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TP 3: Classical Mechanics 2 and Electrodynamics 2 — Sheet 12

Winter Term 2019/2020

Due date: Friday, 24.01.2020, 1pm. Solve the exercises marked with a star* and hand in your solution into the mailbox for TP3. Prepare the other exercises *at home* such that you will be able to present these at the blackboard in the exercise classes of week 4: Tuesday 28.01.2020 or Thursday 30.01.2020.

1. Reflection*

Consider the case of a plane electromagnetic wave with frequency ω which perpendicularly hits an interface of two different dielectric media with optical densities n and n'. Assume that the relative magnetic constant is identical in both media $\mu_r/\mu_r' = 1$.

a) Use the Fresnel Formulas from the lecture and write down the special case for the amplitudes of the electric field of the reflected and transmitted waves.

b) Now, consider the incidence from the vacuum ($n = 1$) onto a metal with the following frequency-dependent refraction index

$$n' = \sqrt{\frac{\epsilon}{\epsilon_0}}\sqrt{1 + i\frac{\sigma}{\epsilon\omega}}$$

where $\epsilon = \epsilon_0\epsilon_r$ is the dielectric constant and σ the conductivity constant in Ohm's law (both are positive $\epsilon > 0$, $\sigma > 0$). Use the result from the previous task to calculate the ratio

$$R = \frac{|\vec{E}_0''|^2}{|\vec{E}_0|^2}$$

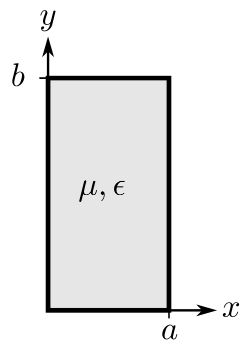
of reflected and incoming intensity of the electric field. (Definition of fields as in the lecture.)

c) Show that one obtains for good conductors $R \approx 1 - 2\omega\delta/c$, where δ is the penetration depth as defined and discussed in the lecture.

2. Waveguides*

In the lecture, we discussed the propagation of electromagnetic waves of given frequency ω inside cylindrical waveguides made of ideal conductors. For waveguides filled with a homogeneous linear medium with dielectric constant ϵ and magnetic permeability μ, we have derived possible solutions of the Maxwell equations starting with the ansatz

$$\vec{E}(t,\vec{r}) = (E'_z(x,y)\vec{e}_z + \vec{E}'_t(x,y))e^{ikz-i\omega t}$$
$$\vec{B}(t,\vec{r}) = (B'_z(x,y)\vec{e}_z + \vec{B}'_t(x,y))e^{ikz-i\omega t}$$



It was derived further that for transverse magnetic (TM) waves in such a wavequide, we have $B'_z = 0$ and the transverse fields $\vec{E}'_t(x,y)$ and $\vec{B}'_t(x,y)$ can be calculated, once the z-components of the electric field $E'_z(x,y)$ is known. This one has to fulfill the Helmholtz equation

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \gamma^2\right) E'_z(x,y) = 0$$

where $\gamma^2 = \mu\epsilon\omega^2 - k^2$ with the boundary condition that the electric field vanishes at the surface of the conductor. Now, consider a rectangular waveguide of dimensions a and b, see drawing.

a) Find the solutions $E'_z(x,y)$ of this boundary value problem. Which are the allowed values for γ?
Hint: Use the ansatz $E'_z(x,y) = E_0 \sin(k_1 x + \phi_1)\sin(k_2 y + \phi_2)$.

b) Which is the minimal frequency ω for a propagating mode to exist?

c) In which frequency range can only one single mode propagate? (Give an inequality.)

d) Calculate the transverse components of the electromagnetic wave $\vec{E}'_t(x,y)$ and $\vec{B}'_t(x,y)$ using the formulas as derived in the lecture and write down the full result for the electric field and magnetic induction.

3. Fourier transform

a) Show that the Fourier transform of $f(\vec{r} - \vec{a})$ for a constant vector $\vec{a}$ is $e^{-i\vec{k}\cdot\vec{a}}\tilde{f}(\vec{k})$ (Shift theorem).

b) Show that the Fourier transform of $f(\alpha\vec{r})$ is $\frac{1}{|\alpha|^d}\tilde{f}(\vec{k}/a)$ where d is the dimension of the vector space and α is a real positive constant.

c) Show that the Fourier transform of $f(D_\phi\vec{r})$ is $\tilde{f}(D_\phi\vec{k})$, where D_ϕ is a rotation matrix.

d) Derive the following property: If and only if $\tilde{f}(k)$ is real for all k, it holds $f(x) = f(-x)$ for all x. (k and x real numbers.)