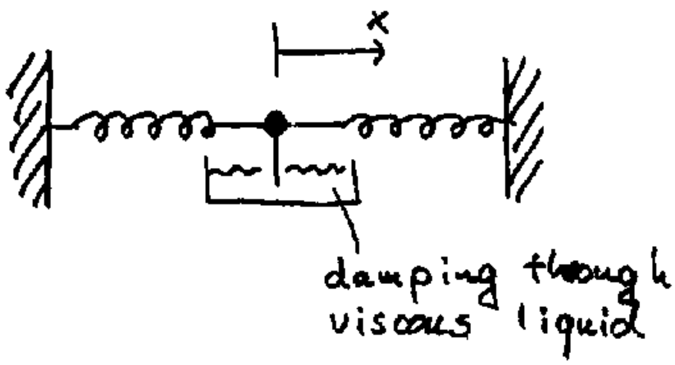
$0 = 2a$

$v_0 = -2b\omega_0 \Rightarrow -2b = \frac{v_0}{\omega_0}$

special solution $x(t) = \frac{v_0}{\omega_0}\sin(\omega_0 t)$

Linear harmonic oscillator with damping

Examples



1)

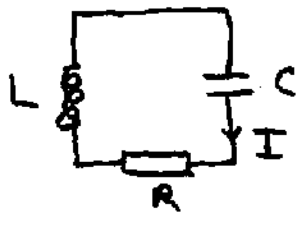
Stoke's Law, Hooke's law

$$m\ddot{x} = -\alpha\dot{x} - kx$$

$$\Rightarrow \ddot{x} + \frac{\alpha}{m}\dot{x} + \frac{k}{m}x = 0$$

2) similar electric circuit

$$L\ddot{I} + R\dot{I} + \frac{1}{C}I = 0$$



inductance
capacity
resistance

general form of homogeneous O.E.

$$\ddot{x} + 2\beta\dot{x} + \omega_0^2 x = 0, \quad \omega_0 = \sqrt{\frac{k}{m}}, \quad \beta = \frac{\alpha}{2m}$$

Ansatz: $x(t) = e^{\lambda t}$, $\dot{x}(t) = \lambda e^{\lambda t}$, $\ddot{x}(t) = \lambda^2 e^{\lambda t}$

$$\Rightarrow \left(\lambda^2 + 2\beta\lambda + \omega_0^2\right) e^{\lambda t} = 0$$

$$\lambda^2 + 2\beta\lambda + \omega_0^2 = 0 \qquad \text{(polynomial equation)}$$

$$\lambda_{1/2} = -\beta \pm \sqrt{\beta^2 - \omega_0^2}$$

1) $\beta^2 < \omega_0^2$ 2 complex solutions
2) $\beta = \omega_0$ 1 solution
3) $\beta^2 > \omega_0^2$ 2 real solution

1) + 3) : two linearly independent solutions
general solution

$$x(t) = a_1 e^{\lambda_1 t} + a_2 e^{\lambda_2 t}$$

2) : need to get another solution

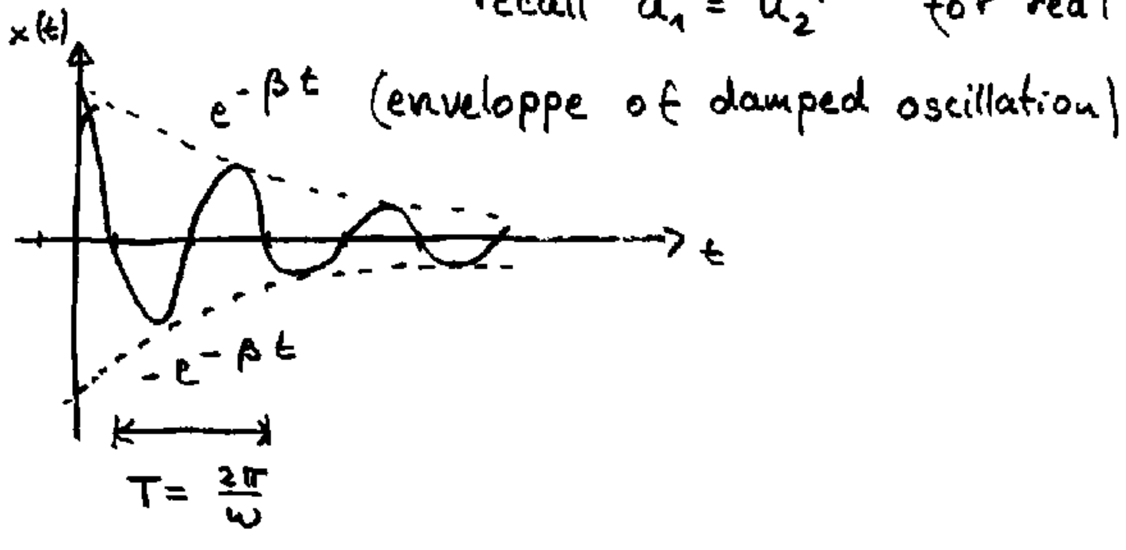
Discussion:

1) Oscillatory Case (Weak damping)
introduce eigenfrequency $\omega = \sqrt{\omega_0^2 - \beta^2}$

$$\lambda_{1/2} = -\beta \pm i\omega$$

$$x(t) = \underbrace{e^{-\beta t}}_{\text{overall damping}} \underbrace{\left(a_1 e^{i\omega t} + a_2 e^{-i\omega t}\right)}_{\text{Solution as discussed without damping, but } \omega_0 \to \omega = \sqrt{\omega_0^2 - \beta^2}}$$

recall $a_1 = a_2^*$ for real $x(t)$



alternative forms of the solution

general initial conditions : $x_0 = x(t=0) = a_1 + a_1^*$
$v_0 = \dot{x}(t=0) = -\beta(a_1 + a_1^*) + i\omega(a_1 - a_1^*)$

$$x_0 = 2 \,\mathrm{Re}\, a_1$$
$$v_0 = -\beta x_0 + i\omega(a_1 - a_1^*) = -\beta x_0 - \omega\, 2 \,\mathrm{Im}\, a_1$$
$$v_0 + \beta x_0 = -\omega\, 2\,\mathrm{Im}\, a_1$$
$$a_1 = \frac{x_0}{2} - i\frac{v_0 + \beta x_0}{2\omega}$$
$$x(t) = e^{-\beta t}\left[x_0 \cos(\omega t) + \frac{v_0 + \beta x_0}{\omega} \sin(\omega t)\right]$$

introducing $A = \sqrt{x_0^2 + \left(\frac{v_0 + \beta x_0}{\omega}\right)^2}$

$$\varphi = \arctan\left(\frac{\omega x_0}{v_0 + \beta x_0}\right)$$

$$x(t) = e^{-\beta t} A \sin(\omega t + \varphi)$$

2) Critical damping : Aperiodic Limiting Case

$\beta = \omega_0$ only one solution : $\lambda = -\beta$

$x(t) = e^{-\beta t}$ extended ansatz $x(t) = e^{-\beta t} f(t)$

derive diff. eq. for f

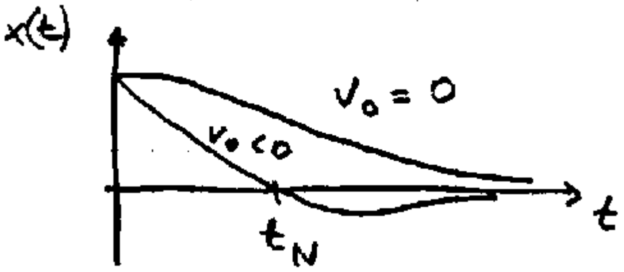
here take suitable limit

$$\omega = \sqrt{\omega_0^2 - \beta^2} \to 0$$

$$\lim_{\omega \to 0} x(t) = \lim_{\omega \to 0}\left[e^{-\beta t}\left\{x_0 \cos(\omega t) + \frac{v_0 + \beta x_0}{\omega}\sin(\omega t)\right\}\right]$$
$$= e^{-\beta t}\left\{x_0 \underbrace{\lim_{\omega \to 0} \cos(\omega t)}_{1} + (v_0 + \beta x_0) \underbrace{\lim_{\omega \to 0} \frac{\sin(\omega t)}{\omega}}_{t \underbrace{\lim_{\omega \to 0} \frac{\sin(\omega t)}{\omega t}}_{1} = t}\right\}$$

have full solution in terms of initial conditions

$$x(t) = e^{-\beta t}\{x_0 + (v_0 + \beta x_0)t\}$$



no oscillations
one zero crossing possible:

$$0 = x_0 + (v_0 + \beta x_0)t_N$$
$$t_N = -\frac{x_0}{v_0 + \beta x_0}$$

3) Strong damping (creeping case)
$\beta > \omega_0$: two real solutions

$$\lambda_{1/2} = -\beta \pm \gamma \qquad \gamma = \sqrt{\beta^2 - \omega_0^2} < \beta \quad \Rightarrow \lambda_{1/2} < 0$$

general solution:

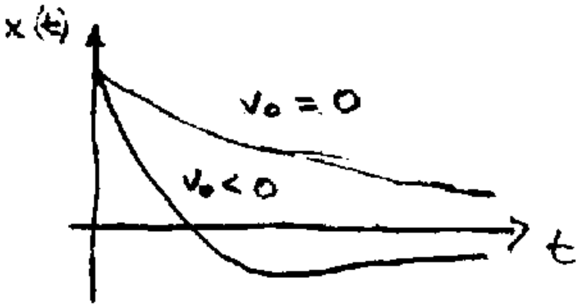
$$x(t) = e^{-\beta t}\left(a_1 e^{\gamma t} + a_2 e^{-\gamma t}\right)$$

rewrite with initial conditions $x(t=0) = x_0$
$\dot{x}(t=0) = v_0$

$$x(t) = e^{-\beta t}\left(x_0 \cosh(\gamma t) + \frac{v_0 + \beta x_0}{\gamma}\sinh(\gamma t)\right)$$

$$a_1 = \frac{1}{2}\left(x_0 + \frac{v_0 + \beta x_0}{\gamma}\right), \quad a_2 = \frac{1}{2}\left(x_0 - \frac{v_0 + \beta x_0}{\gamma}\right)$$

$$\cosh(x) = \frac{1}{2}\left(e^{x} + e^{-x}\right), \quad \sinh(x) = \frac{1}{2}\left(e^{x} - e^{-x}\right)$$



zero crossing still possible, but amplitude larger than for aperiodic case (same initial conditions : $e^{\gamma t} > t$

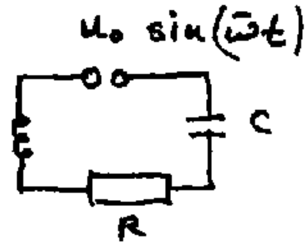
Damped Linear Oscillator under Influence of external Force

1) mechanical oscillator

$$m \ddot{x} + \alpha \dot{x} + K x = F(t)$$

special case: periodic force: $F(t) = mf \cos(\bar{\omega} t)$

2) electrical oscillator



$$L \ddot{I} + R \dot{I} + \frac{1}{C} I = U_0 \bar{\omega} \cos(\bar{\omega} t)$$

inhomogeneous D.E (complex valued)

$$\ddot{z} + 2\beta \dot{z} + \omega_0^2 z = f \, e^{i \bar{\omega} t} \qquad x(t) = \mathrm{Re}\, z(t)$$

general solution: $z(t) = z_{hom}(t) + z_0(t)$ ← special solution

solution of hom. D.E.

consider solution after settling time: $\lim_{t \to \infty} z_{hom}(t) = 0$

ansatz:
$$z_0(t) = A \, e^{i \bar{\omega} t}$$
$$\dot{z}_0(t) = i \bar{\omega} A \, e^{i \bar{\omega} t}$$
$$\ddot{z}(t) = -\bar{\omega}^2 A \, e^{i \bar{\omega} t}$$

insert: $\left(-\bar{\omega}^2 + 2 i \beta \bar{\omega} + \omega_0^2\right) A \, e^{i \bar{\omega} t} = \frac{f}{m} e^{i \bar{\omega} t}$

$$\Rightarrow \left(-\bar{\omega}^2 + 2 i \beta \bar{\omega} + \omega_0^2\right) A = \frac{f}{m}$$

$$A = -f \frac{1}{\bar{\omega}^2 - \omega_0^2 - 2 i \beta \bar{\omega}} = \mathrm{Re}\, A + i \, \mathrm{Im}\, A = |A| e^{i\varphi}$$

$$= -f \frac{\bar{\omega}^2 - \omega_0^2 + 2 i \beta \bar{\omega}}{(\bar{\omega}^2 - \omega_0^2)^2 + 4 \beta^2 \bar{\omega}^2}$$

$$|A| = f \frac{1}{\sqrt{(\bar{\omega}^2 - \omega_0^2)^2 + 4 \beta^2 \bar{\omega}^2}}, \quad \tan \varphi = \frac{\mathrm{Im}\, A}{\mathrm{Re}\, A} = \frac{2 \beta \omega}{\bar{\omega}^2 - \omega_0^2}$$

check for correct $\arg(A)$: $\quad \operatorname{Im} A < 0 \quad$ for $\bar{\omega} > 0$

$\operatorname{Re} A > 0$ or $\operatorname{Re} A < 0$

$$\bar{\varphi} = \arctan\left(\frac{2\beta\bar{\omega}}{\bar{\omega}^2 - \omega_0^2}\right) \qquad \pi < \bar{\varphi} < 2\pi$$

special solution: $\quad z_0(t) = A\, e^{i\bar{\omega} t} = |A|\, e^{i(\bar{\omega} t + \bar{\varphi})}$

$$= x_0(t) + i\, y_0(t)$$

$$x_0(t) = |A| \cos(\bar{\omega} t + \bar{\varphi})$$

general solution: $\quad x(t) = x_{hom}(t) + x_0(t)$

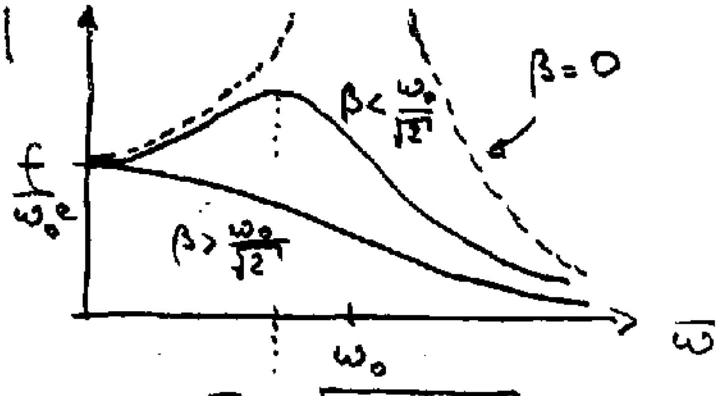
fixed by initial conditions $x(t_0)$, $\dot{x}(t_0)$

remaining for $t \gg \frac{1}{\beta}$

Discussion of special solution:

$$x_0(t) = |A| \cos(\bar{\omega} t + \bar{\varphi})$$

Amplitude:



$\bar{\omega}_m = \sqrt{\omega_0^2 - 2\beta^2}$

(resonance frequency)

$\bar{\omega} \to 0 \qquad A \to \frac{f}{\omega_0^2}$

$\bar{\omega} \to \infty \qquad A \propto \frac{1}{\omega_0^2} \to 0$

$$\frac{d|A|}{d\bar{\omega}} \overset{!}{=} 0$$

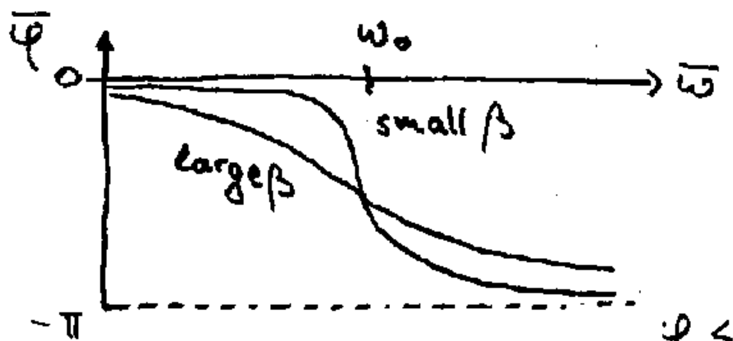
$$(\bar{\omega}^2 - \omega_0^2 + 2\beta^2)\,\omega = 0$$

solutions: $\bar{\omega}_0 = 0$

$$\omega_{1/2} = \pm\sqrt{\omega_0^2 - 2\beta^2}$$

1 real solution for: $\beta > \frac{\omega_0}{2}$

phase shift



$$\pi < \bar{\varphi} < 2\pi$$

$$-\pi < \bar{\varphi} < 0$$

$\varphi < 0$: displacement maximum reached after force reaches max.

