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TP 3: Classical Mechanics 2 and Electrodynamics 2 — Sheet 5

Winter Term 2019/2020

Due date: Friday, 22.11.2019, 1pm. Solve the exercises marked with a star* and hand in your solution into the mailbox for TP3. Prepare the other exercises *at home* such that you will be able to present these at the blackboard in the exercise classes of week 48: Tuesday 26.11.2019 or Thursday 28.11.2019.

1. Legendre transformation (2)*

Given the Lagrange function $L = a\dot{x}^2 + b\frac{\dot{y}}{x} + c\dot{x}\dot{y} + fy^2\dot{x}\dot{z}^2 + g\dot{y} - k\sqrt{x^2+y^2}$, where x, y, z are the cartesian coordinates and the other variables are just constants.

a) Obtain the Hamilton function H.

b) Derive the equations of motion.

c) Give the units of the constants a, b, c, f, g, k.

d) Which quantities are conserved? Explain why.

2. Closed trajectories for special central potentials

Consider a particle of mass m in a spherically symmetric potential of the form $V(r) = Cr^n$, where C is a constant and n an exponent. For attractive potentials, the particle can perform circular motions around the center with a angular momentum L_0. (Which sign does the constant C have depending on the value of the exponent n?) Show that the exponent must fulfill $n = M^2 - 2$ with M being a positive integer number in order to allow *closed* trajectories with fixed angular momentum L_0, but slightly different initial conditions than for the circular motion.

Hints: Obtain the Hamilton function $H = H(r, p_r) = \frac{p_r^2}{2m} + V_{\text{eff}}(r)$ for the particle in the central potential, where $V_{\text{eff}}(r)$ is the effective potential as obtained for a motion with fixed angular momentum L_0 which is an integral of motion for a central potential. For the circular motion, the radial momentum vanishes $p_r = 0$. Show that this leads to $L_0^2 = mCnr_0^{n+2}$, where r_0 is the radius of the circle. Now, express the Hamilton function $H(r, p_r)$ in terms of $r = r_0 + \delta r$ and $p_r = \delta p_r$; assuming that the deviation from the circular motion is small, you need to keep terms up to second order in the small quantities δr and δp_r. Show that the terms linear in the small quantities vanish and compare the Hamilton function $H(\delta r, \delta p_r)$ to the Hamilton function of the harmonic oscillator in order to read off the frequency ω_r of the oscillations in δr to find $\omega_r = \omega\sqrt{n+2}$.

3. Canonical transformations

Given a Hamiltonian $H = \frac{1}{2m}p^2q^4 + \frac{1}{kq^2}$.

a) Calculate the equations of motion in the coordinates (q, p) and and explain which type of differential equation you get.

b) Use the generating function $F_4(p, p') = \sqrt{mk}\frac{p}{p'}$ to induce a canonical transformation and obtain the Hamilton function in the new variables. Give the solutions $q(t)$ and $p(t)$ by use of your knowledge on the solutions in the transformed coordinates together with the canonical transformation.

c) Find a generating function $F_1(q, q')$ that generates the same canonical transformation as $F_4(p, p')$. Explain how you got to your solution.

Exercise sheets available at: `http://www.physik.uni-leipzig.de/~kreisel/en/teach.php`