Theoretical Physics 2 : Electrodynamics

Contents

Literature

W. Nolting, Theoretical Physics 3, Electrodynamics Springer 2016

D.J. Griffiths, Introduction to Electrodynamics, Pearson 1998

J. D. Jackson, Classical Electrodynamics, 3rd edition, Wiley 1998

1) Introduction

historical context

a) Experiment $\rightarrow$ equations for electromagnetic fields

force between charged particles $\rightarrow$ $\vec{\nabla} \cdot \vec{E} = \dfrac{\rho}{\epsilon_0}$

$\oplus$ $\oplus$

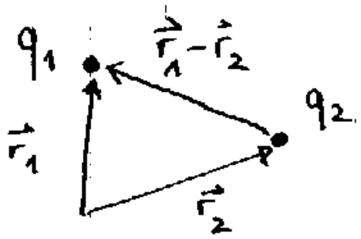
later: discovery of symmetry

b) Einstein and Minkowski: Postulates of Lorentz invariance: theory of special relativity

Postulate (symmetry) $\rightarrow$ equations for electromagnetic fields

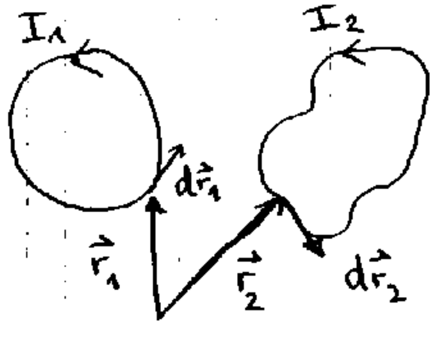
Important experimental discoveries, theories

i) 1785 Coulomb: force between two charges



$$\vec{F}_{12} = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|^2} \frac{\vec{r}_1 - \vec{r}_2}{|\vec{r}_1 - \vec{r}_2|} = \vec{F}_{21}$$

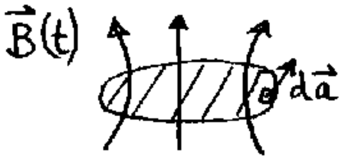
ii) 1825: Ampère: force between two wires with currents I_1 and I_2



$$\vec{F}_{12} = \frac{\mu_0}{4\pi} I_1 I_2 \oint_{C_1}\oint_{C_2} \frac{d\vec{r}_1 \times \left[d\vec{r}_2 \times (\vec{r}_1 - \vec{r}_2) \right]}{|\vec{r}_1 - \vec{r}_2|^3}$$

$$= -\vec{F}_{21}$$

iii) Faraday: relation between time-dependent electric fields and magnetic induction



$$\oint_C \vec{E}\cdot d\vec{r} = -\frac{d}{dt}\int_A \vec{B}\cdot d\vec{a}$$

iv) Maxwell (1864): "On the dynamical theory of the electromagnetic field"

→ formulation of linear, partial differential equations for the electromagnetic fields

v) Hertz (1888): experimental confirmation of Maxwell's equations: discovery of transversal electromagnetic waves

vi) Lorentz, Poincaré, Einstein, Minkowski:
Lorentz invariance of Maxwell's equations ~1905
(consistent with special relativity, postulates of symmetry)

vii) ~ 1950 Feynman, Schwinger, Dyson
QED: Quantum Electro Dynamics
quantum description of electrodynamics and coupling to matter
electromagnetic fields: "photons", matter: electrons ...

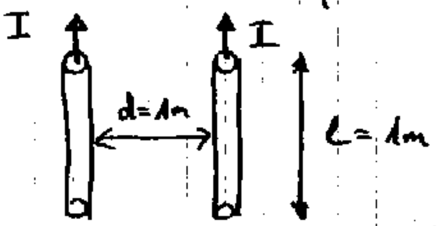
viii) ~ 1960: QCD: Quantum-Chromo Dynamics
unified theory: electromagnetic, weak, strong force

Electromagnetic units

here : International System of units (SI)

important base units

Quantity	Unit	Definition
length	1 m	distance traveled by light in $\frac{1}{299792458}$ second
time	1 s	duration of a period of radiation of Cs-133 atom
mass	1 kg	redefinition (2018) using Planck's constant h
electric current	1 A	current through 2 parallel conductors of 1 A produces force of $2 \cdot 10^{-7}$ Newton per meter length



$$|\vec{F}| = 2 \cdot 10^{-7}\ \text{N}$$

more explicitely : $$\frac{dF}{dl} = 2 k_2 \frac{I^2}{d}$$

$$k_2 = \frac{\mu_0}{4\pi} = 10^{-7} \frac{\text{N}}{\text{A}^2}$$

μ_0 : magnetic constant

Note 1: By fixing the unit of current, also the unit of charge is fixed

Note 2: Coulomb's law $\qquad q_1 \oplus \xleftrightarrow{\ r\ } \ominus\, q_2$

$$F = k_1 \frac{q_1 q_2}{r^2} \qquad k_1 = c^2 k_2 = c^2 10^{-7} \frac{N}{A^2}$$

$$= \frac{1}{4\pi\epsilon_0} \qquad \epsilon_0 = \frac{10^7}{4\pi c^2} \frac{A^2}{N^2}$$