Arbitrary space-dependent force in one dimension

$$F = F(x)$$

$$m\ddot{x} = F(x) \qquad \text{nonlinear D.E.}$$

consider : $V(x) = -\int^{x} F(x')\,dx'$, $\frac{d}{dt}V(x) = -F(x)\,\dot{x}$

$$\dot{x}\ddot{x} = \frac{d}{dt}\left(\frac{\dot{x}^2}{2}\right)$$

$$m\ddot{x}\dot{x} = F(x)\,\dot{x}$$

insert definitions

$$\frac{d}{dt}\left(m\frac{\dot{x}^2}{2}\right) = -\frac{d}{dt}V(x)$$

integration

$$m\frac{\dot{x}^2}{2} = -V(x) + E$$

← integration constant

$$\Rightarrow \quad \dot{x} = \sqrt{\frac{2(E - V(x))}{m}} = \frac{dx}{dt} \qquad dt = \frac{dx}{\sqrt{\frac{2(E-V(x))}{m}}}$$

integration via separation of variables

$$\int_{t_0}^{t} dt = \int_{x_0}^{x} \frac{dx'}{\sqrt{\frac{2(E-V(x'))}{m}}} \quad \Rightarrow \quad t(x) = \int_{x_0}^{x} \frac{dx'}{\sqrt{\frac{2(E-V)}{m}}} + t_0$$

inversion : $x(t)$ solution of D.E. with parameters t_0 and E

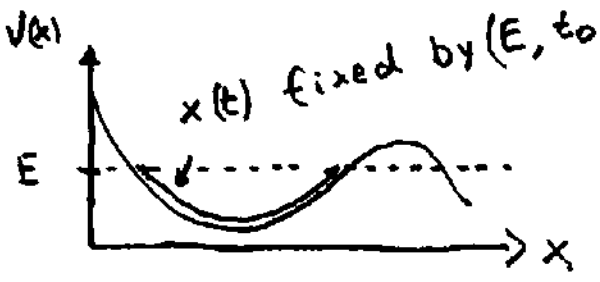
physical interpretation : $T = m\frac{\dot{x}^2}{2}$ (kinetic energy)

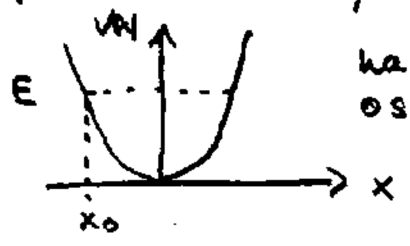
$V(x) = -\int^{x} F(x')\,dx$ (potential energy)

$E = T + V$ (total energy)

$\frac{dE}{dt} = 0$ (energy conservation)

Example $F = -kx$, $V(x) = \frac{k^2}{2}x$





harmonic oscillator

Fundamental concepts and theorems

Work

1 dimension with $F = F(x)$: $dW = -F\,dx$

generalisation : $\vec{F} = \vec{F}(\vec{r}, \dot{\vec{r}}, t)$

$\delta W = -\vec{F} \cdot d\vec{r}$ not a total differential (in general)

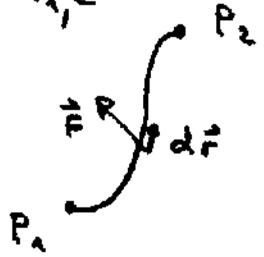
sign convention

$\vec{F} \uparrow\uparrow d\vec{r}$ $\quad dW < 0$

$\vec{F} \downarrow\uparrow d\vec{r}$ $\quad dW > 0$

$\vec{F} \perp d\vec{r}$ $\quad dW = 0$

$$W_{21} = -\int_{P_1, C}^{P_2} \vec{F} \cdot \delta\vec{r}$$



← integration by parametrization of space curve with parameter α

$$W_{21} = \int_{\alpha_1}^{\alpha_2} d\alpha \; \frac{d\vec{r}(\alpha)}{d\alpha} \cdot \vec{F}(\vec{r}(\alpha), \dot{\vec{r}}(\alpha), t)$$

depends on: P_1, P_2, velocity $\dot{\vec{r}}$, form of curve C

Power :

$$P = \frac{\delta W}{dt} = -\vec{F} \cdot \frac{d\vec{r}}{dt}$$

can be obtained from Newton's equation :

$$m\ddot{\vec{r}} = \vec{F}$$
$$m\ddot{\vec{r}} \cdot \dot{\vec{r}} = \vec{F} \cdot \dot{\vec{r}}$$

Units : work $\quad [W] = N\,m = J$

power $\quad [P] = \frac{[W]}{[t]} = \frac{J}{s} = W$

kinetic energy $\quad T = \frac{m}{2}\dot{\vec{r}}^{\,2}$

Newton's equation $\quad \frac{d}{dt}T = m\,\ddot{\vec{r}}\cdot\dot{\vec{r}} = -P = -\frac{\delta W}{dt}$

integration : $\quad W_{21} = -(T_2 - T_1) = T_1 - T_2$

$$= \frac{m}{2}\left(\dot{\vec{r}}^{\,2}(t_1) - \dot{\vec{r}}^{\,2}(t_2)\right)$$

Note : for $W_{21} \neq 0$ follows $|\dot{\vec{r}}(t_1)| \neq |\dot{\vec{r}}(t_2)|$

Potential energy and conservative forces

Potential $V(\vec{r})$: $\quad \frac{dV(\vec{r})}{dt} = -\vec{F}\cdot\frac{d\vec{r}}{dt} = P$

in this case $\quad dV = \delta W \quad$ (total differential of work exsists)

$\vec{F} = F(\vec{r})$: conservative force

calculation of force :

total differential :

$$dV(\vec{r}) = \sum_i \frac{\partial V}{\partial x_i}\,dx_i = \vec{\nabla} V \cdot d\vec{r} \quad \Big| \; \frac{1}{dt}$$

$$\dot{V}(\vec{r}) = \frac{dV}{dt} = \vec{\nabla} V \cdot \frac{d\vec{r}}{dt} = \vec{\nabla} V \cdot \dot{\vec{r}}$$

compare with modified Newton's equation yields

$$\vec{F} = -\vec{\nabla} V$$

consider general force : $\vec{F} = \vec{F}_{cons} + \vec{F}_{diss}$

$$\frac{d}{dt}T = \vec{F}\cdot\dot{\vec{r}} = \left(\vec{F}_{cons} + \vec{F}_{diss}\right)\dot{\vec{r}} = \frac{d}{dt}V + \vec{F}_{diss}\cdot\dot{\vec{r}}$$

$$\frac{d}{dt}(T+V) = \vec{F}_{diss}\cdot\dot{\vec{r}} \qquad E = T+V \quad \text{(Energy)}$$

$$\frac{d}{dt}E = \vec{F}_{diss}\cdot\dot{\vec{r}}$$

1) Energy conservation in absence of dissipative forces

$$\frac{d}{dt}E = 0 \quad , \quad E = m\frac{\dot{\vec{r}}^2}{2} + V(\vec{r}) = \text{const.}$$

2) $\vec{F}_{diss} \uparrow\downarrow \dot{\vec{r}} \quad \Rightarrow \quad \frac{d}{dt}E < 0$

Properties of conservative forces : 1) $\vec{F} = \vec{F}(\vec{r})$

2) $\vec{F}$ conservative $F = -\vec{\nabla}V \Leftrightarrow \vec{\nabla}\times\vec{F} = 0$

$$\vec{F} = -\vec{\nabla}V \quad , \text{ thus } \quad \vec{\nabla}\times\vec{F} = -\vec{\nabla}\times(\vec{\nabla}V)$$

$$\partial_i\partial_j V = \partial_j\partial_i V \qquad = -\sum_{ijk}\epsilon_{ijk}\frac{\partial_i\partial_j V}{\partial x_i\partial x_j}\vec{e}_k$$

$$\epsilon_{ijk} = -\epsilon_{jik} \qquad \overset{!}{=} 0$$

(symmetric and antisymmetric product)

other direction by use of Helmholz decomposition

$$\vec{F}(\vec{r}) = \vec{\nabla}\varphi + \vec{\nabla}\times\vec{f}$$

each field can be decomposed into gradient and curl field

assume $\vec{\nabla}\times\vec{F} = 0$ then:

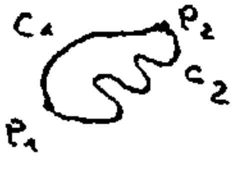
$$\vec{\nabla}\times(\vec{\nabla}\varphi + \vec{\nabla}\times\vec{f}) = \vec{\nabla}\times(\vec{\nabla}\times\vec{f}) = \vec{\nabla}(\vec{\nabla}\cdot\vec{f}) - \Delta\vec{f} = 0$$

only solution $\vec{f} = \text{const}$, but $\vec{\nabla}\times\vec{f} = 0$, thus $\vec{F}(\vec{r})$ only has gradient contribution

3) Work independent of path

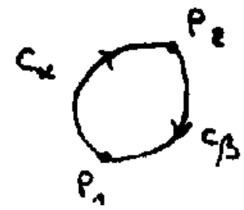
zero for closed loop

$$\vec{F} \text{ conservative} \Leftrightarrow W_{21} = \int_{C_1} d\vec{r}\cdot\vec{F} = \int_{C_2} d\vec{r}\cdot\vec{F} \Leftrightarrow \oint d\vec{r}\cdot\vec{F} = 0$$



total differential $dV = \vec{\nabla} V \cdot d\vec{r} = -\vec{F} \cdot d\vec{r}$

$$\oint d\vec{r} \cdot \vec{F} = \int_{C_\alpha} d\vec{r} \cdot \vec{F} + \int_{C_\beta} d\vec{r} \cdot \vec{F}$$



$$= \int_{P_1}^{P_2} dV + \int_{P_2}^{P_1} dV = V(\vec{r}_2) - V(r_1) + V(r_1) - V(r_2) = 0$$

reversing path : $d\vec{r} = \frac{d\vec{r}}{dt} dt$ reversed

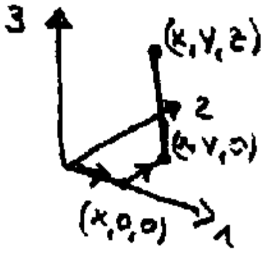
$$\Rightarrow \int_{C_\alpha} d\vec{r} \cdot \vec{F} = \int_{-C_\beta} d\vec{r} \cdot \vec{F}$$

Example : 08.12.2010

isotropic harmonic oscillator

$\vec{F} = -k\,\vec{r}$ conservative? $\vec{\nabla} \times \vec{F} = -k\,\vec{\nabla} \times \vec{r} = 0$

choose suitable path to calculate $V(\vec{r})$:



$$V(\vec{r}) = -\int_0^{\vec{r}} d\vec{r} \cdot \vec{F}(\vec{r}) = k \int_0^{r} d\vec{r} \cdot \vec{r}$$

$$= k \left[\int_0^1 \frac{d\vec{r}_1}{dt} \cdot \vec{r}_1 \, dt + \int_0^1 \frac{d\vec{r}_2}{dt} \cdot \vec{r}_2 \, dt + \int_0^1 \frac{d\vec{r}_3}{dt} \cdot \vec{r}_2 \, dt \right]$$

$$\vec{r}_1 = (xt, 0, 0) \qquad \frac{d\vec{r}_1}{dt} = (x, 0, 0)$$

$$\vec{r}_2 = (0, yt, x) \qquad \frac{d\vec{r}_2}{dt} = (0, y, 0)$$

$$\vec{r}_3 = (y, x, zt) \qquad \frac{d\vec{r}_3}{dt} = (0, 0, z)$$

$$V(\vec{r}) = k \left[\int_0^1 x^2 t\,dt + \int_0^1 y^2 t\,dt + \int_0^1 z^2 t\,dt \right]$$

$$= k (x^2 + y^2 + z^2) \int_0^1 t\,dt = k \frac{x^2 + y^2 + z^2}{2} = \frac{k\,\vec{r}^{\,2}}{2}$$

Angular Momentum, Torque (Moment)

Newton's equation $m\ddot{\vec{r}} = \vec{F}$ multiply with $\vec{r}$

$$m\,\vec{r}\times\ddot{\vec{r}} = \vec{r}\times\vec{F}$$

define: $\vec{M} = \vec{r}\times\vec{F}$ (Torque)

$\vec{L} = m(\vec{r}\times\dot{\vec{r}})$ angular momentum

$\dot{\vec{L}} = m(\underbrace{\dot{\vec{r}}\times\dot{\vec{r}}}_{0} + \vec{r}\times\ddot{\vec{r}}) = m(\vec{r}\times\ddot{\vec{r}})$

angular momentum Law

$$\frac{d}{dt}\vec{L} = \vec{M} \qquad (\text{same form as } \dot{\vec{p}} = \vec{F})$$

conservation of angular momentum:

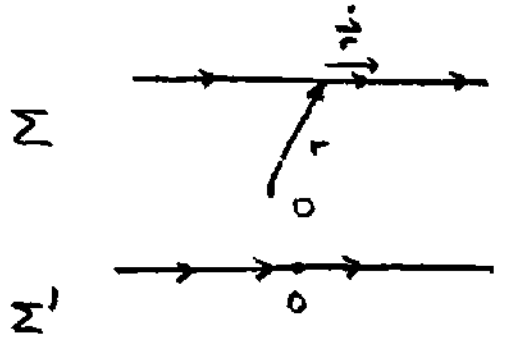
$$\vec{M} = 0\,,\ \frac{d}{dt}\vec{L} = 0 \iff \vec{L} = \text{const}$$

two possibilities a) $\vec{F} = 0$ no force

b) $\vec{F}\parallel\vec{r}$ central force

angular momentum for uniform straight line motion:

$\vec{L} = m\vec{r}\times\dot{\vec{r}}$



$\vec{L}\neq 0$

$\vec{L}' = 0$

$\vec{r}\parallel\dot{\vec{r}}$

in general:

$\vec{r}' = \vec{r} + \vec{a}\,,\ \dot{\vec{r}}' = \dot{\vec{r}}$

$\vec{L} = m\vec{r}\times\dot{\vec{r}}$

$\vec{L}' = m\vec{r}'\times\dot{\vec{r}}' = m(\vec{r}+\vec{a})\times\dot{\vec{r}} = \vec{L} + \vec{a}\times\dot{\vec{r}}$

but:

$$\frac{d\vec{L}'}{dt} = \frac{d\vec{L}}{dt} + \frac{d}{dt}(\vec{a}\times\dot{\vec{r}}) = \frac{d\vec{L}}{dt} + \vec{a}\times\underset{\frac{\vec{F}}{m} = 0 = \dot{\vec{p}}}{\ddot{\vec{r}}} = \frac{d\vec{L}}{dt} = 0$$

conservation of $\vec{L}$ if $\dot{\vec{p}} = 0$

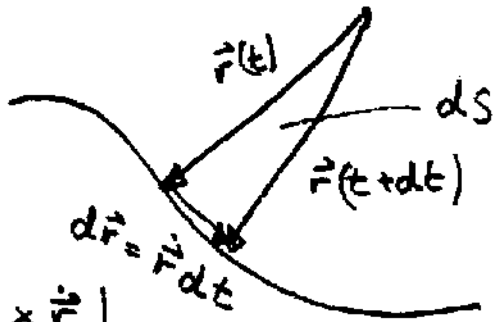
Properties of motion with $\dot{\vec{L}} = 0$, $\vec{L} = \text{const}$

1) movement in plane $\perp \vec{L}$: $\vec{r} \perp \vec{L}$

consider $\vec{r} \cdot \vec{L} = \underbrace{\vec{r} \cdot (\vec{r} \times \dot{\vec{r}})}_{\text{triple dot product}} = 0$

2) area conservation principle : $\vec{L} = \text{const}$

"the position vector of the mass point sweeps equal areas in equal times"



$$dS = \frac{1}{2} \left|\vec{r}(t) \times \vec{r}(t+dt)\right|$$
$$= \frac{1}{2} \left|\vec{r}(t) \times [\vec{r}(t) + \dot{\vec{r}}\,dt]\right|$$
$$= \frac{1}{2} \left|\vec{r}(t) \times \dot{\vec{r}}\,dt\right| = \frac{dt}{2} \left|\vec{r} \times \dot{\vec{r}}\right|$$

$$\frac{dS}{dt} = \frac{1}{2} \left|\vec{r} \times \dot{\vec{r}}\right| = \frac{1}{2m} \left|m\,\vec{r} \times \dot{\vec{r}}\right| = \frac{1}{2m} |\vec{L}| = \text{const}$$

holds for a) $\vec{F} = 0$
b) $\vec{F} \parallel \vec{r}$

Central forces

any central force: $\vec{F} = f(\vec{r}, \dot{\vec{r}}, t)\,\vec{e}_r \Rightarrow \frac{d\vec{L}}{dt} = 0$

$\vec{M} = \vec{r} \times \vec{F} = 0$

necessary condition for conservative force $f(\vec{r}, \cancel{\dot{\vec{r}}}, \cancel{t}) = f(\vec{r})$

Theorem : $\vec{F} = f(r)\,\vec{e}_r \Leftrightarrow \vec{F}$ conservative

$f(r, \cancel{\vartheta}, \cancel{\varphi})$

Proof : a) calculate curl for arbitrary function $f(\vec{r})$

$$0 = \vec{\nabla} \times \vec{F} = \vec{\nabla} \times \left(\frac{f(\vec{r})}{r}\,\vec{r}\right) = \left[\vec{\nabla}\left(\frac{f(\vec{r})}{r}\right)\right] \times \vec{r} + \frac{f(\vec{r})}{r}\,\underbrace{\vec{\nabla} \times \vec{r}}_{0}$$

We obtain $\quad 0 = \left[\vec{\nabla} \frac{f(\vec{r})}{r}\right] \times \vec{r}$, thus $\vec{r} \parallel \vec{\nabla} \frac{f(\vec{r})}{r}$

Recall : gradient field $\perp$ to surface where $\frac{f(\vec{r})}{r}$ constant

→ $\frac{f(\vec{r})}{r} = \text{const}$ has to be a sphere ($\vec{r} \perp$ to this surface everywhere) : $f(\vec{r})$ independent of direction of $\vec{r}$, thus $f(\vec{r}) = f(r)$

b) assume $\vec{F}(\vec{r}) = f(r)\vec{e}_r \qquad \vec{\nabla} \times \vec{F} = 0$ (shown before)

Theorem : a conservative force is central fore if and only if $V(\vec{r}) = V(r)$

a) consider $\vec{F} = -\vec{\nabla} V$ with $V = V(r)$ chain rule

calculate gradient : $\vec{F} = -\vec{\nabla} V = -\frac{dV}{dr} \underbrace{\vec{\nabla} r}_{\frac{\vec{r}}{r} \text{ (shown before)}}$

$$= -\frac{dV}{dr} \vec{e}_r$$

thus $\vec{F} \parallel \vec{r}$ (central force)

b) we show $V(\vec{r}) = V(r)$ assuming : $\vec{F} = f(r)\vec{e}_r$, $\vec{F} = -\vec{\nabla} V$

read equation for components : $\vec{e}_r = \frac{\vec{r}}{r}$

$$\frac{\partial V}{\partial x_i} = -f(r)\frac{x_i}{r} = -f(r)\frac{dr}{dx_i}$$

introduce function $\tilde{f}(r)$ such that $f(r) = \frac{d\tilde{f}}{dr}$

$$\frac{dV}{dx_i} = -\frac{d\tilde{f}}{dr}\frac{dr}{dx_i} = -\frac{d\tilde{f}(r)}{dx_i} \qquad \text{holds for } i = 1,2,3$$

→ $V(\vec{r})$ only depends on r as $\tilde{f}(r)$.

Movement in conservative central forcefields

conservative force $\quad E = T + V = \text{const.}$

central force $\quad \vec{L} = m\,\vec{r} \times \dot{\vec{r}} = \overrightarrow{\text{const}}$

$\Big\}$ 1st integrals of movement (integrating Newton's law)

choose $\vec{L} = L\,\vec{e}_z$ (movement in x-y plane)

describe movement in polar coordinates:

$$\vec{r} = r\,\vec{e}_r$$
$$\dot{\vec{r}} = \dot{r}\,\vec{e}_r + r\dot{\varphi}\,\vec{e}_\varphi$$

calculate angular momentum

$$\vec{L} = m\,\vec{r} \times \dot{\vec{r}} = m r\,\vec{e}_r \times (\dot{r}\,\vec{e}_r + r\dot{\varphi}\,\vec{e}_\varphi)$$
$$= m r^2 \dot{\varphi}\,(\vec{e}_r \times \vec{e}_\varphi) = m r^2 \dot{\varphi}\,\vec{e}_z$$

read off $\quad L = m r^2 \dot{\varphi} = \text{const}$

calculate energy

$$E = \frac{1}{2} m \dot{\vec{r}}^2 + V(r) = \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\varphi}^2\right) + V(r)$$
$$\xrightarrow{m r^2 \dot{\varphi}^2 = \frac{L^2}{2mr^2}} = \frac{1}{2} m \dot{r}^2 + \frac{L^2}{2mr^2} + V(r)$$

define effective potential $U(r) = \frac{L^2}{2mr^2} + V(r)$ for movement in radial direction.

