$$0 = \frac{\partial L}{\partial \alpha}\Big|_{\alpha=0} = \frac{\partial L'}{\partial \alpha}\Big|_{\alpha=0} - \frac{d}{dt}\frac{\partial F}{\partial \alpha}\Big|_{\alpha=0}$$

$$= \frac{d}{dt}\left[\sum_{i=1}^{s} p_i \frac{\partial q_i}{\partial \alpha}\Big|_{\alpha=0} - \frac{\partial F}{\partial \alpha}\Big|_{\alpha=0}\right]$$

thus $\quad I(\vec{q},\dot{\vec{q}},t) = \sum_{i=1}^{s} p_i \frac{\partial q_i}{\partial \alpha}\Big|_{\alpha=0} - \frac{\partial F}{\partial \alpha}\Big|_{\alpha=0}$

is integral of motion

Examples

1) trivial example: cyclic coordinate q_j

transformation $\quad q_j' = q_j - \alpha \quad \leftarrow$ all other unchanged

$\Leftrightarrow \quad q_j = q_j' + \alpha \qquad \frac{\partial q_i}{\partial \alpha}\Big|_{\alpha=0} = 1 \cdot \delta_{ij}$

$$I(\vec{q},\dot{\vec{q}},t) = p_j \cdot 1 = p_j$$

2) particle in potential with cylinder symmetry, but expressed in cartesian coordinates $\quad \vec{q} = (x,y,z)$

$$L = \frac{m}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - V\left(x^2+y^2, z\right)$$

rotation around z-axis

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix}$$

$$L'(\vec{r}', \dot{\vec{r}}', \alpha) = L(\vec{r}', \dot{\vec{r}}') \quad , \text{ i.e. } F = 0$$

$$I = p_x \frac{\partial x}{\partial \alpha}\Big|_{\alpha=0} + p_y \frac{\partial y}{\partial \alpha}\Big|_{\alpha=0}$$

$$= m\dot{x}\,(-x'\sin\alpha - y'\cos\alpha)\Big|_{\alpha=0} + m\dot{y}\,(x'\cos\alpha - y'\sin\alpha)\Big|_{\alpha=0}$$

$$= m(\dot{x}y - y\dot{x}) = L_z$$

Homogeneity of time

transformation $\quad \vec{q} \to \vec{q}\,' = \vec{q}$

$\qquad\qquad\qquad\quad t \to t' = t + t_0$

Lagrange function invariant

$$L(\vec{q}, \dot{\vec{q}}, t+t_0) = L(\vec{q}, \dot{\vec{q}}, t) \qquad \text{for arbitrary } t_0$$

so far: consider transformations of coordinates only (no time transformation), previous result for integral of motion not applicable

directly calculate:

$$\frac{d}{dt} L = \sum_{i=1}^{s} \left(\frac{\partial L}{\partial q_i} \frac{dq_i}{dt} + \frac{\partial L}{\partial \dot{q}_i} \frac{d\dot{q}_i}{dt} \right) = \sum_{i=1}^{s} \left(\frac{\partial L}{\partial q_i} \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i \right)$$

chain rule, $\frac{\partial L}{\partial t} = 0$

equ. of motion

$$= \sum_{i=1}^{s} \left[\left(\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} \right) \dot{q}_i + \frac{\partial L}{\partial \dot{q}_i} \ddot{q}_i \right]$$

product rule

$$= \frac{d}{dt} \left(\sum_{i=1}^{s} \frac{\partial L}{\partial \dot{q}_i} \dot{q}_i \right) = \frac{d}{dt} \left(\sum_{i=1}^{s} p_i \dot{q}_i \right)$$

rewrite as

$$\frac{d}{dt} \left(\sum_{i=1}^{s} p_i \dot{q}_i - L \right) = 0$$

integral of motion $\quad H = \sum_{i=1}^{s} p_i \dot{q}_i - L$

use $p_i = \frac{\partial L}{\partial \dot{q}_i}$ to obtain $\dot{q}_i = \dot{q}_i(\vec{q}, \vec{p})$

to obtain $\quad H(\vec{q}, \vec{p}) \quad$ (Hamilton function)

