integration yields : $y(1+y'^2) = a$

(first order D.E.)

$\Rightarrow \quad y'^2 = \frac{a}{y} - 1$ separation of variables

$y' = \sqrt{\frac{a-y}{y}}$, $dx = \sqrt{\frac{y}{a-y}}\, dy$

$$x_b - x_a = \int_{y_a}^{y_b} dy \sqrt{\frac{y}{a-y}} = 2a \int_{\varphi_a}^{\varphi_b} d\varphi \sin^2\varphi$$

substitution $y = a \sin^2\varphi$

$$\frac{dy}{d\varphi} = 2a \sin\varphi \cos\varphi$$

$$= a \left[\varphi - \sin\varphi \cos\varphi\right]_{\varphi_a}^{\varphi_b}$$

restricting to the case $y_a = 0$, $\varphi_a = 0$
(particle is at rest at the initial position)

$$x_b = a\left(\varphi_b - \tfrac{1}{2} \sin 2\varphi_b\right) - x_a$$

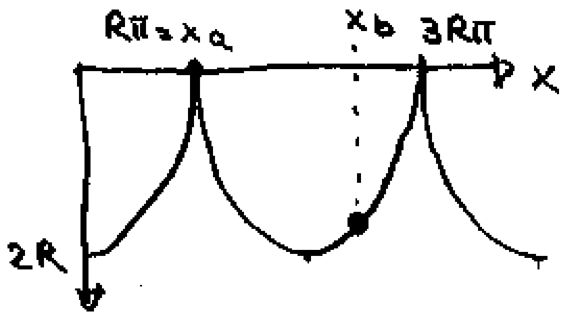
$$y_b = a \sin^2\varphi_b = a\left(1 - \cos^2\varphi_b\right) = a\left(1 - \cos(2\varphi_b)\right)$$

re-define the two parameters

$x_a = R\pi$, $R = \frac{a}{2}$

$\psi = 2\varphi_b + \pi$

$$\left.\begin{aligned} x_b &= R(\psi + \sin\psi) \\ y_b &= R(1 - \cos\psi) \end{aligned}\right\}$$ parametrization of cycloid trajectory

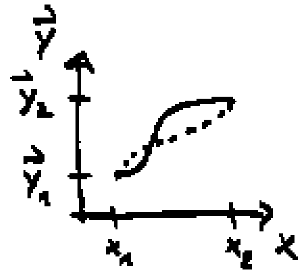


Generalization to multiple dimensions

so far $\quad J[y(x)] = \int_{x_1}^{x_2} f(x, y, y')\, dx$

$$\delta J = 0 \iff \frac{\partial f}{\partial y} - \frac{d}{dx}\frac{\partial f}{\partial y'} = 0$$

Now: $\quad J[\vec{y}(x)] = \int_{x_1}^{x_2} f(x, \vec{y}, \vec{y}')\, dx$



same steps as before, but vector quantities

$$M = \{\vec{y}(x) \in C^2 \mathbb{R}^n \mid \vec{y}(x_1) = \vec{y}_1,\ \vec{y}(x_2) = \vec{y}_2\}$$

competitive set

"number" all components of $\vec{y}$ with a parameter α

$$\vec{y}_\alpha(x) = \vec{y}_0(x) + \vec{\gamma}_\alpha(x) = \vec{y}_0(x) + \alpha \left.\frac{\partial \vec{\gamma}_\alpha}{\partial \alpha}\right|_{\alpha=0}$$

properties of $\vec{\gamma}$: $\quad \vec{\gamma}_0(x) = 0 \quad$ for all x

$\vec{\gamma}_\alpha(x_1) = \vec{\gamma}_\alpha(x_2) = 0 \quad$ no deformation at endpoints

variation of function $\vec{y}(x)$:

$$\delta \vec{y}_0(x) = d\alpha \left.\frac{\partial \vec{\gamma}_\alpha(x)}{\partial \alpha}\right|_{\alpha=0}$$