Solution of the equations of motion

$$\vec{r}(t) = r(t)\,(\cos\varphi(t), \sin\varphi(t), 0)$$

need to obtain $r(t)$ and $\varphi(t)$

similar as for 1D motion:

$$\frac{m}{2}\dot{r}^2 = E - U(r)$$

$$\dot{r}^2 = \frac{2}{m}\left[E - U(r)\right]$$

$$\frac{dr}{dt} = \dot{r} = \sqrt{\frac{2}{m}\left[E - U(r)\right]} \qquad dt = \frac{dr}{\sqrt{\frac{2}{m}\left[E - U(r)\right]}}$$

$$\int_{t_0}^{t} dt' = t - t_0 = \int_{r_0}^{r} dr' \frac{1}{\sqrt{\frac{2}{m}\left[E - U(r')\right]}}$$

from $t(r)$ obtain $r(t)$

also $\varphi(r)$ with one integration

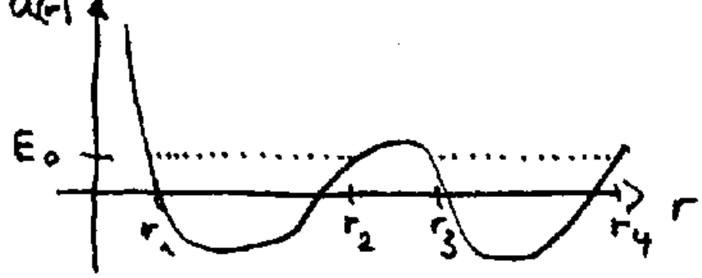
$$\dot{\varphi} = \frac{d\varphi}{dt} = \frac{L}{mr^2} \qquad d\varphi = \frac{L}{mr^2}\underbrace{dt}_{\frac{dr}{\sqrt{\frac{2}{m}\left[E - U(r)\right]}}} = \frac{L}{r^2}\frac{dr}{\sqrt{2m(E-U(r))}}$$

$$\int_{\varphi_0}^{\varphi} d\varphi' = \varphi - \varphi_0 = L\int_{r_0}^{r} \frac{dr'}{r'^2\sqrt{2m\left[E - U(r)\right]}}$$

yields $\varphi(r) = \varphi(r(t)) =: \varphi(t)$ with solution $r(t)$

Remarks:

1) same equations can be obtained from Newton's law
2) movement in effective potential



for real $r(t)$ we need $E \geq U(r)$

1) allowed regions $U(r) \leq E$
2) classically forbidden $U(r) > E$
3) points of return $U(r_i) = E$

Example : motion in attractive potential $V(r) = -\frac{\alpha}{r}$

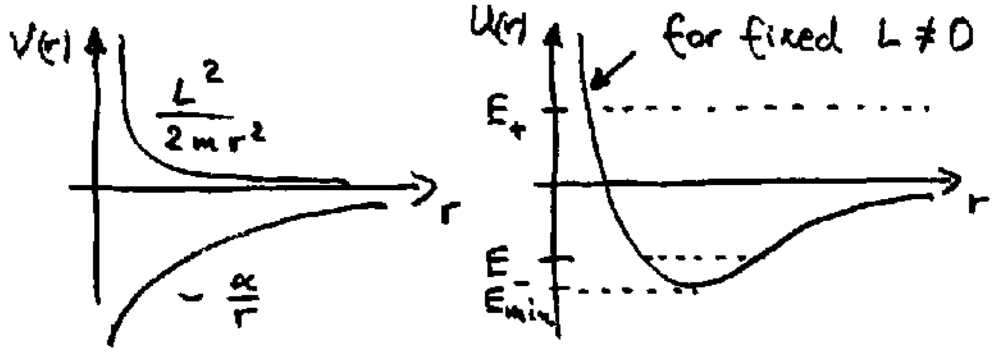
a) Coulomb potential $\alpha = \frac{q_1 q_2}{4\pi\epsilon_0}$ q_i : charges

b) gravitational potential $\alpha = \gamma m M$ γ: gravitational constant

$\gamma \approx 6.676 \cdot 10^{-11} N\left(\frac{m}{kg}\right)^2$

m : mass of planet

M : solar mass



E_+ : scattering states (no return)

E_- : bounded oscillatory motion

E_{min} : motion with fixed r_0 (circle)

$E = 0$ special scattering

motion described by solution of integral

$$\varphi - \varphi_0 = L \int_{r_0}^{r} \frac{dr'}{r'^2} \frac{1}{\sqrt{2m\left(E - \frac{L^2}{2mr'^2} + \gamma \frac{mM}{r'}\right)}}$$

hint : substitute $r = \frac{1}{s}$ to solve

here : derive differential equation for $s(\varphi) = \frac{1}{r(\varphi)}$

$$\frac{ds}{d\varphi} = \frac{d}{d\varphi}\frac{1}{r(\varphi)} = \frac{dt}{d\varphi}\frac{d}{dt}\left(\frac{1}{r}\right) = \frac{1}{\dot\varphi}\frac{dr}{dt}\left(-\frac{1}{r^2}\right) = \frac{-\dot r}{\dot\varphi r^2}$$

$$L = m r^2 \dot\varphi \quad \Leftrightarrow \quad r^2 \dot\varphi = \frac{L}{m} \qquad \frac{ds}{d\varphi} = -\frac{\dot r m}{L}$$

rewrite $\dot r = -\frac{L}{m}\frac{ds}{d\varphi}$

energy conservation:

$$E = \frac{m}{2}\dot{r}^2 + \frac{L^2}{2mr^2} - \gamma\frac{mM}{r} \qquad r = \frac{1}{s}$$

$$\dot{r} = -\frac{L}{m}\frac{ds}{d\varphi}$$

$\frac{d}{dt}E = 0$ but also $\frac{d}{d\varphi}E = 0$

$$0 = \frac{d}{d\varphi}\left[\frac{L^2}{2m}\left[\left(\frac{ds}{d\varphi}\right)^2 + s^2\right] - \gamma m M s\right]$$

$$= \frac{ds}{d\varphi}\left[\frac{L^2}{m}\left(\frac{d^2s}{d\varphi^2} + s\right) - \gamma m M\right]$$

two solutions

1) $\frac{ds}{d\varphi} = 0 \qquad s(\varphi) = \text{const} = \frac{1}{r_0}$ circular motion

2) $0 = \frac{L^2}{m}\left(\frac{d^2s}{d\varphi^2} + s\right) - \gamma m M$

$$\frac{d^2s}{d\varphi^2} + s = \frac{\gamma m^2 M}{L^2} \qquad \text{inhomogeneous D.E.}$$

$$s(\varphi) = s_{hom}(\varphi) + s_0(\varphi) \qquad \text{general solution}$$

$$s_{hom}(\varphi) = \alpha \sin\varphi + \beta\cos\varphi$$

$$s_0(\varphi) = \frac{\gamma m^2 M}{L^2} \qquad \text{(guessed special solution)}$$

$$s(\varphi) = \alpha\sin\varphi + \beta\cos\varphi + \frac{\gamma m^2 M}{L^2}$$

α, β fixed by initial conditions: $\left.\frac{ds}{d\varphi}\right|_{\varphi=0} = 0$

$$\left.\frac{ds}{d\varphi}\right|_{\varphi=0} = \alpha\cos\varphi - \beta\sin\varphi\Big|_{\varphi=0} = \alpha = 0$$

fixes direction of coordinate system

$$\left.\frac{d^2s}{d\varphi^2}\right|_{\varphi=0} = -\beta\cos\varphi\Big|_{\varphi=0} < 0$$

want to have s maximal ($\rightarrow r = \frac{1}{s}$ minimal) at $\varphi = 0$

in summary $\quad S(\varphi) = \beta \cos\varphi + \frac{\gamma m^2 M}{L^2}$

introduce $\quad K = \frac{L^2}{\gamma M m^2} \quad , \quad \beta = \frac{\epsilon}{K} \geq 0$

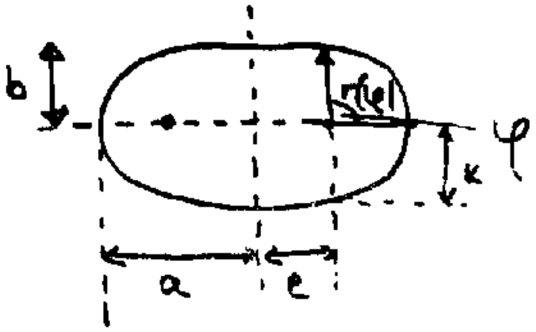
$$\frac{1}{r} = \frac{1}{K}\left(1 + \epsilon \cos\varphi\right) \quad , \quad r(\varphi) = \frac{K}{1+\epsilon\cos\varphi}$$

equation of conic section in polar coordinates

Discussion (4 cases)

1) $\epsilon = 0 \qquad r(\varphi) = K = r_0 \quad$ circle with radius r_0

2) $0 < \epsilon < 1 \quad$ ellipse



a : semi-major axis
b : semi-minor axis

i) closest point

$$r(0) = \frac{K}{1+\epsilon\cos 0} = \frac{K}{1+\epsilon} = a - e$$

ii) farthest point

$$r(\pi) = \frac{K}{1+\epsilon\cos\pi} = \frac{K}{1-\epsilon} = a + e$$

combine equations ⊕/⊖

$$1-\epsilon^2 = \frac{K}{a}$$

$$1-\epsilon^2 = \frac{K\epsilon}{e}$$

numerical eccentricity $\quad \epsilon = \frac{e}{a}$

$$K = \frac{a^2-e^2}{a}$$

iii) $\quad r\left(\frac{\pi}{2}\right) = \frac{K}{1+\epsilon\cos\left(\frac{\pi}{2}\right)} = K$

use normal form of ellipse $\quad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

for $x = e$, $y = K \qquad \frac{e^2}{a^2} + \frac{K^2}{b^2} = 1$

$\qquad \downarrow \epsilon = \frac{e}{a}$

$$\frac{K^2}{b^2} = 1 - \epsilon^2 \quad \downarrow \; 1-\epsilon^2 = \frac{K}{a}$$

$$b^2 = a \cdot k = a^2 - e^2$$

Calculate semi-axis for fixed E, L:

closest point: $\dot r = 0$, $T = 0$

$$E = U(r_0) = \frac{L^2}{2mr_0^2} - \frac{\gamma m M}{r_0} \overset{k=\frac{L^2}{\gamma M m^2}}{=} \gamma m M \left(\frac{k}{2r_0^2} - \frac{1}{r_0} \right)$$

$$= \gamma m M \left(\frac{\frac{a^2-e^2}{a}}{2(a-e)^2} - \frac{1}{a-e} \right) = -\frac{\gamma m M}{2a}$$

$$a = -\frac{\gamma m M}{2E}$$ (enegy determines semi-major axis)

$$b^2 = a \cdot k = -\frac{\gamma m M}{2E} \frac{L^2}{\gamma M m^2} = -\frac{L^2}{2mE}$$

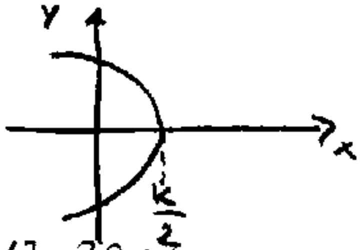
$$b = \frac{L}{\sqrt{-2mE}}$$ (semi-minor axis) $E < 0$ (bounded motion)

3) $\epsilon = 1$ parabola ; transform back to x-y:

$$x = r\cos\varphi = \frac{k\cos\varphi}{1+\cos\varphi} = \frac{k}{2}\left(\frac{1+\cos\varphi}{1+\cos\varphi} - \frac{1-\cos\varphi}{1+\cos\varphi} \right)$$

$$y = r\sin\varphi = \frac{k\sqrt{1-\cos^2\varphi}}{1+\cos\varphi} = k\sqrt{\frac{1-\cos\varphi}{1+\cos\varphi}}$$

$$x = \frac{k}{2} - \frac{k}{2}\underbrace{\frac{1-\cos\varphi}{1+\cos\varphi}}_{\frac{y^2}{k^2}} = \frac{k}{2} - \frac{y^2}{2k}$$



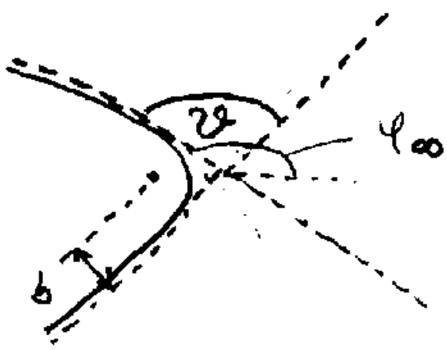
14.12.2016

4) $\epsilon > 1$ Hyperbola

$$r(\varphi) = \frac{k}{1+\epsilon\cos\varphi}$$

b: impact parameter

ϑ: deflection angle



take $r \to \infty$ $\qquad \cos \varphi_\infty = -\frac{1}{\epsilon}$

geometry : $\pi - \vartheta = 2(\pi - \varphi_\infty) \Leftrightarrow \frac{\vartheta}{2} = \varphi_\infty - \frac{\pi}{2}$

$\sin \frac{\vartheta}{2} = \sin\left(\varphi_\infty - \frac{\pi}{2}\right) = -\cos(\varphi_\infty) = \frac{1}{\epsilon}$

calculate impact parameter (energy conservation)

$$E = \frac{1}{2} m \dot{\vec{r}}_\infty^{\,2} = \frac{1}{2} m \dot{r}_\infty^2 \qquad \dot{r}_\infty = \sqrt{\frac{2E}{m}}$$

$r \to \infty,\ U \to 0$ $\qquad$ and use angular momentum:

$$L = m\,|\vec{r} \times \dot{\vec{r}}| = m\,|\vec{r}_\infty \times \dot{\vec{r}}_\infty| = m\, b\, \dot{r}_\infty = b\sqrt{2Em}$$

only contribution of $\vec{r}_\infty \perp$ to $\dot{\vec{r}}_\infty$

$$b = \frac{L}{\sqrt{2Em}}$$

for the deflection angle we can calculate

$$r\left(\frac{\pi}{2}\right) = \frac{K}{1+\epsilon} = r_0 \quad \text{and} \quad \text{we have } \dot{r} = 0$$

$$E = U(r_0) = \gamma M m \left(\frac{K}{2r_0^2} - \frac{1}{r_0}\right) = \gamma M m \left[\frac{(1+\epsilon)^2}{2K} - \frac{1+\epsilon}{K}\right]$$

$$= \gamma M m \frac{\epsilon^2 - 1}{2K} \qquad K = \frac{L^2}{\gamma M m^2}$$

$$\Rightarrow \epsilon^2 - 1 = \frac{2KE}{\gamma M m} = \frac{2L^2 E}{\gamma^2 M^2 m^3} = \frac{4E^2 b^2}{\gamma^2 M^2 m^2} = \frac{1}{\sin^2 \frac{\vartheta}{2}} - 1$$

$b = \frac{L}{\sqrt{2Em}}$ $\qquad \sin\frac{\vartheta}{2} = \frac{1}{\epsilon}$

$$= \frac{1 - \sin^2 \frac{\vartheta}{2}}{\sin^2 \frac{\vartheta}{2}} = \frac{\cos^2 \frac{\vartheta}{2}}{\sin^2 \frac{\vartheta}{2}} = \cot^2 \frac{\vartheta}{2}$$

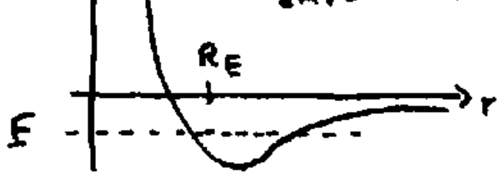
$$\Rightarrow \tan \frac{\vartheta}{2} = \frac{\gamma M m}{2bE}$$

(deflection angle fixed by b, E)

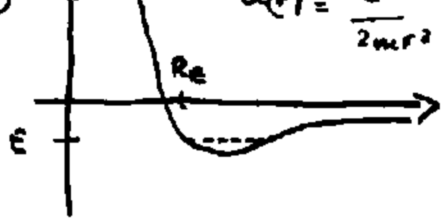
The cosmic velocities

movement in effective potential $U(r) = \frac{L^2}{2mr^2} - \gamma \frac{mM}{r}$

① $U(r) = \frac{L_1^2}{2mr^2} - \gamma \frac{mM}{r}$



② $U(r) = \frac{L'^2}{2mr^2} - \gamma \frac{mM}{r}$



for fixed energy E: satellite falls on earth

$L_2 > L_1$: satellite does not hit surface

$\varphi = 0 : \vec{r} \perp \dot{\vec{r}}$

$$L_1 = |\vec{L}| = |m\, \vec{r} \times \dot{\vec{r}}| = m\, R_E\, V_1$$

$$r(\varphi = 0) = R_E = \frac{K}{1+\epsilon} = \frac{L_1^2}{\gamma m^2 M} \frac{1}{1+\epsilon}$$

$$= \frac{m^2 R_E^2 V_1^2}{\gamma m^2 M} \frac{1}{1+\epsilon}$$

$$V_1^2 = \frac{\gamma M}{R_E}(1+\epsilon) \geq \frac{\gamma M}{R_E}$$

smallest v_1 : $\epsilon = 0$ (circular motion)

Force : $F_G(R_E) = \frac{\gamma m M}{R_E^2} = m g \Rightarrow g = \frac{\gamma M}{R_E^2}$

$$V_1 \geq \sqrt{g R_E} \approx 7.91 \ \frac{km}{s}$$ (first cosmic velocity)

velocity to leave gravitational field
(smallest possible energy : $E = 0$, parabola)

$$L_2 = m R_E V_2$$

$$0 = E = U(R_E) = \frac{L_2^2}{2mR_E^2} - \gamma \frac{Mm}{R_E} \Rightarrow R_E = \frac{L_2^2}{2\gamma m^2 M} \overset{g = \frac{\gamma M}{R_E^2}}{=} \frac{V_2^2}{2g}$$

$L_2 = m R_E V_2$

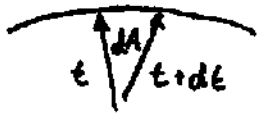
$$V_2 = \sqrt{2 g R_E} = \sqrt{2}\, V_1 \approx 11.2 \ \frac{km}{s}$$

(second cosmic velocity)

Kepler's Laws

1) planets move along ellipses with sun at focal point (conservation of energy and angular momentum)
2) areal velocity is constant : result of $\dot{\vec{L}} = 0$

$$\frac{dA}{dt} = \frac{L}{2m}$$



3) for all planets : $\frac{T^2}{a^3} = \text{const}$ T : orbital period
a : semi-major axis

consider area of ellipse $A = \pi a b$

use 2) : $\frac{dA}{dt} = \frac{L}{2m}$: $A = \frac{dA}{dt} \cdot T = \frac{L}{2m} T$

$$T = \frac{2\pi a b m}{L}$$

$$\frac{T^2}{a^3} = \frac{(2\pi a b m)^2}{L^2 a^3} = \frac{4\pi^2 m^2}{L^2} \frac{b^2}{a} = \frac{4\pi^2 m^2}{L^2} \frac{L^2}{\gamma M m^2} = \frac{4\pi^2}{\gamma M} = \text{const}$$

$$\frac{b^2}{a} = k = \frac{L^2}{\gamma M m^2}$$

Mechanics of many-particle systems

Definitions : N- particle system $\quad i = 1 \dots N$

m_i mass of ith particle

$\vec{r}_i$ position of ith particle

$\vec{F}_i$ total force on particle i

$\vec{F}_i^{ex}$ external force on particle i

$\vec{F}_{ij}$ force executed from particle j on particle i (internal force)

Newton's equation

$$m_i \ddot{\vec{r}}_i = \vec{F}_i = \vec{F}_i^{ex} + \sum_j \vec{F}_{ij}$$

$3N$ coupled D.E (no analytic solution)

third axiom

$$\vec{F}_{ij} = -\vec{F}_{ji}$$
$$\vec{F}_{ii} = 0$$

Next steps: a) derive conservation laws

b) consider $N=2$: analytical solution

Momentum conservation law

add up all equ :
$$\sum_{i=1}^{N} m \ddot{\vec{r}}_i = \sum_{i=1}^{N} \vec{F}_i = \sum_{i=1}^{N} \vec{F}_i^{ex} + \underbrace{\sum_{i,j=1}^{N} \vec{F}_{ij}}_{\frac{1}{2}\sum_{ij}\underbrace{(\vec{F}_{ij}+\vec{F}_{ji})}_{0}}$$

define
$$M = \sum_i m_i \quad \text{(total mass)}$$
$$\vec{R} = \frac{1}{M}\sum_{i=1}^{N} m_i \vec{r}_i \quad \text{(center of mass)}$$
$$\vec{P} = \sum_i m_i \dot{\vec{r}}_i = M \dot{\vec{R}} \quad \text{(total momentum)}$$
$$\vec{F}^{ex} = \sum_{i=1}^{N} \vec{F}_i^{ex} \quad \text{(total external force)}$$
$$\left(\ddot{\vec{R}} = \frac{1}{M}\sum_{i=1}^{N} m_i \ddot{\vec{r}}_i \right)$$

$$\underbrace{\sum_i m_i \ddot{\vec{r}}_i}_{} = \sum_i \vec{F}_i^{ex}$$

$$\dot{\vec{P}} = M \ddot{\vec{R}} \qquad = \vec{F}^{ex} \qquad \text{(center of mass theorem)}$$

momentum conservation law : $\vec{F}^{ex} = 0 \iff \vec{P} = \text{const}$

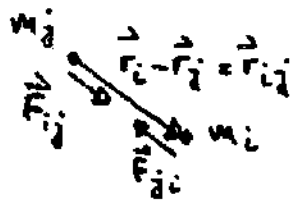
Conservation of angular momentum

define : $\vec{L} = \sum_{i=1}^{N} \vec{L}_i = \sum_{i=1}^{N} m_i \, \vec{r}_i \times \dot{\vec{r}}_i$ (total angular momentum)

$$\dot{\vec{L}} = \sum_{i=1}^{N} m_i \Big[\underbrace{\dot{\vec{r}}_i \times \dot{\vec{r}}_i}_{0} + \vec{r}_i \times \ddot{\vec{r}}_i \Big] \underset{\text{Newton's law}}{=} \sum_{i=1}^{N} \vec{r}_i \times \vec{F}_i$$

$$= \sum_{i=1}^{N} \vec{r}_i \times \vec{F}_i^{ext} + \underbrace{\sum_{i,j=1}^{N} \vec{r}_i \times \vec{F}_{ij}}$$

$$\frac{1}{2} \sum_{i,j=1}^{N} \Big(\vec{r}_i \times \vec{F}_{ij} + \vec{r}_j \times \underbrace{\vec{F}_{ji}}_{-\vec{F}_{ij}} \Big) = \frac{1}{2} \sum_{i,j=1} \underbrace{(\vec{r}_i - \vec{r}_j) \times \vec{F}_{ij}}_{0}$$



Note : $\vec{F}_{ij} = \alpha(\vec{r}_i - \vec{r}_j)$, $\vec{r}_{ij} = \vec{r}_i - \vec{r}_j \parallel \vec{F}_{ij}$
(direction!)