Interpretation of H for conservative systems with holonomic-scleronomic constraints (transformations to generalized coordinates do not depend on time $\vec{q} = \vec{q}(\{\vec{r}_i\}, \cancel{t})$

we obtain $\frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} = p_i$ (conservative)

$$\vec{r}_i = \vec{r}_i(\vec{q}) \qquad i = 1 \dots N$$

write down kinetic energy:

$$T = \sum_{i=1}^{N} \frac{m_i}{2} \dot{\vec{r}}_i^{\,2} = \sum_{i=1}^{N} \sum_{\alpha,\beta=1}^{S} \frac{m_i}{2} \left(\frac{\partial \vec{r}_i}{\partial q_\alpha} \dot{q}_\alpha \right) \cdot \left(\frac{\partial \vec{r}_i}{\partial q_\beta} \dot{q}_\beta \right)$$

$$= \frac{1}{2} \sum_{\alpha,\beta=1}^{S} \mu_{\alpha\beta} \dot{q}_\alpha \dot{q}_\beta$$

$$\mu_{\alpha\beta} = \sum_{i=1}^{N} m_i \frac{\partial \vec{r}_i}{\partial q_\alpha} \cdot \frac{\partial \vec{r}_i}{\partial q_\beta} \qquad \text{(generalized masses)}$$

$$p_i = \frac{\partial T}{\partial \dot{q}_i} = \sum_{\alpha=1}^{S} \mu_{i\alpha} \dot{q}_\alpha$$

examine the term in the Hamilton function

$$\sum_{i=1}^{S} p_i \dot{q}_i = \sum_{i,\alpha=1}^{S} \mu_{i\alpha} \dot{q}_\alpha \dot{q}_i = 2T$$

$$H = \sum_{i=1}^{S} p_i \dot{q}_i - L = 2T - (T - V) = T + V$$

$= E$ (total energy of the system)

Note: for holonomic-rheonomic constraints and $L(\vec{q}, \dot{\vec{q}}, t) = L(\vec{q}, \dot{\vec{q}})$ we still have $\frac{\partial L}{\partial t} = 0$ and $H = \text{const}$, but in general $H \neq E$

Summary : Noether theorem

symmetry : continous coordinate transformation

$\vec{q} \to \vec{q}\,' = \vec{q}\,'(\vec{q}, t, \alpha)$ leaves

$L'(\vec{q}\,', \dot{\vec{q}}\,', t, \alpha) = L(\vec{q}\,', \dot{\vec{q}}\,', t) + \frac{dF(\vec{q}\,', t, \alpha)}{dt}$

invariant

integral of motion

$$I(\vec{q}, \dot{\vec{q}}, t) = \sum_{i=1}^{S} \frac{\partial L}{\partial \dot{q}_i} \frac{\partial q_i}{\partial \alpha}\Big|_{\alpha=0} - \frac{\partial F}{\partial \alpha}\Big|_{\alpha=0}$$

special case : cyclic coordinate

$\frac{\partial L}{\partial q_i} = 0 \qquad p_i = \text{const}$

symmetry	invariance	integral of motion
homogeneity of space	translation $\vec{r} \to \vec{r} + \vec{a}$	momentum
isotropy of space	rotation $\vec{r} \to D\vec{r}$	angular momentum
homogeneity of time	time translation $t \to t + t_0$	Hamilton function

3) Hamilton mechanics

scope:

Lagrange mechanics — Lagrange eqns. of motion

$L(\vec{q}, \dot{\vec{q}}, t)$ — $\frac{\partial L}{\partial q_j} - \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_j} = 0 \quad j = 1 \dots s$

s D.E. of 2nd order

Hamilton mechanics — Hamilton eqns. of motion

$H(\vec{q}, \vec{p}, t)$ — 2s differential equations of 1st order

→ general solution : 2 s parameters (fixed by initial conditions)

Legendre transformation

given : function $f(x) \in C^2\, \mathbb{R}$

want : change of variable x to variable m:

$$m = \frac{df}{dx} \qquad \text{slope of } f \text{ in point } x$$

(assuming $\frac{d^2 f}{dx^2} > 0$) we can invert

$$m(x) = \frac{df(x)}{dx} \quad \text{to} \quad x(m)$$

Definition : Legendre transform $g(m)$ of $f(x)$:

