Theoretical Physics 3 : Mechanics 2
Electrodynamics 2

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Office hours : Fr. 3-4 pm

Exercise classes

Tuesday 13:30-15:00 SR 221
Assem Afanah

Thursday 13:30-15:00 SR 225
Mathias Hänsel

Exercise sheets : given out weekly

Requirement to take part in exam : 50% of points
- some exercises to be handed in
- some exercises to be discussed in the exercise class points granted for willing to present at the blackboard

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Literature:

W. Nolting: 2 Analytical Mechanics
3 Electrodynamics
4 Special Theory of relativity

Honerkamp, Römer: Theoretical Physics, A Classical approach, Springer, 1993

Landau, Lifshitz: Mechanics, Butterworth-Heinemann, 2006

W. Greiner: Systems of particles and Hamilton dynamics, Springer, 2010
Classical Electrodynamics, Springer, 1998

Griffith: Classical electrodynamics, Pearson, 2008

Jackson: Classical electrodynamics, 3rd edition, John Wiley, 1998

Contents

1) Analytical mechanics : short repetition

2) Hamilton principle: Calculus of variations, Lagrange equations, conservation laws (Noether theorem)

3) Hamilton mechanics : Legendre transformation, canonical equations of motion, action principles, Poisson brackets, canonical transformations

4) Hamilton-Jacobi theory

5) Special relativity
Einstein's postulates, Lorentz transformation covariant formulation of classical mechanics

6) Electrodynamics 2
covariant formulation of electrodynamics wave equations, electromagnetic waves, radiation, Lagrange- and Hamilton-function of charged particle and electromagnetic field

1) Analytical mechanics : short repetition

Task: Reformulate Newton's equation of motion $\vec{F} = \dot{\vec{p}}$ such that

a) (geometric) constraints can be taken into account

b) it is invariant under coordinate transformations

- generalized coordinates $\vec{q}(t)$ (describe state of system uniquely)
- choose such that all q_i are independend (possible for holonomic constraints $f(\vec{r}_1, \dots, \vec{r}_N, t) = 0$)
- conservative systems, define Lagrange function

$$L = T - V$$

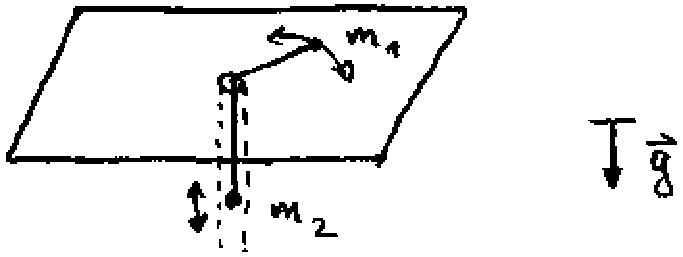
kinetic energy $T = T(\vec{q}, \dot{\vec{q}}, t)$

potential energy $V = V(\vec{q})$

- dynamics described by Lagrange equations

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_j} - \frac{\partial L}{\partial q_j} = 0$$

Example: 2 masses connected by thread of length ℓ
subject to gravitational field
and constraint by plane and tube



constraints:

$$\left.\begin{aligned} r_{1,z} &= 0 \\ r_{2,x} &= 0 \\ r_{2,y} &= 0 \\ \ell - r_{2,z} - \sqrt{r_{1,x}^2 + r_{1,y}^2} &= 0 \end{aligned}\right\} \quad \begin{aligned} S &= 2\cdot 3 - 4 \\ &= 2 \end{aligned}$$

gen. coordinates

generalized coordinates

$$r = \sqrt{r_{1,x}^2 + r_{1,y}^2} \qquad r_{1,x} = r\cos\varphi$$

$$\varphi = \arctan\left(\frac{r_{1,x}}{r_{1,y}}\right) \qquad r_{1,y} = r\sin\varphi$$

$$T = \frac{1}{2}\left(m_1 \dot{\vec{r}}_1^{\,2} + m_2 \dot{\vec{r}}_2^{\,2}\right) = \frac{1}{2} m_1\left(\dot{r}^2 + r^2\dot{\varphi}^2\right) + \frac{1}{2} m_2 \dot{r}^2$$