variation of derivative

$$\delta \vec{y}_0{}'(x) = \delta \frac{d\vec{y}_0}{dx} = \frac{d}{dx}\delta \vec{y}_0 = d\alpha \frac{d}{dx}\left.\frac{\partial \vec{\gamma}_\alpha}{\partial \alpha}\right|_{\alpha=0}$$

variation of functional

$$\delta J = J[\vec{y}_{d\alpha}(x)] - J[\vec{y}_0(x)]$$

$$= \int_{x_1}^{x_2} \left\{ f(x, \vec{y}_{d\alpha}, \vec{y}_{d\alpha}^{\,\prime}) - f(x, \vec{y}_0, \vec{y}_0^{\,\prime}) \right\}$$

Taylor expansion of $f(x, \vec{y}, \vec{y}^{\,\prime})$

$$f(x, \vec{y}_{d\alpha}, \vec{y}_{d\alpha}^{\,\prime}) = f(x, \vec{y}_0, \vec{y}_0^{\,\prime}) + \sum_{i=1}^{s} \frac{\partial f}{\partial y_i} \delta y_0^i + \frac{\partial f}{\partial y_i'} \delta y_0^{\prime i}$$

Variation of functional

$$\delta J = \int_{x_1}^{x_2} dx \sum_{i=1}^{s} \left(\frac{\partial f}{\partial y_i} \frac{\partial \chi_\alpha^i}{\partial \alpha}\Big|_{\alpha=0} + \underbrace{\frac{\partial f}{\partial y_i'} \frac{d}{dx} \frac{\partial \chi_\alpha^i}{\partial \alpha}\Big|_{\alpha=0}}_{\text{integration by parts for each } i=1\ldots s} \right) d\alpha$$

$$\int_{x_1}^{x_2} dx \frac{\partial f}{\partial y_i'} \frac{d}{dx} \frac{\partial \chi_\alpha^i}{\partial \alpha} = \underbrace{\frac{\partial f}{\partial y_i'} \frac{\partial \chi_\alpha^i}{\partial \alpha}\Big|_{x_1}^{x_2}}_{\substack{\text{no variation at endpoints} \\ \text{for each component } i=1\ldots s}} - \int_{x_1}^{x_2} dx \left(\frac{d}{dx} \frac{\partial f}{\partial y_i'} \right) \frac{\partial \chi_\alpha^i}{\partial \alpha}$$

put everything together

$$\delta J = \int_{x_1}^{x_2} dx \sum_{i=1}^{s} \left(\frac{\partial f}{\partial y_i} - \frac{d}{dx} \frac{\partial f}{\partial y_i'} \right) \delta y_0^i = 0$$

want functional to be stationary

variation arbitrary, therefore each term has to vanish

$$\frac{\partial f}{\partial y_i} - \frac{d}{dx} \frac{\partial f}{\partial y_i'} = 0 \qquad i = 1 \ldots s$$

Back to Hamilton's principle: $\delta S = 0$

$$S[\vec{q}(t)] = \int_{t_1}^{t_2} L(\vec{q}(t), \dot{\vec{q}}(t), t)\, dt$$

identify:

$$\left.\begin{array}{l} x \to t \\ \vec{y} \to \vec{q} \\ \vec{y}\,' \to \dot{\vec{q}} \\ f \to L \end{array}\right\} \quad \frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0 \qquad j = 1 \dots s$$

Euler-Lagrange D.E.