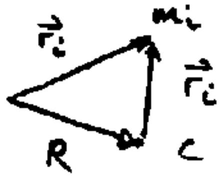
define $\vec{M}_i^{ex} = \vec{r}_i \times \vec{F}_i^{ex}$ (external torque)

$\qquad \vec{M}^{ex} = \sum_{i=1}^{N} \vec{M}_i^{ex}$ (total torque)

$$\frac{d}{dt} \vec{L} = \sum_{i=1}^{N} \vec{r}_i \times \vec{F}_i^{ex} = \vec{M}^{ex} \qquad \text{(angular momentum law)}$$

$\vec{M}^{ex} = 0 \iff \vec{L} = \text{const}$ (angular momentum conservation for closed system $\vec{F}_i^{ex} = 0$)

Decomposition of angular momentum : relative, center of mass



$\vec r_i - \vec R = \Delta \vec r_i$

$$
\begin{aligned}
\vec L &= \sum_{i=1}^{N} m_i \left(\vec r_i \times \dot{\vec r}_i\right) \\
&= \sum_{i=1}^{N} m_i \left(\vec R + \Delta \vec r_i\right) \times \left(\dot{\vec R} + \Delta \dot{\vec r}_i\right) \\
&= \sum_{i=1}^{N} m_i \left(\vec R \times \dot{\vec R} + \vec R \times \Delta \dot{\vec r}_i + \Delta r_i \times \dot{\vec R} + \Delta \vec r_i \times \Delta \dot{\vec r}_i\right) \\
&= M\, \vec R \times \dot{\vec R} + \vec R \times \underbrace{\left(\sum_i m_i \Delta \dot{\vec r}_i\right)}_{\substack{\text{total momentum: } \sum_i m_i (\dot{\vec r}_i - \dot{\vec R}) \\ = \vec P - M\dot{\vec R} = 0}} + \underbrace{\left(\sum_i m_i \Delta \vec r_i\right)}_{\substack{\sum_i m_i (\vec r_i - \vec R) \\ = M\vec R - M\vec R = 0 \\ \text{center of mass}}} \times \dot{\vec R} + \sum_i m_i \Delta \vec r_i \times \Delta \dot{\vec r}_i \\
&= M\, R \times \dot{\vec R} + \sum_i m_i \Delta \vec r_i \times \Delta \dot{\vec r}_i \\
&= \vec L_s + \vec L_r
\end{aligned}
$$

$\vec L_s$: angular momentum of mass centered at $\vec R$

$\vec L_r$: angular momentum of N particles with reference to the center of mass $\vec R$

Conservation of Energy (similar as masspoint)

Newton's law : multiply with $\dot{\vec r}_i$ and sum :

$$
\sum_{i=1}^{N} m_i \ddot{\vec r}_i \cdot \dot{\vec r}_i = \sum_{i=1}^{N} \underbrace{\frac{1}{2} \frac{d}{dt} \left(m_i \dot{\vec r}_i^{\,2}\right)}_{\frac{d}{dt} T_i \text{ (kinetic energy of one particle)}} = \sum_i \dot{\vec r}_i \cdot \vec F_i
$$

define $T = \sum_{i=1}^{N} T_i$

$$
\frac{d}{dt} T = \sum_{i=1}^{N} \dot{\vec r}_i \cdot \vec F_i
$$

Two cases
a) all forces conservative $\vec\nabla_i \times \vec F_i = 0$
b) split up forces $\vec F_i = \vec F_i^{cons} + \vec F_i^{diss}$

a) conservative forces : $\vec{F}_i = -\vec{\nabla}_i V$

$V = V(\vec{r}_1, \dots, \vec{r}_N)$ with $dV = \sum_{i=1}^{N} \vec{\nabla}_i V \cdot d\vec{r}_i$

rewrite : $\sum_{i=1}^{N} \dot{\vec{r}}_i \cdot \vec{F}_i = -\sum_{i=1}^{N} \dot{\vec{r}}_i \cdot \vec{\nabla}_i V = -\frac{dV}{dt}$ (total differential)

in total :

$$\frac{d}{dt}(T+V) = 0 \qquad T+V = E = \text{const}$$

(conservation of energy)

b) dissipative forces

$$\frac{d}{dt}(T+V) = \sum_{i=1}^{N} \vec{F}_i^{\,diss} \cdot \dot{\vec{r}}_i$$

decomposition of potential : internal /external contributions

$$V(\vec{r}_1, \dots, \vec{r}_N) = \sum_{i=1}^{N} V_i^{ex}(\vec{r}_i) + \frac{1}{2} \sum_{i,j} V_{ij}(|\vec{r}_i - \vec{r}_j|)$$

calculate force on particle k :

$$\vec{F}_k = -\vec{\nabla}_k V = -\vec{\nabla}_k V_k^{ex}(\vec{r}_k) - \frac{1}{2} \sum_{i,j} \vec{\nabla}_k V_{ij}(|\vec{r}_i - \vec{r}_j|)$$

we use $\vec{\nabla}_k V_{ij}(|\vec{r}_i - \vec{r}_j|) = \vec{\nabla}_k V_{kj}(|\vec{r}_k - \vec{r}_j|)\,\delta_{ki} + \vec{\nabla}_k V_{ik}(|\vec{r}_i - \vec{r}_k|)\,\delta_{jk}$

symmetry $V_{ij}(|\vec{r}_i - \vec{r}_j|) = V_{ij}(|\vec{r}_j - \vec{r}_i|) = V_{ji}(|\vec{r}_j - \vec{r}_i|)$

it follows

$$\vec{F}_k = \vec{F}_k^{ex} - \sum_{\substack{j=1 \\ j \neq k}}^{N} \underbrace{\vec{\nabla}_k V_{kj}(|\vec{r}_k - \vec{r}_j|)}_{-\vec{F}_{kj}} = \vec{F}_k^{ex} + \sum_{\substack{j=1 \\ j \neq k}}^{N} \vec{F}_{kj}$$

(form of force as initially assumed)

Note : $\vec{F}_{kj}$ is conservative central force , V_{kj} only depends on $|\vec{r}_k - \vec{r}_j|$

3rd axiom:

$$\vec{F}_{kj} = -\vec{\nabla}_k V_{kj}(|\vec{r}_k-\vec{r}_j|) \overset{\text{chain rule}}{=} + \vec{\nabla}_j V_{kj}(|\vec{r}_k-\vec{r}_j|)$$
$$= \vec{\nabla}_j V_{jk}(|\vec{r}_k-\vec{r}_j|) = -\vec{F}_{jk}$$

note: choose constant in potentials V_{kj} such that $V_{ii} = 0$, then $V_{kj}(|\vec{r}_k-\vec{r}_j|) = V_{jk}(|\vec{r}_k-\vec{r}_j|)$

Virial Theorem

(properties of time averaged kinetic and potential energy)

a) conservative forces

b) finite velocity, position

c) closed system

a) multiply Newton's laws with $\vec{r}_i$:

$$\sum_{i=1}^{N} m_i \ddot{\vec{r}}_i \cdot \vec{r}_i = \sum_i \vec{F}_i \cdot \vec{r}_i \qquad \vec{F}_i = -\vec{\nabla}_i V$$

$$\frac{d}{dt}\left(\sum_{i=1}^{N} m_i \dot{\vec{r}}_i \cdot \vec{r}_i\right) - \sum_{i=1}^{N} m_i \dot{\vec{r}}_i^{\,2} = -\sum_i \vec{\nabla}_i V \cdot \vec{r}_i$$

time average of function $f(t)$

$$\langle f \rangle = \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^{\tau} dt\, f(t)$$

first term:

$$\left\langle \frac{d}{dt}\left(\sum_{i=1}^{N} m_i \dot{\vec{r}}_i \cdot \vec{r}_i\right)\right\rangle = \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^{\tau} dt\, \frac{d}{dt}\left(\sum_{i=1}^{N} m_i \dot{\vec{r}}_i \cdot \vec{r}_i\right)$$

$$= \lim_{\tau\to\infty} \frac{1}{\tau} \sum_{i=1}^{N} m_i \dot{\vec{r}}_i \cdot \vec{r}_i \Big|_0^{\tau}$$

b) $\dot{\vec{r}}_i \cdot \vec{r}_i < \infty$ $\quad = 0$

thus

$$-\left\langle \sum_{i=1}^{N} m_i \dot{\vec{r}}_i^{\,2} \right\rangle = -\left\langle \sum_{i=1}^{N} \nabla_i V \cdot \vec{r}_i \right\rangle$$

$$2\langle T\rangle = \left\langle \sum_{i=1}^{N} \vec{\nabla}_i V \cdot \vec{r}_i \right\rangle \quad \text{(Virial theorem)}$$

virial of the forces

c) closed system $\vec{F}_i^{\,ex} = -\vec{\nabla}_i V_i^{ex} = 0$

special case $V(\vec{r}_1, \dots, \vec{r}_n) = \frac{1}{2} \sum_{i,j=1}^{N} V_{ij}(|\vec{r}_i - \vec{r}_j|)$

with $V_{ij}(|\vec{r}_i - \vec{r}_j|) = \alpha_{ij} |\vec{r}_i - \vec{r}_j|^n$

$$\sum_{i=1}^{N} \vec{\nabla}_i V \cdot \vec{r}_i = \frac{1}{2} \sum_{ijk} \vec{\nabla}_i V_{jk}(|\vec{r}_j - \vec{r}_k|) \cdot \vec{r}_i$$
$$= \frac{1}{2} \sum_{ijk} \frac{dV_{jk}}{d|\vec{r}_j - \vec{r}_k|} \vec{\nabla}_i |\vec{r}_j - \vec{r}_k| \cdot \vec{r}_i$$

$$\frac{dV_{jk}}{d|\vec{r}_j - \vec{r}_k|} = n \frac{V_{jk}}{|\vec{r}_j - \vec{r}_k|}, \quad \vec{\nabla}_i |\vec{r}_j - \vec{r}_k| = (\delta_{ij} - \delta_{ik}) \frac{\vec{r}_j - \vec{r}_k}{|\vec{r}_j - \vec{r}_k|}$$

$$\vec{\nabla}|\vec{r}| = \frac{\vec{r}}{|\vec{r}|}$$

$$\sum_{i=1}^{N} \vec{\nabla}_i V \cdot \vec{r}_i = \frac{n}{2} \sum_{j,k=1}^{N} \frac{V_{jk}}{|\vec{r}_j - \vec{r}_k|^2} (\vec{r}_j - \vec{r}_k) \cdot (\vec{r}_j - \vec{r}_k) = \frac{n}{2} \sum_{j,k=1}^{3} V_{jk} = nV$$

Virial theorem: $2\langle T\rangle = n\langle V\rangle$

Examples:

1) harmonic oscillators $V_{ij} = \frac{1}{2} K_{ij} (\vec{r}_i - \vec{r}_j)^2$, $n = 2$

$$\langle T\rangle = \langle V\rangle, \quad \langle E\rangle = \langle T\rangle + \langle V\rangle = 2\langle T\rangle = 2\langle V\rangle$$

2) Coulomb / gravitational potential

$$V_{ij} = -\frac{\alpha}{|\vec{r}_i - \vec{r}_j|} = -\alpha |\vec{r}_i - \vec{r}_j|^{-1}, \quad n = -1$$

$$2\langle T\rangle = -\langle V\rangle$$

$$E = \langle T\rangle + \langle V\rangle = \frac{1}{2}\langle V\rangle = -\langle T\rangle < 0$$

negative energy (result of bounded motion)

Two particle systems ($N=2$)

idea: decouple center of mass and relative motion

$$\vec{R} = \frac{m_1 \vec{r}_1 + m_2 \vec{r}_2}{m_1 + m_2} = \frac{m_1}{M}\vec{r}_1 + \frac{m_2}{M}\vec{r}_2 \qquad \vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r}$$

$$\vec{r} = \vec{r}_1 - \vec{r}_2 \qquad \vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r}$$

equations of motion:

$$m_1 \ddot{\vec{r}}_1 = \vec{F}_1 = \vec{F}_1^{ex} + \vec{F}_{12}$$
$$m_2 \ddot{\vec{r}}_2 = \vec{F}_2 = \vec{F}_2^{ex} + \vec{F}_{21}$$

center of mass theorem $M\ddot{\vec{R}} = \vec{F}^{ex}$ 06.01.2016

$$\ddot{\vec{r}} = \ddot{\vec{r}}_1 - \ddot{\vec{r}}_2 = \frac{\vec{F}_1^{ex}}{m_1} - \frac{\vec{F}_2^{ex}}{m_2} + \frac{\vec{F}_{12}}{m_1} - \frac{\vec{F}_{21}}{m_2}$$
$$= \frac{\vec{F}_1^{ex}}{m_1} - \frac{\vec{F}_2^{ex}}{m_2} + \vec{F}_{12}\left(\frac{1}{m_1} + \frac{1}{m_2}\right)$$
$$= \frac{\vec{F}_1^{ex}}{m_1} - \frac{\vec{F}_2^{ex}}{m_2} + \frac{\vec{F}_{12}}{\mu}$$

reduced mass $\frac{1}{\mu} = \frac{1}{m_1} + \frac{1}{m_2}$, $\mu = \frac{m_1 m_2}{M}$

closed systems: $\vec{F}_i^{ex} = 0$

$$M\ddot{\vec{R}} = 0 \qquad \dot{\vec{P}} = 0, \quad \vec{P} = \text{const}$$

choose inertial system with $\vec{R} = 0$

$$\mu \ddot{\vec{r}} = \vec{F}_{12}$$

(movement in central force $\vec{F}_{12} = f \cdot (\vec{r}_1 - \vec{r}_2) = f \cdot \vec{r}$)

all quantities can be decomposed

angular momentum $\vec{L} = \vec{L}_S + \vec{L}_r = M\vec{R}\times\dot{\vec{R}} + \sum_i m_i \Delta\vec{r}_i \cdot \Delta\dot{\vec{r}}_i$

$$= M R \times \dot{\vec{R}} + m_1\left(\frac{m_2}{M}\right)^2 \vec{r}\cdot\dot{\vec{r}} + m_2\left(\frac{m_1}{M}\right)^2 \vec{r}\cdot\dot{\vec{r}}$$

$$\vec{L} = M\,\vec{R}\times\dot{\vec{R}} + \frac{m_1 m_2}{M}\,\frac{m_2+m_1}{M}\,\vec{r}\times\dot{\vec{r}}$$

$$= M\,\vec{R}\times\dot{\vec{R}} + \mu\,\vec{r}\times\dot{\vec{r}}$$

kinetic energy $\quad T_i = \frac{m_i}{2}\dot{\vec{r}}_i^{\,2}$

$$T = T_1 + T_2 = \frac{m_1}{2}\left[\dot{\vec{R}}^2 + \left(\frac{m_2}{M}\right)^2\dot{\vec{r}}^{\,2} - \frac{m_2}{M}\,\dot{\vec{R}}\cdot\dot{\vec{r}}\right]$$
$$+ \frac{m_2}{2}\left[\dot{\vec{R}}^2 + \left(\frac{m_1}{M}\right)^2\dot{\vec{r}}^{\,2} + \frac{m_1}{M}\,\dot{\vec{R}}\cdot\dot{\vec{r}}\right]$$
$$= \frac{M}{2}\dot{\vec{R}}^2 + \frac{1}{2}\underbrace{\frac{m_1 m_2}{M}}_{\mu}\dot{\vec{r}}^{\,2} = T_S + T_r$$

potential energy (conservative forces)

$$V(\vec{r}_1,\vec{r}_2) = \underbrace{V_1^{ext}(\vec{r}_1) + V_2^{ext}(\vec{r}_2)}_{} + \underbrace{V_{12}(\vec{r})}_{}$$
$$= \quad V_S \quad + \quad V_r$$

total energy

$$E = E_S + E_r \qquad E_S = T_S + V_S\,, \quad E_r = T_r + V_r$$

Example : planetary motion

gravitational potential of two masses :

$$V_{12}(|\vec{r}_1-\vec{r}_2|) = -\gamma\,\frac{m_1 m_2}{|\vec{r}_1-\vec{r}_2|} = -\gamma\,\frac{\mu M}{r}$$

$$\vec{F}_{12} = -\vec{\nabla}_r V_{12} = -\gamma\,\frac{\mu M}{r^2}\,\vec{r}$$

$$\mu\ddot{\vec{r}} = \vec{F}_{12} = -\gamma\,\frac{\mu M}{r^2}\,\vec{e}_r$$

already solved : replace $m \to \mu$

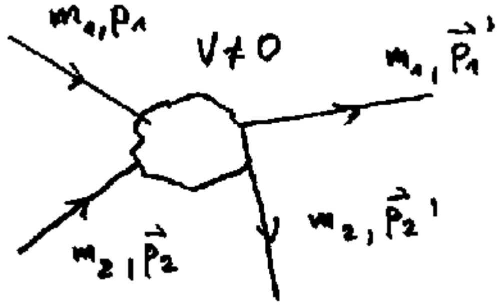
$$r(\varphi) = \frac{k_r}{1+\epsilon_r\cos\varphi} \qquad k_r = \frac{L_r^2}{\gamma\mu^2 M}\,,\quad \epsilon_r = \sqrt{1+\frac{L_r^2\mu E_r}{(\gamma\mu^2 M)^2}}$$

$$\vec{r}_1 = \frac{m_2}{M}\vec{r}$$
$$\vec{r}_2 = -\frac{m_1}{M}\vec{r}$$

$m_1 = \frac{m_2}{2}$

old result for $m_2 \gg m_1$

Two-body collision



deduce $\vec{p}_1{}'$ and $\vec{p}_2{}'$ from $\vec{p}_1$ and $\vec{p}_2$

a) conservation of momentum $\quad \vec{P} = \vec{P}'$

laboratory reference frame Σ_L $\quad \vec{p}_1 + \vec{p}_2 = \vec{p}_1{}' + \vec{p}_2{}'$

center of mass ref. frame Σ_{CM} $\quad \vec{R} = 0, \ \dot{\vec{R}} = 0$

$\Rightarrow \vec{P} = 0$

$\vec{p}_1 = -\vec{p}_2 \ , \ \vec{p}_1{}' = \vec{p}_2{}'$

b) energy conservation (use Σ_{CM})

$$E = \sum_{i=1}^{2} \frac{\vec{p}_i^{\,2}}{2m_i} = \sum_{i=1}^{2} \frac{\vec{p}_i^{\,\prime 2}}{2m_i} + Q$$

change of internal energy

$Q = 0$ elastic collision

$Q \neq 0$ inelastic collision $\quad Q > 0$ kinetic energy converted into internal energy

use $\vec{p}_1^{\,2} = \vec{p}_2^{\,2}$, $\vec{p}_1^{\,\prime 2} = \vec{p}_2^{\,\prime 2}$ $\quad Q < 0$ internal energy converted into kin. energy