$$V = m_2 g (r - \ell)$$

$$L = \frac{m_1 + m_2}{2}\dot{r}^2 + \frac{m_1 r^2 \dot{\varphi}^2}{2} - m_2 g (r - \ell)$$

$$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{r}} - \frac{\partial L}{\partial r} = \frac{d}{dt}\left[(m_1 + m_2)\dot{r}\right] - m_1 r \dot{\varphi}^2 + m_2 g$$

$$= (m_1 + m_2)\ddot{r} - m_1 r \dot{\varphi}^2 + m_2 g$$

$$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} - \frac{\partial L}{\partial \varphi} = \frac{d}{dt}\left[m_1 r^2 \dot{\varphi}\right]$$

cyclic coordinate

$$L_1 = m_1 r^2 \dot{\varphi}$$

(conserved quantity)

$$0 = (m_1 + m_2)\ddot{r} - \frac{L_1^2}{m_1 r^3} + m_2 g$$

(2nd order differential equation, nonlinear)

multiplication with $\dot{r}$ and integration yields energy conservation law (integral of motion)

$$E = \frac{1}{2}(m_1+m_2)\dot{r}^2 + \frac{1}{2} m_1 r^2 \dot{\varphi}^2 + m_2 g r \quad (= T+V)$$

Qualitative discussion of dynamics

stationary solution $\dot{r} = 0$, $r = r^*$

$$\frac{L_1^2}{m_1 r^{*3}} = m_2 g$$

$$\Rightarrow \quad r^* = \sqrt[3]{\frac{L_1^2}{m_1 m_2 g}}$$, L_1 fixed by initial conditions

for $r_0 = r(t=0) > r^*$
$\ddot{r} > 0$: radius grows with time

for $r_0 < r^*$
$\ddot{r} < 0$: radius shrinks

Special case: $L_1 = 0$

$$0 = (m_1 + m_2)\ddot{r} + m_2 g$$

$$\ddot{r} = -\frac{m_2 g}{m_1 + m_2}$$ (constant acceleration)

2) The Hamilton principle

so far: consider differential variational principle $\delta \vec{q}$ virtual displacement
(D'Alembert's principle: total work of virtual displacement vanishes)

now: integral variational principle
vary trajectory between fixed points $\vec{q}(t_1)$ and $\vec{q}(t_2)$

configuration space: s-dimensional space given by generalized coordinates

$\vec{q}(t) = (q_1(t), \dots, q_s(t))$ characterizes state of system

curve given by $\vec{q}(t)$: configuration path

restriction to: holonomic, conservative systems

formally insert given configuration path into Lagrange function

$$L = L(\vec{q}(t), \dot{\vec{q}}(t), t) = \tilde{L}(t)$$

define action (functional)

$$S[\vec{q}(t)] = \int_{t_1}^{t_2} \tilde{L}(t)\, dt$$

units: $[S] = 1\mathrm{J} \cdot 1\mathrm{s}$

competitive set of configuration paths

$$M = \{ \vec{q}(t) \mid \vec{q}(t_1) = \vec{q}_1 \; , \; \vec{q}(t_2) = \vec{q}_2 \}$$

a) all $\vec{q}(t)$ start/end at the same times and have the same endpoints $\vec{q}_1, \vec{q}_2$

b) all $\vec{q}(t)$ are connected by virtual displacements (compatible with constraints) from the real (physical) path

Hamilton's principle

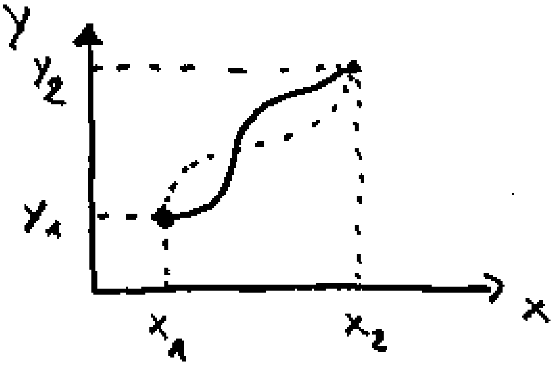
action functional $\quad S[\vec{q}(t)] = \int_{t_1}^{t_2} L(\vec{q}(t), \dot{\vec{q}}(t), t)\,dt$