(electric constant)

Note 3: $\mu_0 \epsilon_0 = \frac{1}{c^2}$

Other systems of units than MKSA:

<u>cgs-system</u> (cm, gramm, second)

here: fix $k_1 = 1$, $k_2 = \frac{1}{c^2}$ in Ampère's and Coulomb's law

advantage: electric field $\vec{E}$ and magnetic induction $\vec{B}$ have the same units

Electromagnetic fields, Maxwell's equations

Scope: derivation of Maxwell's equations from (experimentally) verified laws

Here: presentation of final results

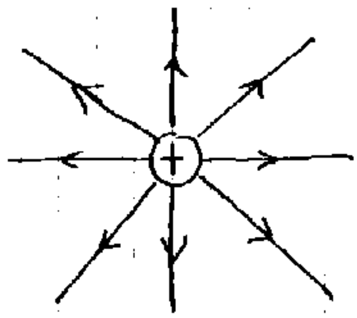
a) electric field $\vec{E}(\vec{r}, t)$

measurement of (static) electric field $\vec{E}(\vec{r})$ by test charge q_{test}

$$\vec{F}(\vec{r}) = q_{test}\, \vec{E}(\vec{r})$$

$$\vec{E}(\vec{r}) = \frac{\vec{F}(\vec{r})}{q_{test}} \qquad \text{units: } [E] = 1\,\frac{N}{C} = 1\,\frac{V}{m}$$

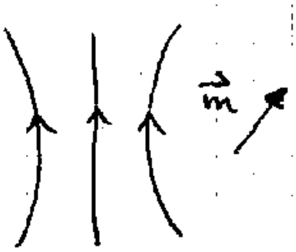
example: electric field of a positive point charge



b) magnetic induction $\vec{B}(\vec{r}, t)$

"there are no magnetic monopoles", thus measurement by influence on magnetic "dipole" $\vec{m}$

torque of $\vec{B}(\vec{r})$ on magnetic dipole



$$\vec{N}(\vec{r}) = \vec{m}_{test} \times \vec{B}(\vec{r})$$

Mathematical methods : Vector analysis

Nabla "operator" $\vec{\nabla} = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$

$$= \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3} \right) = (\partial_1, \partial_2, \partial_3)$$

three distinct "derivatives"

i) Gradient $\vec{\nabla} \varphi = \left(\frac{\partial \varphi}{\partial x}, \frac{\partial \varphi}{\partial y}, \frac{\partial \varphi}{\partial z} \right)$ "vector"

ii) Divergence ("source") $\vec{\nabla} \cdot \vec{a} = \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3} = \sum_{j=1}^{3} \frac{\partial a_j}{\partial x_j}$

iii) Curl $\vec{\nabla} \times \vec{a} = \left(\frac{\partial a_3}{\partial x_2} - \frac{\partial a_2}{\partial x_3}, \frac{\partial a_1}{\partial x_3} - \frac{\partial a_3}{\partial x_1}, \frac{\partial a_2}{\partial x_1} - \frac{\partial a_1}{\partial x_2} \right)$

$$= \sum_{i,j,k=1}^{3} \epsilon_{ijk} \, \partial_i \, a_j \, \vec{e}_k$$

totally antisymmetric ϵ-tensor : values $1, -1, 0$

→ other combinations possible; usually understood by sum notation

<u>Recall:</u> Differential equations

- linear differential equation
- homogeneous / inhomogeneous D. E.
- partial differential equations

Maxwell's equations

M1 $\vec{\nabla} \cdot \vec{B} = 0$ (no magnetic monopoles)

M2 $\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$ (Faraday's law of induction)

M3 $\vec{\nabla} \cdot \vec{D} = \rho$ (Coulomb's law)

M4 $\vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{j}$ (Ampère's law with Maxwell's displacement current

$\vec{E}$: electric field

$\vec{B}$: magnetic induction (flux density)

$\vec{D}$: dielectric displacement

$\vec{H}$: magnetic field (H-field)