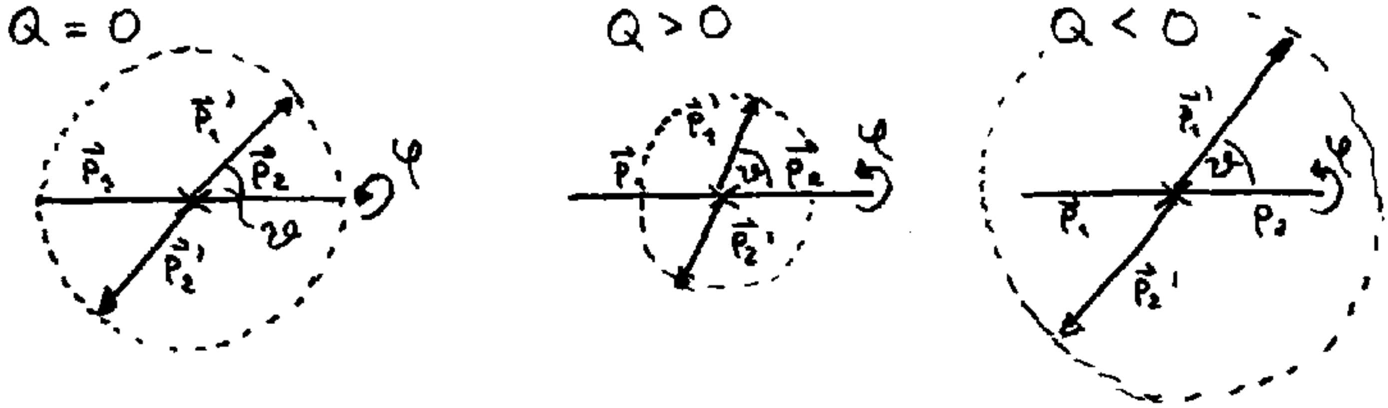
$$E = \frac{1}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \vec{p}_1^{\,2} = \frac{1}{2}\left(\frac{1}{m_1} + \frac{1}{m_2}\right) \vec{p}_1^{\,\prime 2} + Q$$

$$\frac{\vec{p}_1^{\,2}}{2\mu} = \frac{\vec{p}_1^{\,\prime 2}}{2\mu} + Q$$

$$\vec{p}_1^{\,\prime 2} = \vec{p}_1^{\,2} - 2\mu Q$$

$$p_1' = \sqrt{p_1^2 - 2\mu Q}$$

Note: 6 unknown quantities : $\vec{p}_1{}'$ and $\vec{p}_2{}'$
but only $3+1 = 4$ equations :
magnitude p_i' fixed, direction not

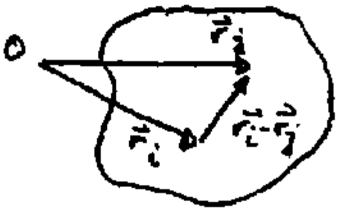


ϑ, φ : fixed by details of potential V_{12}
(compare result from $V_{12} = -\gamma \frac{Mm}{r}$)

The rigid body

Definition : rigid body : system of $N \approx 10^{23}$ mass points with fixed distances

$$r_{ij} = |\vec{r}_i - \vec{r}_j|$$



(not deformable)

possible movements : translation, rotation

How many degrees of freedom?

1 particle — 3 degrees of freedom $\vec{r}_1 = (x_1, y_1, z_1)$

2 particles — $6 - 1 = 5$ degrees of freedom

$\vec{r}_i = (x_i, y_i, z_i)$ — constraint $r_{12} = |\vec{r}_1 - \vec{r}_2|$

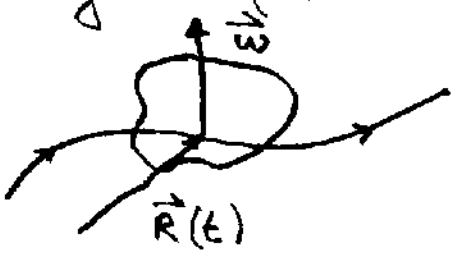
3 particles — $5 + 3 - 2 = 6$ degrees of freedom

$\vec{r}_3 = (x_3, y_3, z_3)$ — $r_{13} = |r_1 - r_3|$, $r_{23} = |\vec{r}_2 - \vec{r}_3|$

N particles — 6 degrees of freedom (each particle added generates 3 more constraints)

fix 6 degrees of freedom

1) Translation : coordinates of fixed point : $\vec{R} = (X, Y, Z)$
2) Rotation : 3 angles fix direction, $\vec{\omega}(t)$



Special cases : spinning top : no translation

3 degrees of freedom

fixed axis (physical pendulum)

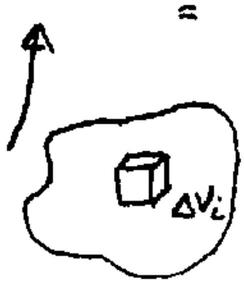
1 degree of freedom : φ

Recall : quantities for N-particle system

mass $\quad M = \sum_{i=1}^{N} m_i \quad = \int \rho(\vec{r})\, dV$

center of mass $\quad \vec{R} = \frac{1}{M} \sum_{i=1}^{N} m_i \vec{r}_i \quad = \frac{1}{M} \int \rho(\vec{r})\, \vec{r}\, dV$

total momentum $\quad \vec{P} = \sum_{i=1}^{N} m_i \dot{\vec{r}}_i \quad = \int \rho(\vec{r})\, \dot{\vec{r}}\, dV$



infinitesimal volume element

Calculation in continuum

$$M = \sum_i \Delta m_i = \sum_i \frac{\Delta m_i}{\Delta V_i} \Delta V_i$$

take limit

mass density

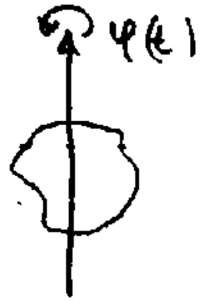
$$\left.\begin{array}{l} \Delta V_i \to 0 \\ \Delta m_i \to 0 \end{array}\right\} \quad \rho(\vec{r}) = \lim_{\Delta V \to 0} \frac{\Delta m(r)}{\Delta V(r)}$$

$$\Delta V \to dV = d^3\vec{r}$$

recall : volume integral $\quad M = \int \rho(\vec{r})\, dV$

$$\overset{\rho = \rho_0 \text{ in } V}{=} \int_V \rho_0\, dV \underset{\text{cube}}{=} \rho_0 \int_{c_1}^{c_2} dz \int_{b_1}^{b_2} dy \int_{a_1}^{a_2} dx$$

Rotation around an axis



degree of freedom $\varphi(t)$

calculate kinetic energy

$$T = \sum_{i=1}^{N} \frac{m_i}{2} \vec{v}_i^{\,2}$$

angular velocity $\omega = \dot{\varphi}$, $\vec{\omega} = (0, 0, \omega)$

rotation of mass point around axis:

$$\dot{\vec{r}}_i = \vec{\omega} \times \vec{r}_i = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ 0 & 0 & \omega \\ x_i & y_i & z_i \end{vmatrix} = \omega\,(-y_i, x_i, 0)$$

$$T = \sum_{i=1}^{N} \frac{m_i}{2} (\vec{\omega} \times \vec{r}_i)^2 = \sum_{i=1}^{N} \frac{m_i}{2} (\hat{\omega} \times \vec{r}_i)^2 \,\omega^2$$

$$= \frac{1}{2} J \omega^2 \qquad J = \sum_{i=1}^{N} m_i (\hat{\omega} \times \vec{r}_i)^2$$

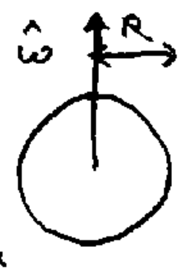
(moment of inertia)

rotation around z-axis

$$J = \sum_{i=1}^{N} m_i (\hat{\omega} \times \vec{r}_i)^2 \overset{\hat{\omega} = (0,0,1)}{=} \sum_{i=1}^{N} m_i (x_i^2 + y_i^2)$$

$$= \int d^3r \; \varrho(\vec{r})\,(x^2 + y^2) \qquad \text{(continuum)}$$

Example : sphere with homogeneous mass distribution



$$\varrho(\vec{r}) = \left\{ \begin{matrix} \varrho_0 & r \le R \\ 0 & r > R \end{matrix} \right\} = \varrho_0\, \underbrace{\Theta(R-r)}_{\text{step function}}$$

use spherical coordinates $\vec{r} = r\,(\sin\vartheta\cos\varphi, \sin\vartheta\sin\varphi, \cos\vartheta)$

$$(\hat{\omega}\times\vec{r})^2 = r^2\left(\sin^2\vartheta\cos^2\varphi + \sin^2\vartheta\sin\varphi\right) = r^2\sin^2\vartheta$$

$$J = \int d^3r\; \rho(\vec{r})\,(\hat{\omega}\times\vec{r})^2 \qquad d^3r = r^2\sin\vartheta\,dr\,d\vartheta\,d\varphi$$

$$= \int_0^\infty dr\, r^4 \int_0^\pi d\vartheta\,\sin^3\vartheta \int_0^{2\pi} d\varphi\; \underbrace{\rho(\vec{r})}_{\rho_0\,\Theta(R-r)}$$

$$= \rho_0 \int_0^R dr\, r^4 \int_0^\pi d\vartheta\,\sin^3\vartheta \int_0^{2\pi} d\varphi \qquad x = \cos\vartheta$$

$$= \rho_0 \left.\frac{r^5}{5}\right|_0^R \int_{-1}^1 dx\,(1-x^2)\; \varphi\Big|_0^{2\pi}$$

$$= \rho_0\frac{R^5}{5}\left(2-\frac{2}{3}\right)2\pi = \underbrace{\frac{4\pi}{3}R^3}_{V}\rho_0\,\frac{2}{5}R^2 = M\,\frac{2}{5}R^2$$

Rotation around axis in conservative force:

energy conservation

$$E = T + V = \frac{1}{2}J\omega^2 + V = \frac{1}{2}J\dot{\varphi}^2 + V(\varphi)$$

compare $E = \frac{m}{2}\dot{x}^2 + V(x)$ (1 dimensional motion in potential)

solution by separation of variables

$$t(\varphi) = t - t_0 = \int_{\varphi_0}^{\varphi} d\varphi' \frac{1}{\sqrt{\frac{2}{J}\left(E - V(\varphi')\right)}}$$

inversion: $\varphi(t)$

Angular momentum law

$\vec{\omega} \parallel \vec{L}$

in general : angular momentum not parallel to rotation axis

$\vec{\omega} \nparallel \vec{L}$

consider : component parallel to $\vec{\omega}$:

$$L_\omega = \hat{\omega} \cdot \vec{L} = \hat{\omega} \cdot \sum_{i=1}^{N} m_i \, \vec{r}_i \times \dot{\vec{r}}_i$$

$$= \sum_{i=1}^{N} m_i \underbrace{(\vec{r}_i \times \dot{\vec{r}}_i) \cdot \hat{\omega}}_{(\hat{\omega} \times \vec{r}_i) \cdot \dot{\vec{r}}_i} \underset{\dot{\vec{r}}_i = \vec{\omega} \times \vec{r}_i}{=} \sum_{i=1}^{N} m_i \, (\hat{\omega} \times \vec{r}_i) \cdot (\vec{\omega} \times \vec{r}_i)$$

$$= \omega \sum_{i=1}^{N} m_i \, (\hat{\omega} \times \vec{r}_i)^2 = \omega J = \dot{\varphi} J$$

general angular momentum law $\frac{d}{dt} \vec{L} = \vec{M}^{ex} = \sum_i \vec{r}_i \times \vec{F}_i$

multiply with $\hat{\omega} = \overrightarrow{const}$

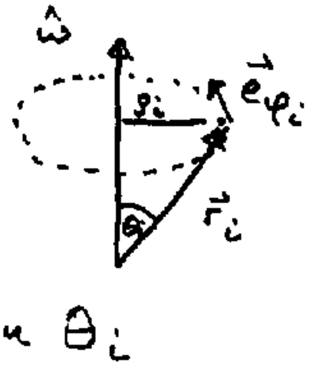
$$\frac{d}{dt} (\hat{\omega} \cdot \vec{L}) = \hat{\omega} \cdot \vec{M}^{ex}$$

$$\frac{d}{dt} (\dot{\varphi} J) = \sum_i (\vec{r}_i \times \vec{F}_i^{ex}) \cdot \hat{\omega}$$

$$\ddot{\varphi} J = \sum_i (\hat{\omega} \times \vec{r}_i) \cdot \vec{F}_i^{ex}$$

evaluate $\hat{\omega} \times \vec{r}_i = \vec{e}_z \times \vec{r}_i = (-y_i, x_i, 0)$

in cylindridrical coordinates



$$\hat{\omega} \times \vec{r}_i = (-y_i, x_i, 0)$$
$$= \varrho_i (-\sin\varphi, \cos\varphi, 0)$$
$$= \varrho_i \, \vec{e}_\varphi \qquad , \quad \varrho_i = r_i \sin\theta_i$$

$$\ddot{\varphi} J = \sum_i \varrho_i \, \vec{e}_\varphi \cdot \vec{F}_i^{ex}$$

special case: $\vec{M}^{ex} = 0$, $\hat{\omega} \cdot \vec{M}^{ex} = 0$

$$J \ddot{\varphi} = 0 \qquad \dot{\varphi} = \omega = \text{const}$$

$$\omega J = L_\omega = \text{const}$$

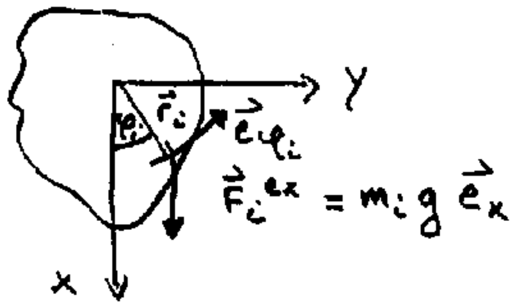
$$T = \tfrac{1}{2} J \omega = \text{const}$$

Analogies between translational and rotational motion

Translation		Rotation	
position	x	angle	φ
mass	m	moment of inertia	J
velocity	$v = \dot{x}$	angular velocity	$\omega = \varphi$
momentum	$p = m v$	angular momentum	$L_\omega = J\omega$
force	F	torque	M^{ex}_ω
kinetic energy	$T = \frac{1}{2} m v^2$		$T = \frac{1}{2} J \omega$
equation of motion	$F = m \ddot{x}$		$M^{ex}_\omega = J \ddot{\varphi}$

Physical pendulum

rotation $\vec{\omega} \parallel \vec{e}_z$



$$J \ddot{\varphi} = \sum_i \varrho_i \underbrace{\vec{F}_i^{ex} \cdot \vec{e}_{\varphi_i}}_{-m_i g \sin\varphi_i}$$

$\vec{e}_{\varphi_i} = (-\sin\varphi_i, \cos\varphi_i, 0$

$$= -\sum_i \underbrace{\varrho_i \sin\varphi_i}_{y_i \ (\text{geometry})} m_i g = -g \sum_i y_i m_i$$

$\measuredangle$ $\vec{R} = \frac{1}{M} \sum_i \vec{r}_i m_i$, $R_y = \frac{1}{M} \sum_i y_i m_i$

$$= -g M R_y$$

$R_y = R \sin\varphi$, $R = |\vec{R}|$

$$= -g M R \sin\varphi$$

$$\frac{J}{MR}\,\ddot{\varphi} + g\sin\varphi = 0 \qquad\qquad l\ddot{\varphi} + g\sin\varphi = 0$$

(mathematical pendulum)

motion equivalent to mathematical pendulum with

$$l = \frac{J}{MR}$$

small angles $\varphi \approx \sin\varphi$, solution

$$\varphi(t) = A\cos(\omega t) + B\sin(\omega t), \quad \omega = \sqrt{\frac{gMR}{J}}$$

alternative derivation: energy conservation

calculate potential $V = \sum_i V_i \overset{V_i = -m_i g x_i}{=} -g\sum_i m_i x_i$

$$= -Mg R_x = -MgR\cos\varphi$$

$$E = T + V = \frac{1}{2}J\dot{\varphi}^2 - MgR\cos\varphi$$

$$\frac{dE}{dt} = \dot{\varphi}\left(J\ddot{\varphi} + gMR\sin\varphi\right) = 0$$

12.01.2017

Steiner's theorem

Note: $J = \sum_i m_i (x_i^2 + y_i^2)$ depends on choice of rotation axis

decompose $\vec{r}_i = \Delta\vec{r}_i + \vec{R}$, $x_i = x_i' + R_x$

$y_i = y_i' + R_y$

$$J = \sum_i m_i(x_i^2 + y_i^2) = \sum_i m_i\left[(x_i' + R_x)^2 + (y_i' + R_y)^2\right]$$

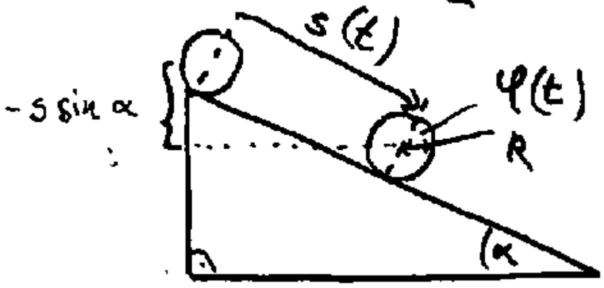
$$= \underbrace{\sum_i m_i (x_i'^2 + y_i'^2)}_{} + \underbrace{\sum_i m_i}_{M}(R_x^2 + R_y^2) + 2R_x\underbrace{\sum_i m_i x_i'}_{0} + 2R_y\underbrace{\sum_i m_i y_i'}_{0}$$

