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TP 3: Classical Mechanics 2 and Electrodynamics 2 — Sheet 0

Winter Term 2019/2020

Due date: will be discussed only in the exercise classes of week 43: Tuesday 22.10.2019 or Thursday 24.10.2019

1. Differential equations

a) Given the first order differential equation for the function $x(t)$ as

$$\dot{x} = x^2 \,.$$

Solve this differential equation by separation of variables for two different initial conditions (1) $x(0) = 1$ and (2) $x(2) = -1$.

b) Solve the inhomogeneous linear differential equation for $x(t)$

$$\dot{x} + 2x = t$$

using the initial condition $x(0) = 0$.
Hint: Determine the general solution of the homogeneous $x_h(t)$ equation and find a special (particular) solution $x_p(t)$ to the inhomogeneous equation by making an ansatz $x_p(t) = c(t)x_h(t)$ and derive a differential equation for $c(t)$.

c) Given the differential equation

$$\dot{x} = (x^2 - \lambda)^2 - \lambda^2 = f(x)$$

for the real-valued function $x(t)$ with a real constant λ.
Find the fixed-points x^* of this differential equation, i.e. the stationary solutions $x(t) = x_0$. Distinguish between $\lambda \leq 0$ and $\lambda > 0$. Draw the function $f(x)$ for $\lambda = -1$ and $\lambda = 1$ and mark the fixed points for these cases. Consider the sign of $f(x^* \pm \epsilon)$ with a small $\epsilon > 0$ to discuss in which direction $x(t)$ flows if starting with the initial conditions $x(0) = x^* \pm \epsilon$.

2. Potential and conservative forces

a) Given the following potential (in spherical coordinates) (r, ϑ, φ)

$$V = E_0(1 - e^{-a(r-r_0)})^2 - E_0$$

Calculate the force $\vec{F}$ given by this potential.
Information: The potential is called Morse potential and can be used to describe the energy of a bond in a diatomic molecule.

b) Given the following forces

- $\vec{F} = (ax + by^2)\vec{e}_x + (az + 2bxy)\vec{e}_y + (ay + bz^2)\vec{e}_z$ in cartesian coordinates (x, y, z) where a and b are constants.
- $\vec{F} = \vec{a} \times \vec{x}$ with the constant vector $\vec{a}$
- $\vec{F} = \vec{a}|\vec{x}|$ with the constant vector $\vec{a}$
- $\vec{F} = \vec{a}(\vec{a} \cdot \vec{x})$ with the constant vector $\vec{a}$

Check whether the force is conservative and if yes, find the corresponding potential.

3. Tautochrone problem

The parametrization of a cycloid in the (x, y) plane is given by

$$x = r(\psi + \sin\psi)$$

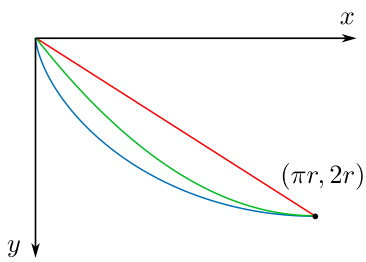
$$y = r(1 - \cos\psi).$$

As it has been discussed in the lecture, the time for a particle to move along a cycloid under the influence of the (constant) gravitational force is stationary. Consider the case where a particle starts (at rest) at the point $(x, y) = 0$ and reaches the point $(x, y) = r(\pi, 2)$ and calculate the time needed along

a) a straight line (movement of particle along inclined plane)

b) a parabola

c) the cycloid.



Hint: In the lecture, the functional

$$J[y(x)] = \frac{1}{\sqrt{2g}} \int_{x_a}^{x_b} dx \sqrt{\frac{1 + y'^2}{y}}$$

was derived to calculate the time a particle takes to move along the curve $y(x)$ from $x = x_a$ to $x = x_b$. Find the functional form $y(x)$ of the straight line and parabola and calculate the integral. For the case of the cycloid, you might use the parametrization of the cycloid as a function of ψ and use the expression of the arc length $ds = \sqrt{(dx)^2 + (dy)^2}$.

Exercises available at: `http://www.physik.uni-leipzig.de/~kreisel/en/teach.php`