$$g(m) = x(m)\, m - f(x(m))$$

$$= x(m) \left.\frac{df}{dx}\right|_{x = x(m)} - f(x(m))$$

mathematical properties

$$\frac{dg}{dm} = x(m) + \frac{dx}{dm} m - \frac{df}{dx}\frac{dx}{dm} = x(m)$$

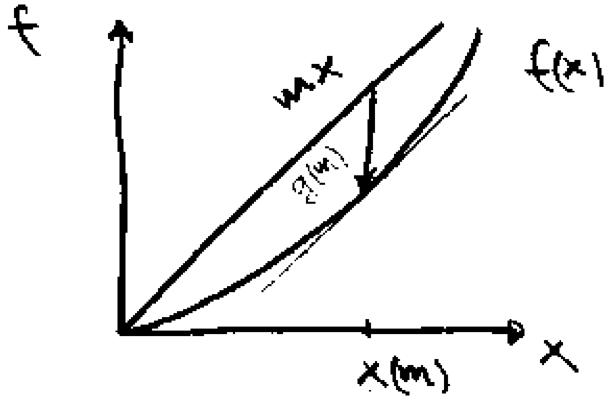
$$\frac{d^2g}{dm^2} = \frac{dx}{dm} = \frac{1}{\frac{dm}{dx}} = \frac{1}{\frac{d^2f}{dx^2}} > 0 \qquad g(m) \text{ is a convex function}$$

differentials

$$\left.\begin{aligned} df &= \frac{df}{dx}\,dx = m(x)\,dx \\ dg &= dx\,m + x\,dm - \frac{df}{dx}\,dx = x(m)\,dm \end{aligned}\right\} \text{exchange of } x \leftrightarrow m$$

geometric interpretation

we have $\frac{d}{dx}\left(m x - f(x)\right) = m - \frac{df}{dx} = 0$



$x(m)$: x-value that minimizes or maximizes the distance between $f(x)$ and the straight line $m \cdot x$

$g(m)$ is the extremal value

Legendre transformation of the Lagrange function with respect to $\dot{q}_i$ $(i = 1 \dots s)$

$$H(\vec{q}, \vec{p}, t) = \sum_{i=1}^{s} \dot{q}_i(\vec{q}, \vec{p}, t)\, p_i - L\left(\vec{q}, \dot{\vec{q}}(\vec{q}, \vec{p}, t), t\right)$$

(generalized coordinates are unchanged)

Canonical equations of motion:

calculate total differential

$$dH = \sum_{i=1}^{s}\left(\dot q_i\, dp_i + p_i\, d\dot q_i\right) - \sum_{i=1}^{s}\left(\frac{\partial L}{\partial q_i}dq_i + \overbrace{\frac{\partial L}{\partial \dot q_i}}^{p_i} d\dot q_i\right) - \frac{\partial L}{\partial t}dt$$

$$= \sum_{i=1}^{s}\left(\dot q_i\, dp_i - \underbrace{\frac{\partial L}{\partial q_i}} dq_i\right) - \frac{\partial L}{\partial t}dt$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot q_i} = \frac{d}{dt} p_i = \dot p_i$$

$$= \sum_{i=1}^{s}\left(\dot q_i\, dp_i - \dot p_i\, dq_i\right) - \frac{\partial L}{\partial t}dt$$

read off partial derivatives

$$\frac{\partial H}{\partial q_i} = -\dot p_i \qquad \frac{\partial H}{\partial p_i} = \dot q_i \qquad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

(canonical equations)

Trajectory in phase space $(\vec q, \vec p)$ fixes dynamics of system

2s equations for 2s variables

calculate time derivative:

$$\frac{d}{dt} H(\vec q, \vec p, t) = \sum_{i=1}^{s}\left(\frac{\partial H}{\partial q_i}\dot q_i + \frac{\partial H}{\partial p_i}\dot p_i\right) + \frac{\partial H}{\partial t}$$

canonical equations

$$= \sum_{i=1}^{s}\left(-\dot p_i \dot q_i + \dot q_i \dot p_i\right) + \frac{\partial H}{\partial t}$$

$$= \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

from $\frac{\partial H}{\partial t} = 0$ follows $\frac{dH}{dt} = 0$ (integral of motion) cyclic coordinates

$$\frac{\partial L}{\partial q_j} = 0 \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot q_j} = 0 \quad , \quad p_j = \frac{\partial L}{\partial \dot q_j} = \text{const}$$

Hamilton $0 = \frac{\partial H}{\partial q_j} = -\dot p_j \qquad \dot p_j = 0$

(one canonical equation is trivial)