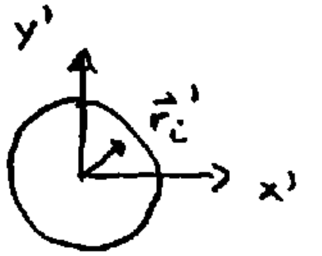
$$= J_S + MS^2 \geq J_S$$

(moment of inertia for axis through $\vec{R}$) (center of mass in coordinate system at center of mass)

Rolling motion

Example: homogeneous cylinder rolling off a plane





each mass point $\quad \dot{\vec{r}}_i = \dot{\vec{r}}_{iT} + \dot{\vec{r}}_{iR}$

$\dot{\vec{r}}_{iR} = \vec{\omega} \times \vec{r}_i\,' \qquad \vec{\omega} \parallel \vec{e}_z$

no sliding : condition

$$\dot{s} = -R\dot{\varphi} = -R\omega \quad \rightarrow \quad \dot{\vec{r}}_i = \dot{s}\,\vec{e}_s + \vec{\omega} \times \vec{r}_i$$

kinetic energy

$$T = \frac{1}{2} \sum_i m_i \dot{\vec{r}}_i^{\,2} = \frac{1}{2} \sum_i m_i \left(\dot{s}\,\vec{e}_s + \vec{\omega} \times \vec{r}_i \right)^2$$

$$= \frac{1}{2} \sum_i m_i \left[\dot{s}^2 + (\vec{\omega} \times \vec{r}_i\,')^2 + 2\dot{s}\,\vec{e}_s \cdot (\vec{\omega} \times \vec{r}_i\,') \right]$$

$$= \frac{1}{2} M \dot{s}^2 + \frac{1}{2} J \omega^2 + \dot{s}\,\vec{e}_s \cdot \Big(\vec{\omega} \times \underbrace{\sum_i m_i \vec{r}_i\,'}_{M \vec{R}\,' = 0} \Big)$$

$$= \frac{1}{2} M \dot{s}^2 + \frac{1}{2} J \omega^2$$

$\qquad \dot{s}^2 = R^2 \omega^2$

$\qquad J = \frac{1}{2} M R^2 \quad$ (no proof)

$$= \frac{3}{4} M R^2 \omega^2$$

potential energy (gravitational potential)

$$V = \sum_i V_i = \sum_i m_i g x_i$$

$$= g M R_x = -g M s \sin\alpha$$

choice of Koordinate system

$$E = T + V = \frac{3}{4} M R^2 \dot{\varphi}^2 - g M s \sin\alpha$$

$$0 = \frac{d}{dt} E = \frac{3}{4} M R^2 \, 2 \dot{\varphi} \ddot{\varphi} + M g \sin\alpha R \dot{\varphi} \qquad \dot{s} = -\dot{\varphi} R$$

$$\Rightarrow \quad \ddot{\varphi} = -\frac{2}{3} \frac{g \sin\alpha}{R} \quad , \quad \ddot{s} = \frac{2}{3} g \sin\alpha$$

$$\varphi(t) = -\frac{1}{3} \frac{g \sin\alpha}{R} t^2 \qquad \begin{array}{l} \varphi(0) = 0 \\ \dot{\varphi}(0) = 0 \end{array}$$

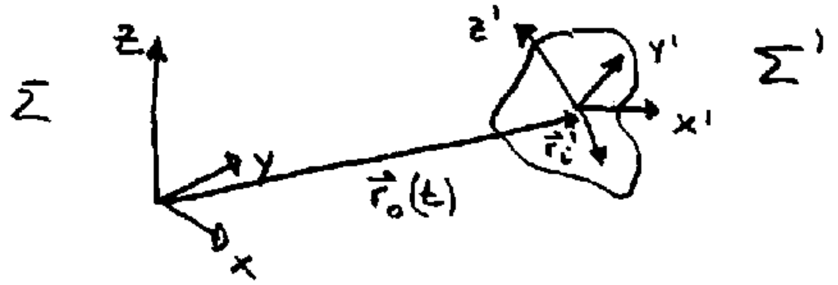
Generalization : moment of inertia

physical pendulum : $\hat{\omega} = \text{const}$ $\quad T = \frac{1}{2} J \omega^2$

spinning top $\quad \hat{\omega} = \hat{\omega}(t)$ $\quad T = ?$

(inertial tensor)

calculate velocity using coordinate system fixed at center of mass position



$$\vec{r}_i = \vec{r}_0 + \vec{r}_i{}' \qquad \vec{r}_0 = \sum_{\alpha=1}^{3} x_{\alpha 0} \vec{e}_\alpha$$

$$\vec{r}_i{}'(t) = \sum_{\alpha=1}^{3} x_{i\alpha}' \, \vec{e}_\alpha{}'(t)$$

recall : time derivative operator in rotating reference frame $\left.\frac{d}{dt}\right|_{\Sigma} = \left.\frac{d}{dt}\right|_{\Sigma'} + \vec{\omega} \times$

$$\dot{\vec{r}}_i = \dot{\vec{r}}_0 + \underbrace{\left.\frac{d}{dt} \vec{r}_i{}'\right|_{\Sigma'}}_{0} + \vec{\omega} \times \vec{r}_i{}' = \dot{\vec{r}}_0 + \vec{\omega} \times \vec{r}_i{}'$$

choice of coordinate system

kinetic energy:

$$T = \frac{1}{2}\sum_i m_i \dot{\vec{r}}_i^{\,2} = \frac{1}{2}\sum_i m_i\left(\dot{\vec{r}}_0 + \vec{\omega}\times\vec{r}_i{}'\right)^2$$

$$= \frac{1}{2}\sum_i m_i \dot{\vec{r}}_0^{\,2} + \frac{1}{2}\sum_i m_i\left(\vec{\omega}\times\vec{r}_i{}'\right)^2 + \sum_i m_i \dot{\vec{r}}_0\cdot\left(\vec{\omega}\times\vec{r}_i{}'\right)$$

$$= \frac{1}{2}\dot{\vec{r}}_0^{\,2}\sum_{i=1}^{N} m_i + \frac{1}{2}\sum_i m_i\left(\vec{\omega}\times\vec{r}_i{}'\right)^2 + \dot{\vec{r}}_0\cdot\Big(\vec{\omega}\times\underbrace{\sum_i m_i\vec{r}_i{}'}_{M\vec{R}'=0}\Big)$$

$$= T_T + T_R$$

consider rotation part:

$$T_R = \frac{1}{2}\sum_i m_i\left(\vec{\omega}\times\vec{r}_i{}'\right)^2$$

$$= \frac{1}{2}\sum_i m_i\Big(\sum_{nlm}\epsilon_{nlm}\,\omega_l\, x_{im}'\,\vec{e}_n\Big)\cdot\Big(\sum_{trs}\epsilon_{trs}\,\omega_r\, x_{is}'\,\vec{e}_t\Big)$$

$\vec{e}_n\cdot\vec{e}_t = \delta_{nt}$

$$\overset{\downarrow}{=} \frac{1}{2}\sum_{\substack{i\\ nlmtrs}} m_i\,\epsilon_{nlm}\,\epsilon_{trs}\,\omega_l\,\omega_r\,x_{im}'\,x_{is}'\,\delta_{nt}$$

$$= \frac{1}{2}\sum_{\substack{i\\ nlmtr}} m_i\,\underbrace{\epsilon_{nlm}\,\epsilon_{nrs}}_{\delta_{lr}\delta_{ms}-\delta_{ls}\delta_{mr}}\,\omega_l\,\omega_r\,x_{im}'\,x_{is}'$$

$$= \frac{1}{2}\sum_{\substack{i\\ lmtr}} m_i\left(\delta_{lr}\delta_{ms}-\delta_{ls}\delta_{mr}\right)\omega_l\,\omega_r\,x_{im}'\,x_{is}'$$

$$= \frac{1}{2}\sum_{\substack{i\\ lm}} m_i\left(\omega_l^2\, x_{im}'^{\,2} - \omega_l\,\omega_m\, x_{il}'\,x_{im}'\right)$$

$$= \frac{1}{2}\sum_{lm}\omega_l\,\omega_m\underbrace{\sum_i m_i\left(\delta_{lm}\,\vec{r}_i{}'^{\,2} - x_{il}'\,x_{im}'\right)}_{J_{lm} = \sum_i m_i\left(\delta_{lm}\vec{r}_i{}'^{\,2} - x_{il}'x_{im}'\right)}$$

$\sum_m x_{im}'^{\,2} = \vec{r}_i{}'^{\,2}$

rewrite kinetic energy $T_R = \frac{1}{2} \vec{\omega}^T \underline{J} \vec{\omega}$

$$= \frac{1}{2} \sum_{\ell m} \omega_\ell J_{\ell m} \omega_m$$

inertial tensor

$$J_{\ell m} = \begin{pmatrix} \sum_i m_i \left(x_{i2}'^2 + x_{i3}'^2\right) & -\sum_i m_i x_{i1}' x_{i2}' & -\sum_i m_i x_{i1}' x_{i3}' \\ -\sum_i m_i x_{i2}' x_{i1}' & \sum_i m_i \left(x_{i1}'^2 + x_{i3}'^2\right) & -\sum_i m_i x_{i2}' x_{i3}' \\ -\sum_i m_i x_{i3}' x_{i1}' & -\sum m_i x_{i3}' x_{i2}' & \sum_i m_i \left(x_{i1}'^2 + x_{i2}'^2\right) \end{pmatrix}$$

symmetric 3×3 matrix

continuum $\quad J_{\ell m} = \int d^3\vec{r} \; \rho(\vec{r}) \left(\delta_{\ell m} \vec{r}^{\,2} - x_\ell x_m \right)$

Properties of inertial tensor

1) Tensor of k-th rank in an n-dimensional space

a) n^k number of elements $\{F_{i_1 \dots i_k}\}$, $i_j = 1 \dots n$

b) transforms itself with respect to all k indices like a vector under coordinate rotations

Examples:

Tensor of 0-th rank : scalar (invariant)

Tensor of 1-st rank : vector $\quad \bar{x}_j = \sum d_{jk} x_k$

$$\vec{\bar{x}} = D \vec{x}$$

Tensor of 2nd rank : $\quad \bar{F}_{jk} = \sum_{\ell m} d_{j\ell} d_{km} F_{\ell m}$

$$= \sum_{\ell m} d_{j\ell} F_{\ell m} d^T_{mk}$$

$$= \left(D F D^T \right)_{jk}$$

Use transformation of $J_{\ell m}$:

kinetic energy in rotated system , $\vec{\bar{\omega}} = D\vec{\omega}$

$$\bar{T}_R = \frac{1}{2}\,\vec{\bar{\omega}}^T\,\underline{\bar{J}}\,\vec{\bar{\omega}} = \frac{1}{2}\,(D\vec{\omega})^T\,D\,\underline{J}\,D^T\,D\,\vec{\omega}$$

$$= \frac{1}{2}\,\vec{\omega}^T\,\underbrace{D^T D}_{E}\,\underline{J}\,\underbrace{D^T D}_{E}\,\vec{\omega} = \frac{1}{2}\,\vec{\omega}^T\,\underline{J}\,\vec{\omega}$$

$$= T_R$$

(kinetic energy invariant)

2) Inertial tensor vs. moment of inertia

rotation around fixed axis

$$T_R = \frac{1}{2}\,J\,\omega^2 \quad , \quad \vec{\omega} = \omega\,\hat{\omega} = \omega\,\vec{n}\ ,\ |\vec{n}| = 1$$

calculate with inertial tensor:

$$T_R = \frac{1}{2}\sum_{\ell m}\omega_\ell\,J_{\ell m}\,\omega_m = \frac{1}{2}\,\omega^2\,\underbrace{\sum_{\ell m} n_\ell\,J_{\ell m}\,n_m}_{J}$$

$$J = \sum_{\ell m} n_\ell\,J_{\ell m}\,n_m$$

example $\quad \vec{n} = \begin{pmatrix}0\\0\\1\end{pmatrix} \quad J = J_{33} = \sum_i m_i\,(x_i^2 + y_i^2)$

3) principal axis of inertia

$$J_{\ell m} = \int d^3r\ \ \rho(\vec{r})\,(\,\delta_{\ell m}\,\vec{r} - x_\ell x_m\,)$$

symmetric $J_{\ell m} = J_{m\ell}$
real $J_{\ell m} = J_{\ell m}^{*}$

can be shown (reference to linear algebra):
Exists a special rotation D such that

$$\bar{J} = D^T\,J\,D = \mathrm{diag}\,(A,B,C) = \begin{pmatrix}A&0&0\\0&B&0\\0&0&C\end{pmatrix}$$

D : principal axis transformation

A, B, C : principal moments ; eigenvalues of J_{km}

$\vec{\bar{e}}_i$: principal axis of inertia $\quad \vec{\bar{e}}_i = D\,\vec{e}_i$

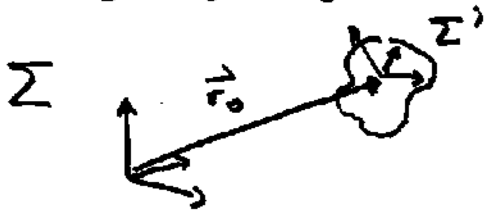
Denotations

$A \neq B \neq C$	asymmetric top	
$A = B \neq C$	symmetric top	oblate $C > A = B$ prolate $C < A = B$
$A = B = C$	spherical top	

Angular momentum of rigid body

recall : fixed axis $\hat{\omega} = \text{const} \qquad L_\omega = J\,\omega$

$$\vec{L} = \sum_i m_i \left(\vec{r}_i \times \dot{\vec{r}}_i\right) \qquad \text{use } \vec{r}_i = \vec{r}_0 + \vec{r}_i{}'$$



$$= \sum_i m_i \left(\vec{r}_0 + \vec{r}_i{}'\right) \times \left(\dot{\vec{r}}_0 + \vec{\omega} \times \vec{r}_i{}'\right)$$

$$= \sum_i m_i \Big[\vec{r}_0 \times \dot{\vec{r}}_0 + \vec{r}_0 \times (\vec{\omega} \times \vec{r}_i{}') + \vec{r}_i{}' \times \dot{\vec{r}}_0 + \vec{r}_i{}' \times (\vec{\omega} \times \vec{r}_i{}') \Big] \qquad \dot{\vec{r}}_i = \vec{v}_0 + \vec{\omega} \times \vec{r}_i{}'$$

$$= M\, \vec{r}_0 \times \dot{\vec{r}}_0 + \vec{v}_0 \times \Big(\vec{\omega} \times \underbrace{\Big(\sum_i m_i \vec{r}_i{}'\Big)}_{\vec{R}' = 0}\Big) + \underbrace{\Big(\sum_i m_i \vec{r}_i{}'\Big)}_{\vec{R}' = 0} \times \dot{\vec{r}}_0$$

$$+ \sum_i m_i\, \vec{r}_i{}' \times \left(\vec{\omega} \times \vec{r}_i{}'\right)$$

$$= \vec{L}_S + \vec{L}_R \qquad \vec{L}_S = M\, \vec{r}_0 \times \dot{\vec{r}}_0 = M\, \vec{R} \times \dot{\vec{R}}$$

$$\vec{L}_R = \sum_i m_i\, \vec{r}_i{}' \times \left(\vec{\omega} \times \vec{r}_i{}'\right)$$

$a \times (\vec{b} \times \vec{c}) = b(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$

$$= \sum_i m_i \left[\vec{\omega}\, \vec{r}_i{}^2 - \vec{r}_i \left(\vec{\omega}\cdot\vec{r}_i\right)\right]$$

projection on rotation axis

$$L_\omega = \hat{\omega} \cdot \vec{L}_R = \sum_i m_i \left[\omega \vec{r}_i^{\,2} - (\hat{\omega} \cdot \vec{r}_i)(\vec{\omega} \cdot \vec{r}_i) \right]$$

$$= \frac{1}{\omega} \sum_i m_i \underbrace{\left[\omega^2 \vec{r}_i^{\,2} - (\vec{\omega} \cdot \vec{r}_i)^2 \right]}_{(\vec{\omega} \times \vec{r}_i)^2} = \frac{2 T_R}{\omega}$$

$$\Rightarrow T_R = \frac{1}{2} \omega L_\omega = \frac{1}{2} \vec{\omega} \cdot \vec{L}_R$$

$$= \frac{1}{2} \vec{\omega}^T \underline{J} \vec{\omega}$$

$\alpha < \frac{\pi}{2}$: $T_R > 0$

comparison : $\vec{L}_R = \underline{J} \vec{\omega}$

can also be seen directly from

$$L_{Rl} = \sum_i m_i \left(\omega_l \vec{r}_i^{\,2} - x_{il} (\vec{\omega} \cdot \vec{r}_i) \right)$$

$$= \sum_m J_{lm} \omega_m$$

in system of principal axis of inertia: $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$

$$\vec{L}_R = (A \omega_1, B \omega_2, C \omega_3)$$

consider rotation with $\vec{\omega} \parallel \vec{e}_1, \vec{e}_2, \vec{e}_3$

$$\left. \vec{L}_R = A \vec{\omega} \quad \text{or} \quad \vec{L}_R = B \vec{\omega} \quad \text{or} \quad \vec{L}_R = C \vec{\omega} \right\} \vec{L} \parallel \omega$$

coordinate transformation : keeps relative directions

find principal axis by solving

$$\vec{L}_R = \underline{J} \vec{\omega} = J \vec{\omega}$$

$$\underline{J} \vec{\omega} - J \vec{\omega} = 0$$

$$(\underline{J} - \underline{E} J) \vec{\omega} = 0$$

eigenvalue equation

J : eigenvalue

$\vec{\omega}$: eigenvector

non-trivial solution for

$$\det(\underline{J} - \underline{E} J) = 0$$

polynomial equation in $\mathcal{J}$: $\alpha \mathcal{J}^3 + \beta \mathcal{J}^2 + \gamma \mathcal{J} + \delta = 0$

3 solutions $\{\mathcal{J}_1, \mathcal{J}_2, \mathcal{J}_3\} = \{A, B, C\}$

proof : consider $D \underline{\mathcal{J}} D^T = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}$, $D D^T = \mathbb{1}$

$$0 = \det\left(\underline{\mathcal{J}} - \underline{\underline{E}}\,\mathcal{J}\right) = \det\left(\underline{\mathcal{J}} - \underline{\underline{E}}\,\mathcal{J}\right) \det D \det D^T$$

$1 = \det(D D^T) = \det D \det D^T$

$\det(AB) = \det A \cdot \det B$

$$= \det\left[D\left(\underline{\mathcal{J}} - \underline{\underline{E}}\,\mathcal{J}\right) D^T\right] = \det\left(\underbrace{D \underline{\mathcal{J}} D^T}_{\begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}} - \underbrace{D \underline{E} D^T}_{\mathbb{1}}\mathcal{J}\right)$$

$$= \det \begin{pmatrix} A-\mathcal{J} & 0 & 0 \\ 0 & B-\mathcal{J} & 0 \\ 0 & 0 & C-\mathcal{J} \end{pmatrix} = (A-\mathcal{J})(B-\mathcal{J})(C-\mathcal{J})$$