(conservative system, holonomic constraints)

Non-holonomic systems

Hamilton's principle

$$0 = \delta S = \int_{t_1}^{t_2} dt \sum_{m=1}^{s} \left(\frac{\partial L}{\partial q_m} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_m} \right) \delta q_m$$

variation of gen. coordinates not independend

need: constraints in differential form

$$\sum_{m=1}^{s} a_{im}\, dq_m + b_{it}\, dt = 0 \qquad i = 1 \dots p$$

p constraints in differential form

example: rolling cylinder

two generalized coordinates fix position of cylinder (rigid body)

$\vec{q} = (X, \varphi)$

X: center of mass coordinate in x direction (other components fixed $Y = Z = 0$)

φ: angle for rotations around symmetry axis (no other rotations allowed $\vartheta = \psi = 0$)

no sliding : $dX = R\,d\varphi$

$\Leftrightarrow \quad dX - R\,d\varphi = 0$

read off $\quad a_x = 1 \;,\; a_\varphi = -R \;,\; b_t = 0$

now: consider variation at fixed t (virtual displacement)

$$\sum_{m=1}^{s} a_{im}\,\delta q_m = 0 \qquad \text{(constraints)}$$

multiply with constant λ_i,
sum over all $i = 1,\dots,p$ and integrate over t:

$$\int_{t_1}^{t_2} \sum_{i=1}^{p} \lambda_i \sum_{m=1}^{s} a_{im}\,\delta q_m = 0$$

add this to Hamilto's principle

$$0 = \int_{t_1}^{t_2} dt \sum_{m=1}^{s} \left(\frac{\partial L}{\partial q_m} - \frac{d}{dt}\frac{\partial L}{\partial \dot q_m} + \sum_{i=1}^{p} \lambda_i a_{im} \right) \delta q_m$$

now: choose the $i = 1 \dots p$ λ_i (lagrange multiplier) such that

$$\frac{\partial L}{\partial q_m} - \frac{d}{dt}\frac{\partial L}{\partial \dot q_m} + \sum_{i=1}^{p} \lambda_i a_{im} = 0 \qquad m = 1 \dots s$$

for all dependend coordinates, for all other coordinates, the same equation also holds

Conservation laws

description of mechanical system by generalized coordinates $\vec{q}(t)$ and generalized velocities $\dot{\vec{q}}(t)$

Def: Integral of motion

Constant function : $F_r\left(\vec{q}(t), \dot{\vec{q}}(t)\right) = C_r \qquad r = 1 \dots R$

all C_r are fixed by initial conditions $C_r = F_r\left(\vec{q}(0), \dot{\vec{q}}(0)\right)$

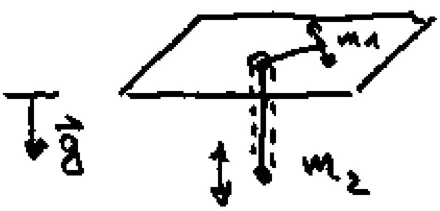
example: cyclic coordinates yield integrals of motion

$$q_i \text{ cyclic} \quad \Leftrightarrow \quad \frac{\partial L}{\partial q_i} = 0 \qquad \text{(Definition)}$$

$$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} = \frac{d}{dt} p_i$$

$$\frac{\partial L}{\partial \dot{q}_i} = p_i \qquad \text{generalized momentum}$$

in this case : generalized momentum is constant



two masses, connected by thread and constraint to plane/tube
$\vec{q} = (r, \varphi)$

$$L = \frac{m_1 + m_2}{2}\dot{r}^2 + \frac{m_1 r^2 \dot{\varphi}^2}{2} - m_2\, g\,(r - l)$$

$$\frac{\partial L}{\partial \varphi} = 0 \qquad p_\varphi = \frac{\partial L}{\partial \dot{\varphi}} = m_1 r^2 \dot{\varphi}^2 \qquad \text{(integral of motion)}$$

φ is cyclic (angular momentum)

(optimal : choose all coordinates cyclic, theory to find these : Hamilton-Jacobi Theory, Ch. 4)

Noether theorem:
from basic symmetries of a system follow integrals of motion

here: consider transformation of coordinates that can be parametrized by a parameter α