$\delta S = 0$ $\qquad$ action functional is extremal for the physical path (in other words: variation of S vanishes)

- contains the whole classical mechanics (of conservative, holonomic systems), independent of coordinates
- principle has applications outside mechanical systems
- competitive set has significance in quantum physics (path integral formulation)

Calculus of variation

Given functional $J[y(x)] = \int_{x_1}^{x_2} f(x,y,y')\,dx$



competitive set

$M = \{ y(x) \in C^2\ \mathbb{R} \mid y(x_1) = y_1,\ y(x_2) = y_2 \}$

two times continously differentiable

Question: Find function $y(x)$ which turns $J[y(x)]$ extremal (stationary)

Find extrema of a function $J(y_1, \ldots, y_m)$ for $m \to \infty$

Strategy: "number" all $y(x)$ with a parameter α

$$y_\alpha(x) = y_0(x) + \gamma_\alpha(x)$$

function we want to find

continous function that deforms y_0 to y_α

Assume that the continous function $\gamma_\alpha(x)$ can be Taylor expanded:

$$\gamma_\alpha(x) = \gamma_0(x) + \alpha \left. \frac{\partial \gamma_\alpha(x)}{\partial \alpha} \right|_{\alpha=0} + O(\alpha^2)$$

Furthermore we have the properties

$$\gamma_0(x) = \gamma_{\alpha=0}(x) = 0 \quad \text{for all } x$$

(no deformation at $\alpha=0$)

$$\gamma_\alpha(x_1) = \gamma_\alpha(x_2) = 0 \quad \text{for all } \alpha$$

(no deformation allowed)

$$y_\alpha(x) = y_0(x) + \gamma_\alpha(x) \qquad \downarrow \; \gamma_0(x) = 0$$
$$= y_0(x) + \alpha \left.\frac{\partial \gamma_\alpha(x)}{\partial \alpha}\right|_{\alpha=0}$$

variation of the function given by : $\alpha \to d\alpha$

$$\delta y_0(x) = y_{d\alpha}(x) - y_0(x)$$
$$= d\alpha \left.\frac{\partial \gamma_\alpha(x)}{\partial \alpha}\right|_{\alpha=0}$$

Variation of the derivative of the function

$$\delta y_0'(x) = \delta \frac{d y_0(x)}{dx} = \frac{d}{dx} \delta y_0(x)$$

variation at fixed x, therefore derivative and variation commute

$$= \frac{d}{dx}\left(d\alpha \left.\frac{\partial \gamma_\alpha(x)}{\partial \alpha}\right|_{\alpha=0}\right) = d\alpha \frac{d}{dx} \left.\frac{\partial \gamma_\alpha(x)}{\partial \alpha}\right|_{\alpha=0}$$

Variation of the functional

$$\delta J = J[y_{d\alpha}(x)] - J[y_0(x)] \quad \text{(change of } J \text{ upon variation of } y\text{)}$$
$$= \int_{x_1}^{x_2} \left[f(x, y_{d\alpha}, y'_{d\alpha}) - f(x, y_0, y_0') \right]$$

Evaluate integrand by Taylor expansion of $f(x, y_{d\alpha}, y'_{d\alpha})$; only terms of $O(d\alpha)$ needed :

$$f(x, y_{d\alpha}, y'_{d\alpha}) = f(x, y_0, y_0') + \frac{\partial f}{\partial y}(x, y_0, y_0')\, \overbrace{\delta y_0}^{y_{d\alpha} - y_0}$$
$$+ \frac{\partial f}{\partial y'}(x, y_0, y_0')\, \delta y_0' + O\big((d\alpha)^2\big)$$
$$= f(x, y_0, y_0') + \left(\frac{\partial f}{\partial y} \frac{\partial \gamma_\alpha}{\partial \alpha}\Big|_{\alpha=0} + \frac{\partial f}{\partial y'} \frac{d}{dx} \frac{\partial \gamma_\alpha}{\partial \alpha}\Big|_{\alpha=0} \right) d\alpha$$

