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TP 3: Classical Mechanics 2 and Electrodynamics 2 — Sheet 11

Winter Term 2019/2020

Due date: Friday, 17.01.2020, 1pm. Solve the exercises marked with a star* and hand in your solution into the mailbox for TP3. Prepare the other exercises *at home* such that you will be able to present these at the blackboard in the exercise classes of week 4: Tuesday 21.01.2020 or Thursday 23.01.2020.

1. An electron in a beam of electrons*

a) Calculate the electric field $\vec{E}(\vec{r})$ in the interior of a long cylinder with constant charge density ρ_0. Use the Maxwell equation $\vec{\nabla} \cdot \vec{E} = \frac{\rho(\vec{r})}{\epsilon_0}$ together with the symmetry of the system. Argue in which direction the electric field points and which are the variables that $|\vec{E}|$ depends on. Then use Gauß theorem.

b) The same system can also be viewed from another system of reference as a cylindrical beam of electrons which moves with a constant velocity v and has a constant charge density in the laboratory system of reference Σ. Obtain the electric field $\vec{E}$ and $\vec{B}$ in the laboratory system.

c) Which force acts on an electron inside the electron beam? Obtain the force in the laboratory system of reference and the system of reference where the electron is at rest. Explain the difference.

d) In order to keep the shape of the electron beam (for example to store and accelerate the electrons in a electron storage ring) one can additionally generate a magnetic induction $\vec{B}_0$ in the laboratory frame. Find an expression for that field such that the electron beam does not divert.

2. Wave packets

Consider a wave packet $F(x) = \int_{-\infty}^{\infty} a(k)e^{ikx}dk$ with the weight function $a(k)$.
Note that the wave packet and the weight function are just related by the Fourier transform, i.e. it holds $a(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(x)e^{-ikx}dx$. Furthermore, we define (for integer n) the moments $\langle x^n \rangle$ and $\langle k^n \rangle$ as

$$\langle x^n \rangle = \frac{\int_{-\infty}^{\infty} x^n |F(x)|^2 dx}{\int_{-\infty}^{\infty} |F(x)|^2 dx} \qquad \langle k^n \rangle = \frac{\int_{-\infty}^{\infty} k^n |a(k)|^2 dk}{\int_{-\infty}^{\infty} |a(k)|^2 dk}\,.$$

a) Given the wave packet $F(x) = e^{-ik_0 x} f(x)$ with envelope function $f(x) = e^{-|x|/a}$. Calculate the corresponding weight function $a(k)$. Note that k_0 and a are real positive constants. Make a drawing of the real part $\mathrm{Re}F(x)$ and $a(k)$ for $k_0 a = 5$; use convenient scales for the axes.

b) Calculate the uncertainties $\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2}$ and $\Delta k = \sqrt{\langle k^2 \rangle - \langle k \rangle^2}$ and the product $\Delta x \Delta k$. Verify that the uncertainty principle $\Delta x \Delta k \geq \frac{1}{2}$ is fulfilled.

Hint:

$$\int_0^\infty \frac{du}{(1+u^2)^2} = \int_0^\infty \frac{du u^2}{(1+u^2)^2} = \frac{\pi}{4}$$

which can be shown using the substitution $u = \tan v$.

3. Surface wave

Given an interface between the vacuum at ($z < 0$) and a (dielectric) medium at $z > 0$. The dielectric medium is characterized by a (frequency dependent) relative dielectric constant ϵ_r and magnetic permeability μ_0. Consider in this geometry an electromagnetic surface wave with magnetic induction

$$\vec{B}(t, \vec{r}) = B_0 \vec{e}_y e^{ikx - i\omega t} f(z)$$

with the function

$$f(z) = \begin{cases} \exp(\kappa z) & z < 0 \\ \exp(-\kappa' z) & z > 0 \end{cases}$$

a) Obtain k, κ and κ' as a function of ω.
Hint: Use an ansatz for the electric field of the form

$$\vec{E}(t, \vec{r}) = \begin{cases} \vec{E}_0 e^{ikx - i\omega t + \kappa z} & z < 0 \\ \vec{E}_0' e^{ikx - i\omega t - \kappa' z} & z > 0 \end{cases}$$

and use the Maxwell equations together with the conditions for the fields on the interface to show that this ansatz fulfills the differential equations when fixing the unknown quantities correctly.

b) For which values of ϵ do these surface waves exist? Note that a “surface wave” is characterized by a decay of the amplitude away from the surface.