$\vec{q} \to \vec{q}\,'(\vec{q}, t, \alpha)$ for $\alpha = 0$ $\vec{q} = \vec{q}\,'$ (no transformation)

examples
1) translation $\vec{r} \to \vec{r}\,' = \vec{r} + \alpha \vec{a}$

2) rotation around z-axis $\vec{r} \to \vec{r}\,' = D_\alpha \vec{r}$ $\quad D_\alpha = \begin{pmatrix} \cos\alpha & \sin\alpha & 0 \\ -\sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix}$

transformation of action

$$S[\vec{q}] = \int_{t_1}^{t_2} dt\, L(\vec{q}, \dot{\vec{q}}, t)$$

$$S'[\vec{q}\,'] = \int_{t_1}^{t_2} dt\, L'(\vec{q}\,', \dot{\vec{q}}\,', t)$$

coordinate transformation leaves action invariant (symmetry) $S[\vec{q}] = S'[\vec{q}\,']$

choose $t_1 \to t_2$, integrands need to be identical

$$L(\vec{q}, \dot{\vec{q}}, t) = L'(\vec{q}\,', \dot{\vec{q}}\,', t) = L'(\vec{q}\,', \dot{\vec{q}}\,', t, \alpha)$$

cannot depend on α $\qquad$ can depend on α explicitely use inverse transformation $\vec{q} = \vec{q}\,(\vec{q}\,', t, \alpha)$

$$L'(\vec{q}\,', \dot{\vec{q}}\,', t, \alpha) = L\left(\vec{q}\,(\vec{q}\,', t, \alpha),\ \dot{\vec{q}}\,(\vec{q}\,', t, \alpha),\ t\right)$$

and calculate partial derivative at constant $\vec{q}\,'$ and $\dot{\vec{q}}\,'$

$$\frac{\partial L'}{\partial \alpha} = \sum_{i=1}^{s}\left\{\frac{\partial L}{\partial q_i}\,\frac{\partial q_i(\vec{q}, t, \alpha)}{\partial \alpha} + \frac{\partial L}{\partial \dot{q}_i}\,\frac{\partial}{\partial \alpha}\left[\frac{d}{dt}\,q_i(\vec{q}, t, \alpha)\right]\right\}$$

$$\underset{\text{equations of motion}}{=} \sum_{i=1}^{s}\left\{\left(\frac{d}{dt}\,\frac{\partial L}{\partial \dot{q}_i}\right)\frac{\partial q_i}{\partial \alpha} + \frac{\partial L}{\partial \dot{q}_i}\,\frac{d}{dt}\,\frac{\partial q_i}{\partial \alpha}\right\}$$

$$\underset{\text{product rule}}{=} \frac{d}{dt}\left[\sum_{i=1}^{s}\,\frac{\partial L}{\partial \dot{q}_i}\,\frac{\partial q_i}{\partial \alpha}\right]$$

$$= \frac{d}{dt}\left[\sum_{i=1}^{s}\,p_i\,\frac{\partial q_i}{\partial \alpha}\right]$$

also holds after setting $\alpha = 0$

$$\left.\frac{\partial L'}{\partial \alpha}\right|_{\alpha=0} = \frac{d}{dt}\sum_{i=1}^{s} p_i\,\left.\underbrace{\frac{\partial q_i(\vec{q}\,', t, \alpha)}{\partial \alpha}}_{\substack{\text{can easily be expressed}\\ \text{by } \vec{q} = \vec{q}\,'|_{\alpha=0}}}\right|_{\alpha=0}$$

equations of motion are forminvariant, thus

$$L'(\vec{q}\,', \dot{\vec{q}}\,', t, \alpha) = L(\vec{q}\,', \dot{\vec{q}}\,', t) + \frac{d}{dt}\,F(\vec{q}\,', t, \alpha)$$

with function $F(\vec{q}\,', t, \alpha)$, see exercises

$$\Leftrightarrow \quad \underset{\substack{\uparrow\\ \text{cannot explicitely depend on } \alpha}}{L(\vec{q}\,', \dot{\vec{q}}\,', t)} = L'(\vec{q}\,', \dot{\vec{q}}\,', t, \alpha) - \frac{d}{dt}\,F(\vec{q}\,', t, \alpha)$$