Variation of functional

$$\delta J = \int_{x_1}^{x_2} dx \left(\frac{\partial f}{\partial y} \frac{\partial \gamma_\alpha}{\partial \alpha}\Big|_{\alpha=0} + \underbrace{\frac{\partial f}{\partial y'} \frac{d}{dx} \frac{\partial \gamma_\alpha}{\partial \alpha}\Big|_{\alpha=0}}_{\text{integration by parts}} \right) d\alpha$$

$$\int_{x_1}^{x_2} dx\, \frac{\partial f}{\partial y'} \frac{d}{dx} \frac{\partial \gamma_\alpha}{\partial \alpha} = \underbrace{\frac{\partial f}{\partial y'} \frac{\partial \gamma_\alpha}{\partial \alpha}\Big|_{x_1}^{x_2}} - \int_{x_1}^{x_2} dx \left(\frac{d}{dx} \frac{\partial f}{\partial y'} \right) \frac{\partial \gamma_\alpha}{\partial \alpha}$$

no variation at the endpoints $\quad \frac{\partial \gamma_\alpha}{\partial \alpha}\Big|_{x=x_1} = 0 \,, \quad \frac{\partial \gamma_\alpha}{\partial \alpha}\Big|_{x=x_2} = 0$

put everything together

$$\delta J = \int_{x_1}^{x_2} dx \left(\frac{\partial f}{\partial y} - \frac{d}{dx} \frac{\partial f}{\partial y'} \right) \underbrace{\frac{\partial \gamma_\alpha}{\partial \alpha}\Big|_{\alpha=0}}_{\delta y_0} d\alpha$$

$$= \frac{d J(\alpha)}{d\alpha}\Big|_{\alpha=0} d\alpha$$

formally interpret first term as derivative

We want the functional to be stationary

$$\delta J[y_0(x)] = \frac{d J(\alpha)}{d\alpha}\Big|_{\alpha=0} d\alpha = 0$$

$$\left.\frac{d\mathcal{J}(\alpha)}{d\alpha}\right|_{\alpha=0} = \int_{x_1}^{x_2} dx \left(\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'}\right) \left.\frac{\partial \chi_\alpha}{\partial \alpha}\right|_{\alpha=0}$$

vanishes if integrand vanishes:

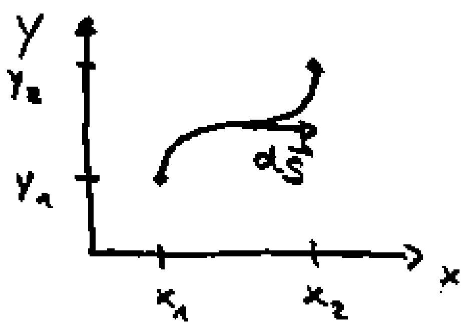
$$\delta \mathcal{J} = 0 \Leftrightarrow \frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

Euler - differential equation

(Leonhard Euler, 1707-1783)

Examples

1) shortest connection between two points on a plane



element of arc:

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx = \sqrt{1 + y'^2}\, dx$$

set up functional

$$\mathcal{J}[y(x)] = \int_{(x_1,y_1)}^{(x_2,y_2)} ds = \int_{x_1}^{x_2} dx \underbrace{\sqrt{1+y'^2}}_{}$$

$$f(x,y,y') = \sqrt{1+y'^2}$$

we want the arc length to be stationary: $\delta \mathcal{J}[y(x)] = 0$

required to fulfill: $\frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$

(Euler DE)

$$0 = \frac{\partial}{\partial y}\sqrt{1+y'^2} - \frac{d}{dx}\frac{\partial}{\partial y}\sqrt{1+y'^2}$$
$$= 0 - \frac{d}{dx}\frac{y'}{\sqrt{1+y'^2}}$$

one integration: $\frac{y'}{\sqrt{1+y'^2}} = C_0 \Rightarrow y' = \overbrace{\frac{C_0}{\sqrt{1-C_0^2}}}^{a}$