ρ: charge density

$\vec{j}$: current density

material equations

$\vec{B} = \mu_0 (\vec{H} + \vec{M})$

μ_0: magnetic constant

$\vec{M}$: magnetization

"linear" media $\vec{M} \sim \vec{H}$

$\vec{D} = \epsilon_0 \vec{E} + \vec{P}$

ϵ_0: electric constant

$\vec{P}$: dielectric polarization

linear media $\vec{P} \sim \vec{E}$

Note 1) In vacuum : $\vec{P} = 0$, $\vec{M} = 0$

$$\vec{D} = \epsilon_0 \vec{E} \quad , \quad \vec{B} = \mu_0 \vec{H}$$

Maxwell's equations are linear differential equations:
superposition principle :
Given $\vec{E}_1$ as electric field of ϱ_1 and
$\vec{E}_2$ as electric field of ϱ_2

Q: What is the field of $\varrho = \varrho_1 + \varrho_2$?

Note 2) in presence of matter : equations are not necessarily linear (not discussed further)

Note 3) Maxwell's equations contain "charge conservation", i.e. continuity equation

$$\partial_t \varrho(\vec{r},t) + \vec{\nabla} \cdot \vec{j}(\vec{r},t) = 0$$

proof by use of Maxwell's equations

M3 $\vec{\nabla} \cdot \vec{D} = \varrho$) take derivative

$$\frac{\partial}{\partial t}\left(\vec{\nabla} \cdot \vec{D}\right) = \frac{\partial}{\partial t} \varrho$$

Schwartz theorem : $\partial_1 \partial_2 f = \partial_2 \partial_1 f$

$$\vec{\nabla} \cdot \left(\frac{\partial \vec{D}}{\partial t}\right) = \frac{\partial}{\partial t} \varrho$$

M4 $\vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{j}$) take derivative

$$\underbrace{\vec{\nabla} \cdot \left(\vec{\nabla} \times \vec{H}\right)}_{= \, ?} - \vec{\nabla} \cdot \left(\frac{\partial \vec{D}}{\partial t}\right) = \vec{\nabla} \cdot \vec{j} \qquad (*)$$

$$\vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{H}\right) = \sum_{ijk=1}^{3} \partial_k \, \epsilon_{ijk} \, \partial_i H_j$$

$$= \frac{1}{2} \sum_{ijk} \Big(\epsilon_{ijk} \, \partial_k \partial_i H_j + \underbrace{\epsilon_{ijk}}_{-\epsilon_{kji}} \underbrace{\partial_k \partial_i}_{\partial_i \partial_k} H_j \Big) = 0$$

we get from (*):

$$\vec{\nabla}\cdot\vec{j} + \underbrace{\vec{\nabla}\cdot\frac{\partial \vec{D}}{\partial t}}_{\frac{\partial \varrho}{\partial t}} = 0 \quad \checkmark$$

Note 4) Description of matter in electromagnetic field needs additionally

Newton's equations for masses m_i associated with (point) charges q_i

$$m \, \ddot{\vec{r}}_i = \vec{F}_i$$

Lorentz force $\quad \vec{F}_i = q_i \big(\vec{E} + \underbrace{\vec{v}_i}_{\dot{\vec{r}}_i \text{ (velocity)}} \times \vec{B} \big)$

Outlook (next steps)

Derivation / interpretation of Maxwell's eqns.

i) Electrostatics $\quad \vec{H} = 0 \;,\; \vec{B} = 0$

$\quad \frac{\partial}{\partial t}(\ldots) = 0$

ii) Magnetostatics $\quad \vec{E} = 0 \;,\; \vec{D} = 0$

iii) general case : wave equations

Mathematical methods : The fundamental theorem of calculus (+generalization)

$$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a) \qquad \text{derivative } f'(x) = \frac{df}{dx}$$

vector analysis : 3 different "types" of derivatives

a) gradient
b) divergence
c) curl

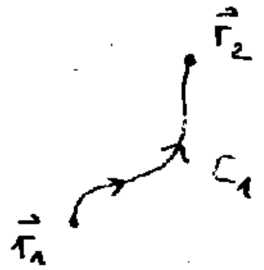
} 3 different gen. fundamental theorems of calculus

a) gradient : given field $V(x,y,z)$

$$\vec{F} = -\vec{\nabla} V \qquad \text{(conservative force)}$$

line integral :

$$\int_{C_1} \vec{F} \cdot d\vec{r} = V(\vec{r}_1) - V(\vec{r}_2)$$



$$-\int_{C_1} (\vec{\nabla} V) \cdot d\vec{r} = V(\vec{r}_1) - V(\vec{r}_2)$$

other consequences:

i) integral is independent of path

$$\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$$

(work is independent of path)

ii) integral over closed loop vanishes

$$\oint \vec{F} \cdot d\vec{r} = 0 \quad ; \quad \oint (\vec{\nabla} V)\, d\vec{r} = 0$$