$\Rightarrow A = \mathcal{J}$ or $B = \mathcal{J}$ or $C = \mathcal{J}$

Theory of spinning top

angular momentum law in Σ' (fixed to rigid body)

$$\left.\frac{d}{dt}\right|_{\Sigma} \vec{L}_R = \vec{M} \qquad \left.\frac{d}{dt}\right|_{\Sigma} = \left.\frac{d}{dt}\right|_{\Sigma'} + \vec{\omega}\times$$

$$\left.\frac{d}{dt}\right|_{\Sigma'} \vec{L}_R + \vec{\omega} \times \vec{L}_R = \vec{M} \qquad \vec{L}_R = (A\omega_\xi, B\omega_\eta, C\omega_\zeta)$$

$$A\dot{\omega}_\xi + (C-B)\,\omega_\eta\omega_\zeta = M_\xi \qquad \vec{\omega} = (\omega_\xi, \omega_\eta, \omega_\zeta)$$

$$B\dot{\omega}_\eta + (A-C)\,\omega_\xi\omega_\zeta = M_\eta \qquad \vec{M} = (M_\xi, M_\eta, M_\zeta)$$

$$C\dot{\omega}_\zeta + (B-A)\,\omega_\xi\omega_\eta = M_\zeta$$

free motion $\vec{M} = 0$, $\frac{d}{dt} T_R = 0$ $|\vec{L}_R| = \text{const}$

$$A\dot{\omega}_\xi + (C-B)\,\omega_\eta\omega_\zeta = 0$$
$$B\dot{\omega}_\eta + (A-C)\,\omega_\xi\omega_\zeta = 0$$
$$C\dot{\omega}_\zeta + (B-A)\,\omega_\xi\omega_\eta = 0$$

non-linear coupled D.E. for ω_i

Discussion :

1) rotation around principal axis

$\vec{\omega} = \omega_\xi \vec{e}_\xi = \text{const}$ is solution $\quad \dot{\omega}_\xi = 0$

$$\omega_\eta = \omega_\zeta = 0$$

2) for $A = B = C \equiv J$

$$\left.\begin{aligned} J\,\dot{\omega}_\xi &= 0 \\ J\,\dot{\omega}_\eta &= 0 \\ J\,\dot{\omega}_\zeta &= 0 \end{aligned}\right\} \quad \vec{\omega} = \text{const} \text{ is solution}$$

3) general case $\quad$ consider $\vec{\omega} = (\omega_0 + p(t),\, q(t),\, r(t))$

$$\omega_0 \gg p, q, r$$

$$A\dot{p} + (C - B)\, q r = 0$$
$$B\dot{q} + (A - C)(\omega_0 + p)\, r = 0$$
$$C\dot{r} + (B - A)(\omega_0 + p)\, q = 0$$

approximation $\quad A\dot{p} \approx 0 \quad p = \text{const}$

$$\left.\begin{aligned} B\dot{q} + (A - C)\,\omega_0 r &= 0 \\ C\dot{r} + (B - A)\,\omega_0 q &= 0 \end{aligned}\right\} \text{coupled linear D.E.}$$

decouple

$$\ddot{q} + D^2 q = 0 \qquad D^2 = \frac{(A-B)(A-C)}{BC}\,\omega_0^2$$
$$\ddot{r} + D^2 r = 0$$

solutions : oscillatory if $D \in \mathbb{R}$, $D^2 > 0$

$D^2 > 0$ if a) $A > B$, $A > C$

b) $A < B$, $A < C$

Lagrange Mechanics

Newton's equations for N-particle system

$$m_i \ddot{\vec{r}}_i = \vec{F}_i^{(ex)} + \sum_{i \neq j} \vec{F}_{ij} = \vec{F}_i$$

1) Constraints of the system example : rigid body $|\vec{r}_{ij}| = const$

2) constraints employed by forces of constraint

$$m_i \ddot{\vec{r}}_i = \underset{\text{driving force}}{\vec{K}_i} + \underset{\text{constraint force}}{\vec{Z}_i}$$

Problems

a) constraint forces unknown (before solving)

b) coordinates are not independent

→ Solution : Lagrange mechanics

Classification of constraints, generalized coordinates

A) Holonomic constraints

$$f_\nu (\vec{r}_1, \dots, \vec{r}_N, t) = 0 \qquad \nu = 1 \dots p$$

system has $S = 3N - p$ (independent coordinates)

examples

1) Dumbbell 

$|\vec{r}_{ij}| = \ell \Leftrightarrow |\vec{r}_{ij}| - \ell = 0$

$S = 3 \cdot 2 - 1 = 5$ independent coordinates

2) rigid body

3) particle on the x-y plane : $z = 0$

a) holonomic-scleronomic constraints

$$\frac{\partial f_r}{\partial t} = 0 \qquad \text{(time-independent)}$$

b) holonomic-rheonomic constraints

$$\frac{\partial f_r}{\partial t} \neq 0 \qquad \text{time-dependent}$$

example: particle in elevator with $z(t) = v_0 t$

$$f(z,t) = z(t) - v_0 t = 0$$

$\to$ Introduction of $S = 3N - p$ $\overset{\text{generalized}}{\text{coordinates}}$ fixes configuration uniquely $\vec{q} = (q_1, \dots, q_s)$; $\vec{r} = \vec{r}(\{q_i\}, t)$

q_i : independent from each other, not necessarily lengths (example particle on sphere $q_1 = \varphi$, $q_2 = \vartheta$)
choice of q_i not unique, but number $i = 1 \dots S$

B) Non-holonomic constraints
(do not reduce degrees of freedom)

a) constraints as inequalities
example: particle inside sphere $|\vec{r}| - R \leq 0$
(still 3 degrees of freedom)

b) constraints in differential, non-integrable form

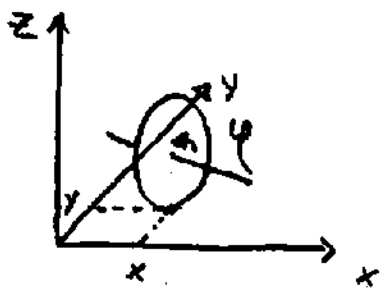
$$\sum_{m=1}^{3N} f_{im}\, dx_m + f_{it}\, dt = 0 \qquad i = 1 \dots p$$

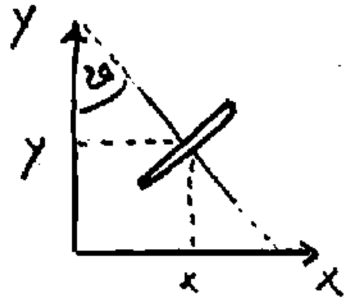
(not a total differential)
There is no function $F_i(\{x_m\}, t)$ with

$$\frac{\partial F_i}{\partial x_m} = f_{im} \quad \text{and} \quad \frac{\partial F_i}{\partial t} = f_{it}$$

otherwise: $F_i(\{x_m\}, t) = \text{const}$ would be holonomic constraint !

Example: rolling wheel





rigid body: 6 degrees of freedom (3 angles, center of mass $\vec{R}$)

2 holonomic-skleronomic constraints

$\psi = 0$ (wheel does not tilt)

$z_0 - R = 0$ $\qquad$ R: radius of wheel

another constraint: velocity ↔ angular velocity $\dot{\varphi}$

$|\vec{v}| = R\dot{\varphi}$ $\qquad$ $\left.\begin{array}{l} v_x = \dot{x} = v\cos\vartheta \\ v_y = \dot{y} = v\sin\vartheta \end{array}\right\}$ geometry

rewrite constraint $\qquad$ $\dot{x} - R\dot{\varphi}\cos\vartheta = 0$

$\dot{y} - R\dot{\varphi}\sin\vartheta = 0$

differential form $\qquad$ $dx - R\cos\vartheta\, d\varphi = 0$

$dy - R\sin\vartheta\, d\varphi = 0$

not integrable: need time-dependence $\vartheta(t)$

→ all 4 coordinates are independent

Note: possible to rewrite as constraint for velocities

$$\sum_{i=1}^{3N} g_{im}\dot{x}_m + g_{it} = 0$$

(not-holonomic constraint)

D'Alembert's principle

goal: eliminate constraint forces from equations of motion

$$m_i \ddot{\vec{r}}_i = \vec{K}_i + \vec{Z}_i$$

driving force (gravitation, spring force, friction)

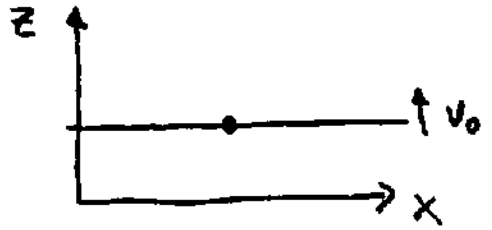
constraint force (thread tension, force on surface)

Definition: virtual displacement $\delta \vec{r}$

infinitestimal chance of coordinates (compatible with constraints), instantaneously executed: $\delta t = 0$

notation $\quad \delta$: virtual, $\quad d$: real $\quad \Big\}$ normal differential (mathematically)

Example: particle in an elevator



$$z(t) = v_0 t$$

real displacement $\quad d\vec{r} = \begin{pmatrix} dx \\ dz \end{pmatrix} = \begin{pmatrix} dx \\ v_0 dt \end{pmatrix} \quad dt \neq 0$

virtual displacement $\quad \delta \vec{r} = \begin{pmatrix} \delta x \\ \delta z \end{pmatrix} = \begin{pmatrix} \delta x \\ \delta t\, v_0 \end{pmatrix} \underset{\delta t = 0}{=} \begin{pmatrix} \delta x \\ 0 \end{pmatrix}$

Definition: Virtual work

$$\delta W_i = -\vec{F}_i \cdot \delta \vec{r}_i \qquad \vec{F}_i = \vec{K}_i + \vec{Z}_i$$

$$= -\vec{K}_i \cdot \delta \vec{r}_i - \vec{Z}_i \cdot \delta \vec{r}_i$$

goal: eliminate $\vec{Z}_i$ from $m_i \ddot{\vec{r}}_i = \vec{K}_i + \vec{Z}_i$

Axiom: Principle of virtual work

$$\sum_i \vec{Z}_i \cdot \delta\vec{r}_i = 0$$

constraint forces do not execute any work for thought movements

multiply Newton's equation with $\delta\vec{r}_i$:

$$\sum_{i=1}^{N} m_i \ddot{\vec{r}}_i \cdot \delta\vec{r}_i = \sum_{i=1}^{N}\left(\vec{K}_i\cdot\delta\vec{r}_i + \vec{Z}_i\cdot\delta\vec{r}_i\right) \overset{\text{axiom}}{=} \sum_i \vec{K}_i\cdot\delta\vec{r}_i$$

$$\Rightarrow \quad \sum_{i=1}^{N}\left(\vec{K}_i - \dot{\vec{p}}_i\right)\cdot\delta\vec{r}_i = 0 \qquad \text{(D'Alembert's principle)}$$

advantage: no constraint forces

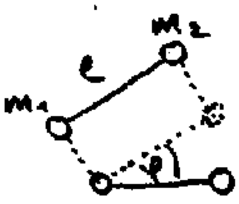
disadvantage: $\delta\vec{r}_i$ depend on each other; introduce generalized coordinates (Lagrange formalism)

Examples (principle of virtual work)

1) particle on surface (also moving)

$$\vec{Z} \perp d\vec{r} \qquad \delta W = \vec{Z}\cdot\delta\vec{r} = 0$$

2) Dumbbell



virtual displacements

$$\delta\vec{r}_1$$

$$\delta\vec{r}_2 = \delta\vec{r}_1 + \delta\vec{\varphi}\times(\vec{r}_2-\vec{r}_1)$$

(rotation around m_1)

$$\delta W = \sum_{i=1}^{2} \vec{Z}_i\cdot\delta\vec{r}_i$$

$$= \vec{Z}_1\cdot\delta\vec{r}_1 + \vec{Z}_2\cdot\left(\delta\vec{r}_1 + \delta\vec{\varphi}\times(\vec{r}_2-\vec{r}_1)\right)$$

$$= \delta\vec{r}_1\cdot\underbrace{\left(\vec{Z}_1+\vec{Z}_2\right)}_{0} + \vec{Z}_2\cdot\delta\vec{\varphi}\times\left(\vec{r}_2-\vec{r}_1\right) \qquad \vec{Z}_1 = -\vec{Z}_2$$

$$= \delta\vec{\varphi}\cdot \overset{0}{\vec{Z}_2\times\left(\vec{r}_2-\vec{r}_1\right)} = 0 \qquad \vec{Z}_2 \parallel \vec{r}_2-\vec{r}_1$$

Note: friction forces do not obey principle

$$\delta W = -\vec{F}_{R_i} \cdot \delta \vec{r}_i \qquad\qquad \vec{F}_R = -\mu |\vec{Z}_i| \, \hat{v}_i$$

$$= \mu |\vec{Z}_i| \, \hat{v}_i \cdot \delta \vec{r}_i$$

sliding friction with constraint force $\vec{Z}_i$

Next step: eliminate constraints, introduce generalized coordinates

$$\vec{r}_i = \vec{r}_i (q_1, \dots, q_S, t) \quad ,$$

total differential

$$d\vec{r}_i = \sum_{j=1}^{S} \frac{\partial \vec{r}_i}{\partial q_s} \, dq_s + \frac{\partial \vec{r}_i}{\partial t} dt$$

calculate velocities (later reference)

$$\dot{\vec{r}}_i = \sum_{j=1}^{S} \frac{\partial \vec{r}_i}{\partial q_j} \frac{dq_j}{dt} + \frac{\partial \vec{r}_i}{\partial t} = \sum_{j=1}^{S} \frac{\partial \vec{r}_i}{\partial q_j} \dot{q}_s + \frac{\partial \vec{r}_i}{\partial t}$$

$$\frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_l} = \sum_{j=1}^{S} \frac{\partial \vec{r}_i}{\partial q_j} \frac{\partial \dot{q}_j}{\partial \dot{q}_l}$$

$$\Rightarrow \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j} = \frac{\partial \vec{r}_i}{\partial q_j}$$

virtual displacement: $\delta t = 0$

$$\delta \vec{r}_i = \sum_{j=1}^{S} \frac{\partial \vec{r}_i}{\partial q_j} \, \delta q_j$$

consider

$$- \delta W_K = \sum_{i=1}^{N} \vec{K}_i \cdot \delta \vec{r}_i = \sum_{i=1}^{N} \sum_{j=1}^{S} \vec{K}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \, \delta q_j$$

$$= \sum_{j=1}^{S} Q_j \, \delta q_j \qquad \text{with} \quad Q_j = \sum_{i=1}^{N} \vec{K}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j}$$

generalized forces

$[Q_j]$ not necessarily "force"

consider conservative system:

$$\vec{K}_i = -\vec{\nabla}_i V(\vec{r}_1, \dots, \vec{r}_N)$$

generalized forces

chain rule

$$Q_j = -\sum_{i=1}^{N} \vec{\nabla}_i V \cdot \frac{\partial \vec{r}_i}{\partial q_j} = -\frac{\partial V}{\partial q_j} \qquad j = 1 \dots s$$

next: rewrite second term $\dot{\vec{p}}_i \cdot \delta \vec{r}_i$ $\quad \frac{\partial \dot{q}_j}{\partial q_j} = 0$

$$\frac{d}{dt} \frac{\partial \vec{r}_i}{\partial q_j} = \sum_{l=1}^{s} \frac{\partial^2 \vec{r}_i}{\partial q_l \partial q_j} \dot{q}_l + \frac{\partial^2 \vec{r}_i}{\partial t \partial q_j} = \frac{\partial}{\partial q_j} \left(\sum_{l=1}^{s} \frac{\partial \vec{r}_i}{\partial q_l} \dot{q}_l + \frac{\partial \vec{r}_i}{\partial t} \right)$$

$$= \frac{\partial \dot{\vec{r}}_i}{\partial q_j}$$

chain rule

$$\sum_i \dot{\vec{p}}_i \cdot \delta \vec{r}_i = \sum_{i=1}^{N} \sum_{j=1}^{s} m_i \ddot{\vec{r}}_i \cdot \frac{\partial \vec{r}_i}{\partial q_j} \delta q_j \qquad \frac{\partial \vec{r}_i}{\partial q_j} = \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j}$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{s} m_i \left\{ \frac{d}{dt} \left(\dot{\vec{r}}_i \cdot \frac{\partial \dot{\vec{r}}_i}{\partial \dot{q}_j} \right) - \dot{\vec{r}}_i \cdot \frac{d}{dt} \frac{\partial \vec{r}_i}{\partial q_j} \right\} \delta q_j$$

$$= \sum_{i=1}^{N} \sum_{j=1}^{s} m_i \left(\frac{1}{2} \frac{d}{dt} \frac{\partial}{\partial \dot{q}_j} \left(\dot{\vec{r}}_i^{\,2} \right) - \frac{1}{2} \frac{\partial}{\partial q_j} \dot{\vec{r}}_i^{\,2} \right) \delta q_j$$

$$= \sum_{i=1}^{s} \left(\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} \right) \delta q_j$$

kinetic energy
$T = \frac{1}{2} \sum_{i=1}^{N} m_i \dot{\vec{r}}_i^{\,2}$

Insert in D'Alembert principle

$$\sum_{i=1}^{s} \Big(\underbrace{\frac{d}{dt} \frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j}}_{\dot{\vec{p}}_i \cdot \delta \vec{r}_i} \cdots \underbrace{- Q_j}_{-\vec{K}_i \cdot \delta \vec{r}_i} \Big) \delta q_j = 0$$

Special cases

1) systems with holonomic constraints (contained in q_j)

all coordinates independent, δq_j independent
→ set all δq_j, except one to zero
→ each term has to vanish

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j \qquad j = 1 \ldots s$$

$0 = \sum_i (\dot{\vec{p}}_i - \vec{K}_i)\cdot\delta\vec{r}_i$
$\Rightarrow \dot{p}_i^\alpha = K_i^\alpha$
(no constrain')

2) conservative systems

Note $\frac{\partial V}{\partial \dot{q}_j} = 0$, and we rewrite

$$\sum_{j=1}^{s}\left[\frac{d}{dt}\frac{\partial}{\partial \dot{q}_j}(T-V) - \frac{\partial}{\partial q_j}(T-V)\right]\delta q_j = 0$$

define $L(\{q_i\},\{\dot{q}_i\},t) = T(\{q_i\},\{\dot{q}_i\},t) - V(\{q_i\},t)$

Lagrange function

$$\sum_{j=1}^{s}\left[\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j}\right]\delta q_j = 0$$

1) + 2) δq_j independent

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \qquad j = 1 \ldots s$$

(Lagrange equations of motion of 2nd kind)

s differential equations of 2nd order
(to be shown)
→ general solution has $2s$ parameters, fixed for example by initial conditions

Properties of the Lagrange function

1) $L = T - V$ $\qquad T = \frac{1}{2}\sum_i m_i \dot{\vec{r}}_i^{\,2}$, $\dot{\vec{r}}_i = \sum_{j=1}^{S} \frac{\partial \vec{r}_i}{\partial q_j}\dot{q}_j + \frac{\partial \vec{r}_i}{\partial t}$

$$T = \frac{1}{2}\sum_{jl} \mu_{jl}\,\dot{q}_j\,\dot{q}_l + \sum_{j=1}^{S} \alpha_j\,\dot{q}_j + \alpha$$

$$\alpha = \frac{1}{2}\sum_{i=1}^{N} m_i \left(\frac{\partial \vec{r}_i}{\partial t}\right)^2$$

$$\alpha_j = \sum_{i=1}^{N} m_i \left(\frac{\partial \vec{r}_i}{\partial t}\right)\cdot\left(\frac{\partial \vec{r}_i}{\partial q_j}\right)$$

$$\mu_{jl} = \sum_{i=1}^{N} m_i \left(\frac{\partial \vec{r}_i}{\partial q_j}\right)\cdot\left(\frac{\partial \vec{r}_i}{\partial q_l}\right) \qquad \text{generalized masses}$$