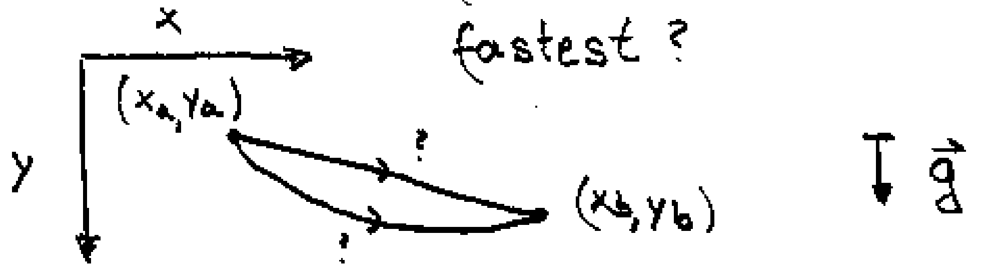
1st order differential equation, has solution:

$y(x) = ax + b$ two parameters a and b fixed by the conditions $y(x_1) = y_1$, $y(x_2) = y_2$

parametrizes straight line

2) Brachistochrone problem

Question: Along which curve does a particle subject to constant gravitational force reach a lower endpoint fastest?



some insights from mechanics
conservative force $\to$ energy conservation

$$E = T + V$$

conveniently set: $0 = E = \frac{1}{2}mv^2 - mgy$

$$v = \sqrt{2gy}$$

Calculation of total time :

$$J[y(x)] = \int_{t_a}^{t_b} dt = \int_{(x_a,y_a)}^{(x_b,y_b)} \frac{ds}{v}$$

calculate time using velocity $\quad ds = v\,dt$

$ds = \sqrt{1+y'^2}\,dx$

$$= \int_{x_a}^{x_b} dx \frac{\sqrt{1+y'^2}}{\sqrt{2gy}} = \frac{1}{2g} \int_{x_a}^{x_b} dx \sqrt{\frac{1+y'^2}{y}}$$

$$f(x,y,y') = \sqrt{\frac{1+y'^2}{y}}$$

calculate partial derivatives

$$\frac{\partial f}{\partial y} = -\frac{\sqrt{1+y'^2}}{2y^{3/2}} \quad , \quad \frac{\partial f}{\partial y'} = \frac{1}{\sqrt{y}} \frac{y'}{\sqrt{1+y'^2}}$$

$$\frac{d}{dx}\frac{\partial f}{\partial y'} = -\frac{y'^2}{2y^{3/2}\sqrt{1+y'^2}} + \frac{y''}{\sqrt{y(1+y'^2)}} - \frac{y'^2 y''}{\sqrt{y}\,(1+y'^2)^{3/2}}$$

Euler equation $\quad \frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$

$$-\frac{\sqrt{1+y'^2}}{2y^{3/2}} = -\frac{y'^2}{2y^{3/2}\sqrt{1+y'^2}} + \frac{y''}{\sqrt{y(1+y'^2)}} - \frac{y'^2 y''}{\sqrt{y}\,(1+y'^2)^{3/2}}$$

$$-(1+y'^2) = -y'^2 + 2yy'' - 2y\frac{y'^2 y''}{1+y'^2}$$

$$0 = 1 + 2yy' - 2y\frac{y'^2y''}{1+y'^2} = 1 + \frac{2yy''}{1+y'^2}$$

$$0 = 1 + y'^2 + 2yy'' \quad \Big\} \text{ multiply by } y'$$

for $y' \neq 0$, this can be rewritten as

$$\frac{d}{dx}\left[y(1+y'^2)\right] = 0$$