General scheme to solve using Hamilton formalism

1) Fix generalized coordinates
(taking constraints into account)
2) Find transformation to generalized coordinates
$\vec r_i = \vec r_i(\vec q, t)$
3) Find Lagrange function $L = T - V$
$L = L(\vec q, \dot{\vec q}, t)$
4) calculate generalized momenta
$p_j = \frac{\partial L}{\partial \dot q_j} \qquad j = 1 \dots s$
5) invert equations $\dot q_j = \dot q_j(\vec q, \vec p, t)$
6) rewrite Lagrange function
$L = L(\vec q, \dot{\vec q}(\vec q, \vec p, t), t)$
7) obtain Hamilton function (Legendre transformation)
$$H(\vec q, \vec p, t) = \sum_{i=1}^{s} \dot q_i p_i - L$$
8) set up canonical equations (and solve D.E.)
$$\dot q_j = \frac{\partial H}{\partial p_j} \qquad \dot p_j = -\frac{\partial H}{\partial q_j} \qquad \frac{\partial H}{\partial t} = -\frac{\partial L}{\partial t}$$

Example : harmonic oscillator (in one dimension)

1) + 2) $x = q$ (generalized coordinate)

3) Lagrange function

$$L = T - V = \frac{m}{2}\dot{q}^2 - \frac{1}{2}kq^2$$

4) generalized momentum

$$p = \frac{\partial L}{\partial \dot{q}} = m\dot{q}$$

5) invert equation $\dot{q}(q,p,t) = \frac{p}{m}$

6) $L = \frac{m}{2}\left(\frac{p}{m}\right)^2 - \frac{1}{2}kq^2 = \frac{p^2}{2m} - \frac{1}{2}kq^2$

7) Hamilton function

$$H(q,p,t) = \frac{p}{m}p - \left(\frac{p^2}{2m} - \frac{1}{2}kq^2\right)$$

$$= \frac{p^2}{2m} + \frac{1}{2}kq^2$$

8) set up canonical equations

$$\dot{q} = \frac{\partial H}{\partial p} = \frac{p}{m}$$

$$\dot{p} = -\frac{\partial H}{\partial q} = -kq$$

$\frac{\partial H}{\partial t} = 0$, $H = \text{const}$ "scleronomic constraints" $H = E$ (total energy)

solution of canonical equations

take derivative $\ddot{q} = \frac{\dot{p}}{m} \Leftrightarrow \dot{p} = \ddot{q}m$

insert into other D.E. $\ddot{q}m = -kq$

$$\Leftrightarrow \ddot{q} + \frac{k}{m}q = 0$$

define $\omega_0^2 = \frac{k}{m}$ $\quad \ddot{q} + \omega_0^2 q = 0$

2nd order. DE

solutions $q(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$

A, B: fixed by initial conditions

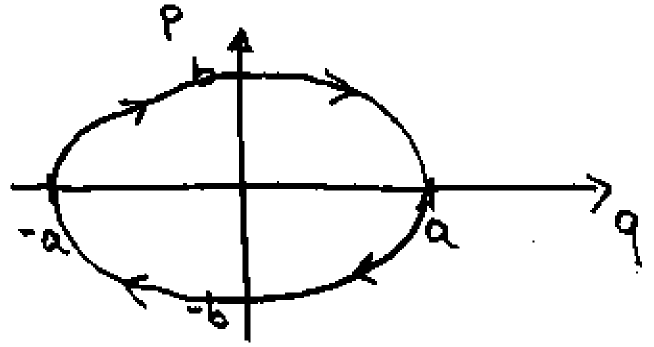
$$p(t) = m\dot{q} = -Am\omega_0 \sin(\omega_0 t) + Bm\omega_0 \cos(\omega_0 t)$$

trajectories in phase space $\vec{\pi}(t) = (q(t), p(t))$

$$E = H = \frac{p^2}{2m} + \frac{1}{2}kq^2 \iff 1 = \frac{p^2}{2mE} + \frac{q^2}{2\frac{E}{k}}$$

equation of ellipse with half axis

$$a = \sqrt{2mE} \quad , \quad b = \sqrt{2\frac{E}{k}}$$



for fixed E closed ellipses in the q-p plane

Modified Hamilton's principle

$$\delta S = 0 \qquad S = \int_{t_1}^{t_2} dt\, \mathcal{L}(t)\, dt$$

rewrite $\quad H = \sum_{i=1}^{s} p_i \dot{q}_i - L \iff L = \sum_{i=1}^{s} p_i \dot{q}_i - H$

then

$$0 = \delta S = \int_{t_1}^{t_2} \left(\sum_{i=1}^{s} p_i \dot{q}_i - H \right) dt$$

now: variation in phase space