Institut für Theoretische Physik
Universität Leipzig

Dr. A. Kreisel
Rohan Kulkarni

TP 3: Classical Mechanics 2 and Electrodynamics 2 — Sheet 1

Winter Term 2019/2020

Due date: Friday, 25.10.2019, 1pm. Solve the exercises marked with a star* and hand in your solution into the mailbox for TP3. Prepare the other exercises *at home* such that you will be able to discussed these in the exercise classes of week 44: Tuesday 29.10.2019 (both groups).

1. Functional Derivative*

The *functional derivative* for a functional $F[f(x)]$ with respect to the function $f(y)$ can be understood as

$$\frac{\delta F[f(x)]}{\delta f\,(y)} = \lim_{\epsilon\to 0} \frac{F\left[f\,(x)+\epsilon\delta\,(x-y)\right]-F\left[f\,(x)\right]}{\epsilon}\,,$$

where $\delta(x)$ is the Dirac delta function.
Mathematically, this works if the functional $F\left[f\,(x)+\epsilon\delta\,(x-y)\right]$ can be expanded as a series in leading power of ϵ.

a) Consider the functional $I[f(x)] = \int_{-1}^{1} f(x)\mathrm{d}x$ and calculate the second functional derivative

(1)
$$\frac{\delta^2 I[f^3(x)]}{\delta f(x_0)\delta f(x_1)}\,.$$

b) Consider the functional $H[f(x)] = \int G(x,y)f(y)\mathrm{d}y$ with another function $G(x,y)$. Calculate the derivative

(2)
$$\frac{\delta H[f(x)]}{\delta f(z)}\,.$$

2. Extremal Work?

Show that for a given force field $\vec{F}(\vec{r})$, the work is extremal only if the force field is conservative.

Hint: The work between two points P_1 and P_2 is given as $W = \int_{P_1}^{P_2} \vec{F}(\vec{r})\cdot d\vec{r}$. Formulate an equation in the language of the calculus of variation and replace the line element $d\vec{r}$ by $v dt$ where v is the velocity such that the integral is now over the time variable. Next, use the Euler-Lagrange equations to obtain the desired result.

3. Geodesic*

a) Show that a particle subject to constraint forces (but not subject to any additional forces) has to move on curved paths which are geodesic.
   A geodesic γ is the shortest path connecting two points which is consistent with the constrains. A definition of a geodesic is given by

$$\delta \int_\gamma ds = 0\,.$$

   Note that s is the length of the space curve γ, see for example Nolting 1, chapter 1.4.3.
   Hint: Formulate the Hamilton principle for this problem. Explain and use $\frac{ds}{dt} = const.$

b) Show further that for the case of a particle on a sphere, these paths are great circles.
   A great circle is a circle on the surface of a sphere which has as center the same point as the center of the sphere.
   Hint: Consider the arc length ds for the particle on the sphere and think about how this can be minimized. Choose a suitable direction of your coordinate system for two given points on the sphere.

Note: In general relativity, the *metric* is defined such that the space curves of particles which move under the influence of graviation force, become geodetic lines. The movement can then be understood as force-free, and the the gravitation just induces a curved space.