Note: skleronomic constraints $\frac{\partial \vec{r}_i}{\partial t} = 0$, $\alpha = 0$; $\alpha_j = 0$

rewrite

$$L = T - V = L_2 + L_1 + L_0 \qquad L_2 = \frac{1}{2}\sum_{jl}^{S} \mu_{jl}\,\dot{q}_j\,\dot{q}_l$$

$$L_1 = \sum_{j=1}^{S} \alpha_j\,\dot{q}_j$$

L_n are homogeneous functions of the generalized velocities of order n

$$L_0 = \alpha - V(q_1, \dots, q_S, t)$$

Definition: $f(x_1, \dots, x_n)$ is homogeneous of order n if it holds $\quad f(a x_1, \dots, a x_n) = a^n\, f(x_1, \dots, x_n)$

2) Lagrange equations are forminvariant under point transformations

$$(q_1, \dots, q_S) \leftrightarrow (\bar{q}_1, \dots, \bar{q}_S)$$

choice of $\{q_j\}$ arbitrary, only number fixed

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0 \qquad \text{holds for } \{q_j\} \text{ and } \{\bar{q}_j\}$$

→ change of coordinate system (cartesian ↔ curvilinear, inertial ↔ non-inertia) : no extra terms (compare: pseudo forces)

Note: Newton's equations not form invariant

$$m\ddot{x} = -\frac{\partial V}{\partial x} \qquad m\ddot{y} = -\frac{\partial V}{\partial y}$$

polar coordinates:

$$m(\ddot{r} - r\dot{\varphi}^2) = -\frac{\partial V}{\partial r}$$

$$m(r\ddot{\varphi} + 2\dot{\varphi}\dot{r}) = -\frac{1}{r}\frac{\partial V}{\partial \varphi}$$

Applications of the Lagrange equations of 2nd kind

Procedure

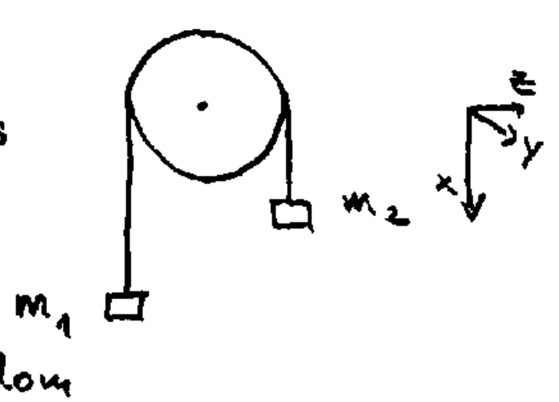
1) Formulate constraints (holonomic)
2) fix $S = 3N - p$ generalized coordinates and transformation
3) write down Lagrange function $L = T(\vec{q}, \dot{\vec{q}}, t) - V(\vec{q}, t)$
4) derive Lagrange equations of motion

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$$

solve S differenti' equations

5) transform to particle coordinates $\vec{r}_i = \vec{r}_i(\vec{q}(t), t)$ and discuss solutions

Atwood's free fall machine



1) holonomic - scleronomic constraints

$y_1 = y_2 = z_1 = z_2 = 0$

$x_1 + x_2 = \ell = \text{const}$

$S = 6 - 5 = 1$ degree of freedom

2) choose : generalized coordinate

$$q = x_1 \qquad \text{constraint gives } x_2 = \ell - q$$

3) kinetic energy $\dot{x}_1 = \dot{q}$

$\dot{x}_2 = -\dot{q}$

$$T = \frac{1}{2}\left(m_1 \dot{x}_1^2 + m_2 \dot{x}_2^2\right) = \frac{1}{2}(m_1 + m_2)\,\dot{q}^2$$

potential energy

$$V = -m_1 g x_1 - m_2 g x_2 = -g\left[m_1 q + m_2(\ell - q)\right]$$
$$= -g\,(m_1 - m_2)\,q - g m_2 \ell$$

4) $$L = T - V = \frac{1}{2}(m_1+m_2)\,\dot{q}^2 + g(m_1 - m_2)\,q + \underbrace{g m_2 \ell}_{\text{can be omitted}}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0$$

$$(m_1 + m_2)\,\ddot{q} - g(m_1 - m_2) = 0$$

$$\Rightarrow \quad \ddot{q} = \frac{m_1 - m_2}{m_1 + m_2}\, g$$

solution $q(t) = q_0 + \tilde{q}_0 t + \frac{1}{2}\frac{m_1 - m_2}{m_1 + m_2} g$

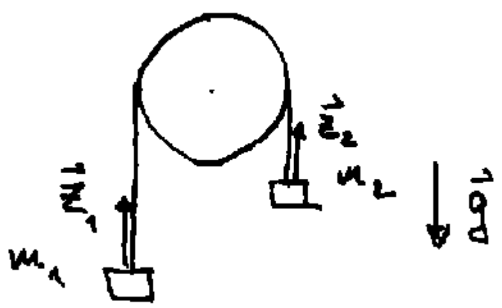
(free fall with reduced m)

5) Transformation back :

$$x_1 = q(t) \quad , \quad x_2 = \ell - q(t)$$

6) constraint forces by use of Newto's equations

$$m_1 \ddot{x}_1 = m_1 g + Z_1$$
$$m_2 \ddot{x}_2 = m_2 g + Z_2$$



subtracting equations

$$m_1 \ddot{x}_1 - m_2 \ddot{x}_2 = (m_1 - m_2)\,g + Z_1 - Z_2$$

use $\ddot{x}_1 = \ddot{q}$, $\ddot{x}_2 = -\ddot{q}$ $\qquad \ddot{q} = \frac{m_1 - m_2}{m_1 + m_2} g$

$$(m_1 + m_2)\ddot{q} = (m_1 - m_2) g + Z_1 - Z_2$$

$$(m_1 + m_2)\frac{m_1 - m_2}{m_1 + m_2} g = (m_1 - m_2) g + Z_1 - Z_2$$

$$\Rightarrow \quad 0 = Z_1 - Z_2 \quad , \quad Z_1 = Z_2$$

adding equations

$$m_1 \ddot{x}_1 + m_2 \ddot{x}_2 = (m_1 + m_2) g + 2Z$$

$$(m_1 - m_2)\ddot{q} = (m_1 + m_2) g + 2Z$$

$$Z = \frac{1}{2}\left((m_1 - m_2)\ddot{q} - (m_1 + m_2) g\right) = \frac{(m_1 - m_2)^2 - (m_1 + m_2)^2}{2(m_1 + m_2)} g$$

$$= -2 \frac{m_1 m_2}{m_1 + m_2} g$$

N particles in conservative force field (no constraints)

1) + 2) trivial $\quad q_1 = x_1 \; , q_2 = y_1 \; , q_3 = z_1$
$\quad q_4 = x_2 \; , q_5 = y_2 \; , q_6 = z_2 \; \ldots$

3) $L(\vec{q}, \dot{\vec{q}}, t) = T - V$
$\qquad = \frac{1}{2}\sum_{i=1}^{N} m_i (\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2) - V$

4) $\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0$

$$\frac{d}{dt}(m_i \dot{x}_i) + \frac{\partial V}{\partial x_i} = 0 \quad \Rightarrow \quad m_i \ddot{x}_i = -\frac{\partial V}{\partial x_i}$$

(Newton's equation)

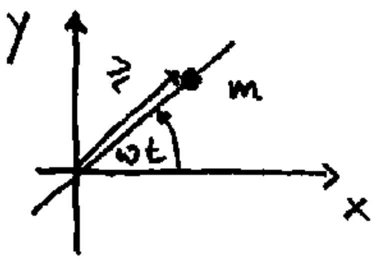
compare: D'Alembert

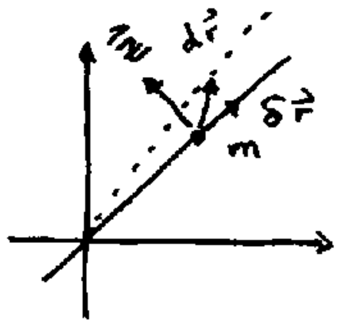
$$\sum_{i=1}^{N} (\vec{K}_i - \dot{\vec{p}}_i)\,\delta\vec{r}_i = 0 \qquad (\text{all } \delta r_i^{\alpha} \text{ independent})$$

$$\Rightarrow \quad K_i - \dot{\vec{p}}_i = 0$$

$$\vec{K}_i = m_i \ddot{\vec{r}}_i$$

Gliding bead on rotating wire





1) 2 constraints

$z = 0$ holonomic scleronomic

$y = x \tan(\omega t)$ holonomic rheonomic

$dW_z = -\vec{Z}\cdot d\vec{r} < 0$

$\delta W_z = -\vec{Z}\cdot\delta\vec{r} = 0$

2) generalized coordinate: $S = 3 - 2 = 1$

$r = q$

transformation:

$x = q\cos(\omega t)$ $\quad z = 0$

$y = q\sin(\omega t)$

3) Lagrange function $L(q,\dot{q},t) = T - V$

$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2) = \frac{m}{2}(\dot{q}^2 + q^2\omega^2)$ $\quad V = 0$

$L = \frac{m}{2}(\dot{q}^2 + q^2\omega^2) = L_2 + L_0$

4) Lagrange equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}} - \frac{\partial L}{\partial q} = 0 \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}} = m\ddot{q}$$

$$\ddot{q} - \omega^2 q = 0 \qquad \frac{\partial L}{\partial q} = m q \omega^2$$

general solution $\quad q(t) = A e^{\omega t} + B e^{-\omega t}$

$\dot{q}(t) = (A e^{\omega t} - B e^{-\omega t})\omega$

initial conditions: $q(0) = r_0 > 0$ $\quad r_0 = A + B$

$\dot{q}(0) = 0$ $\quad 0 = A - B$

$$q(t) = \frac{r_0}{2}\left(e^{\omega t} + e^{-\omega t}\right) = r_0 \cosh(\omega t)$$

$$\dot{q}(t) = \omega r_0 \sinh(\omega t)$$

5) Transformation back, discussion

$$x = r_0 \cosh(\omega t) \cos(\omega t)$$

$$y = r_0 \cosh(\omega t) \sin(\omega t) \qquad z = 0$$

$$\ddot{q} = \omega^2 r_0 \cos(\omega t) > 0 \qquad \text{(increasing acceleration)}$$

$$E = T = \frac{1}{2} m\left(\dot{q}^2 + \omega^2 q^2\right) = \frac{1}{2} m \omega^2 r_0^2 \left[1 + 2\sinh^2(\omega t)\right]$$

(increasing energy)

Generalized momentum and cyclic coordinates

Definition: generalized momentum

$$p_i = \frac{\partial L}{\partial \dot{q}_i} \qquad \dot{p}_i = \frac{\partial L}{\partial q_i} \quad \text{(Lagrange equ. 2nd kind)}$$

Definition: cyclic coordinate

$$\frac{\partial L}{\partial q_i} = 0 \quad \Rightarrow \quad \dot{p}_i = 0, \quad p_i = \text{const}$$

(conservation law)

→ choose q_i such that maximal number cyclic

Example 1) free particle $V = 0$

$$L = T - V = T = \frac{1}{2} m\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) = \frac{1}{2} m \sum_{i=1}^{3} \dot{q}_i^2$$

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = m \dot{q}_i \qquad \dot{p}_i = 0$$

$$p_i = m \dot{q}_i = \text{const.}$$

(momentum conservation)

Example 2: kepler problem

1) no constraints
2) suitable coordinates: spherical coordinates

$(q_1, q_2, q_3) = (r, \vartheta, \varphi)$

$x = r \sin\vartheta \cos\varphi$
$y = r \sin\vartheta \cos\varphi$
$z = r\cos\vartheta$

3) Lagrange function

$$L = T - V$$

$$= \frac{1}{2} m \left(\dot{r}^2 + r^2 \dot{\vartheta}^2 + r^2 \sin^2\vartheta\, \dot{\varphi}^2\right) + \frac{\gamma m M}{r}$$

4) Lagrange equations

$$\frac{\partial L}{\partial \varphi} = 0 \qquad \text{cyclic coordinate}$$

$$p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m r^2 \sin^2\vartheta\, \dot{\varphi} = \text{const}$$

recall z-component of angular momentum

$$L_z = m r^2 \sin^2\vartheta\, \dot{\varphi}$$

Note: direction of z axis arbitrary, choose

$\vec{e}_z \parallel \vec{L}$, $\quad \vec{L} = \vec{r} \times \vec{p} = \text{const}$

(motion in x-y plane, $\vec{r} \cdot \vec{L} = 0$)

In this case $\vartheta = \frac{\pi}{2}$, $\dot{\vartheta} = 0$

Lagrange function simplifies:

$$L = \frac{m}{2}\left(\dot{r}^2 + r^2 \dot{\varphi}^2\right) + \frac{\gamma m M}{r}$$

remaining equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = 0 \qquad m\ddot{r} - m r \dot{\varphi}^2 + \frac{\gamma m M}{r^2} = 0$$

(identical to result from Newton)

Non-holonomic systems (or: no use of holonomic constraints)

+ constraints in differential form

Method: Lagrange equation of 1st kind,
Lagrange multiplier

1) constraints: totally p'
p constraints in differential form

$$\sum_{m=1}^{3N} f_{im}(\vec{r}_1,\dots,\vec{r}_N,t)\,dx_m + f_{it}(\vec{r}_1,\dots,\vec{r}_N,t)\,dt = 0 \qquad i=1\dots p$$

$p'-p$ holonomic constraints

$$f_r(\vec{r}_1,\dots,\vec{r}_N,t) = 0 \qquad r = p+1,\dots,p'$$

2) a) introduce $j = 3N-(p'-p)$ generalized coordinates

$$\vec{r}_i = \vec{r}_i(q_1,\dots,q_j,t) \qquad i=1\dots N$$

$(q_1,\dots,q_j)$ not all indep.

b) rewrite p constraints

$$\sum_{m=1}^{j} a_{im}(q_1,\dots,q_j,t)\,dq_m + b_{it}(q_1,\dots,q_j,t)\,dt = 0$$

$$i=1\dots p$$

3) rewrite constraints for virtual displacements

$$\sum_{m=1}^{j} a_{im}(q_1,\dots,q_j,t)\,\delta q_m = 0 \qquad i=1\dots p$$

4) introduce Lagrange multiplier $\lambda_i(t)$ ← fixed later

$$\lambda_i(t)\sum_{m=1}^{j} a_{im}(q_1,\dots,q_j,t)\,\delta q_m = 0$$

sum equations:

$$\sum_{i=1}^{p} \lambda_i \sum_{m=1}^{j} a_{im}(q_1,\dots,q_j,t)\,\delta q_m = 0$$

5) D'Alembert's principle conservative Systems

$$\sum_{m=1}^{j}\left(\frac{\partial L}{\partial q_m} - \frac{d}{dt}\frac{\partial L}{\partial \dot q_m}\right)\delta q_m = 0 \qquad L = T - V$$

add equ.!

$$\sum_{m=1}^{j}\left(\frac{\partial L}{\partial q_m} - \frac{d}{dt}\frac{\partial L}{\partial \dot q_m} + \sum_{i=1}^{p}\lambda_i a_{im}\right)\delta q_m = 0$$

6) Now we have

$q_1,\dots,q_{j-p}$ independent coordinates : $j-p$

$q_{j-p+1},\dots,q_j$ dependent coordinates : p

fix λ_i such that the p equation.. hold:

$$\frac{\partial L}{\partial q_m} - \frac{d}{dt}\frac{\partial L}{\partial \dot q_m} + \sum_{i=1}^{p}\lambda_i a_{im} = 0 \qquad m = j-p+1,\dots,j$$

δq_m independent for $m = 1 \dots j-p$

$$\sum_{m=1}^{j-p}\left(\frac{\partial L}{\partial q_m} - \frac{d}{dt}\frac{\partial L}{\partial \dot q_m} + \sum_{i=1}^{p}\lambda_i a_{im}\right)\delta q_m = 0$$

For all $m = 1 \dots j$ holds

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q_m} - \frac{\partial L}{\partial q_m} = \sum_{i=1}^{p}\lambda_i a_{im}$$

rewrite differential constraints

$$\sum_{m=1}^{j} a_{im}\dot q_m + b_{it} = 0$$

$j+p$ eqns. for $j+p$ unknowns: j coordinates q_m, p multipliers λ_i

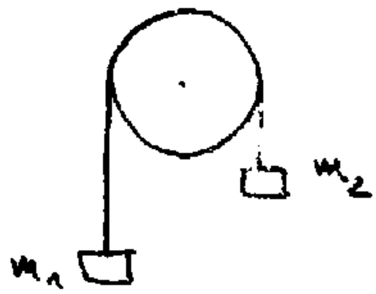
(constraints for generalized velocities)

Interpretation of λ_i : $L = T - V$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} - \underbrace{\left(\frac{d}{dt}\frac{\partial V}{\partial \dot{q}_j} - \frac{\partial V}{\partial q_j}\right)}_{Q_j \text{ generalized forces}} = \sum_{i=1}^{p} \lambda_i a_{ij}$$

$$\Rightarrow \frac{d}{dt}\frac{\partial T}{\partial \dot{q}_j} - \frac{\partial T}{\partial q_j} = Q_j + \underbrace{\sum_{i=1}^{p} \lambda_i a_{ij}}_{\text{generalized constraint forces } \mathcal{Z}_j}$$

Example : Atwood's free fall machine



1) constraints : tolally $5 = p'$

1 constraint in differential form

$dx_1 + dx_2 = 0$

4 holonomic constraints $y_1 = y_2 = z_1 = z_2 = 0$

2) $j = 6 - 4 = 2$ generalized coordinates

$q_1 = x_1$, $q_2 = x_2$

rewrite constraints

$dq_1 + dq_2 = 0$ $a_{11} = 1$, $a_{12} = 1$, $b_{1t} = 0$

4) introduce Lagrange multiplier λ_1

$$\left.\begin{aligned} \mathcal{Z}_1 &= \lambda_1 a_{11} = \lambda_1 \\ \mathcal{Z}_2 &= \lambda_1 a_{12} = \lambda_1 \end{aligned}\right\} \text{thread forces on } m_1 \text{ and } m_2$$

5) Lagrange function

$$L = T - V = \frac{1}{2}\left(m_1 \dot{q}_1^2 + m_2 \dot{q}_2^2\right) + g\left(m_1 q_1 + m_2 q_2\right)$$

6) Lagrange equations + constraints for velocities

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_m} - \frac{\partial L}{\partial q_m} = \lambda_1 \qquad m = 1,2$$

$$m_m \ddot{q}_m - m_m \quad = \lambda_1$$

$$dq_1 + dq_2 = 0 \quad \Rightarrow \quad \dot{q}_1 + \dot{q}_2 = 0$$

Solve 2+1 equations : use $\ddot{q}_1 + \ddot{q}_2 = 0$ in

$$m_1 \ddot{q}_1 - m_1 g = m_2 \ddot{q}_2 - m_2 g$$

$$\ddot{q}_1 = \frac{m_1 - m_2}{m_1 + m_2}\, g \qquad \ddot{q}_2 = -\frac{m_1 - m_2}{m_1 + m_2}\, g$$

$$\lambda_1 = -2g\,\frac{m_1 m_2}{m_1 + m_2} = z_1 = z_2$$ (thread tension)