a) consider the total differential of $F_1(\vec{q}, \vec{q}', t)$

$$dF_1 = \sum_{j=1}^{s}\left(\frac{\partial F_1}{\partial q_j} dq_j + \frac{\partial F}{\partial q_j'} dq_j'\right) + \frac{\partial F_1}{\partial t} dt$$

we can also obtain the total differential from:

$$\frac{dF_1}{dt} = \sum_{i=1}^{s}\left(p_i \frac{dq_i}{dt} - p_i' \frac{dq_i'}{dt}\right) + (H' - H)$$

$$dF_1 = \sum_{i=1}^{s}\left(p_i dq_i - p_i' dq_i'\right) + (H' - H)\, dt$$

comparing coefficients yields

$$p_i = \frac{\partial F_1}{\partial q_j} = p_i(\vec{q}, \vec{q}', t)$$ can be inverted to $q_i' = q_i'(\vec{q}, \vec{p}, t)$

$$p_i' = -\frac{\partial F_1}{\partial q_j'} = p_i'(\vec{q}, \vec{q}', t)$$

inserting $q_i'(\vec{q}, \vec{p}, t)$ yields the phase transformation

Furthermore, we have

$$H' - H = \frac{\partial F_1}{\partial t} \quad \Leftrightarrow \quad H' = H + \frac{\partial F_1}{\partial t}$$

the Hamilton function is uniquely fixed

b) start from the action

$$S = \int_{t_1}^{t_2} dt \left[\sum_{i=1}^{s} p_i \dot{q}_i - H\right]$$

$$= \int_{t_1}^{t_2} dt \left[\sum_{i=1}^{s} p_i' \dot{q}_i' - H'\right] + \underbrace{\int_{t_1}^{t_2} dt \frac{dF_1}{dt}}_{F_1(\vec{q}_2, \vec{q}_2'(t_2), t_2) - F_1(\vec{q}_2, \vec{q}_2'(t_1), t_1)}$$

note: the quantities in the new "coordinates" $\vec{q}_2'(t_2)$ and $\vec{q}_2'(t_1)$, are not fixed, because these depend on $\vec{q}_i' = \vec{q}_i'(\vec{q}_i, \vec{p}(t_i), t_i)$ and $\vec{p}$ is varied at t_i

variation of action

$$0 = \delta S = \delta\, F_1(\vec{q}_1, \vec{q}'(t_1), t_1) - \delta(F_2\ \vec{q}_2, \vec{q}'(t_2), t_2)$$
$$+ \int_{t_1}^{t_2} dt \sum_{i=1}^{S} \left(\delta p_i'\, \dot{q}_i' + p_i\, \delta\dot{q}_i' - \frac{\partial H'}{\partial q_j'}\, \delta q_j' - \frac{\partial H'}{\partial p_j'}\, \delta p_j'\right)$$

terms containing the variation of velocities $\delta\dot{q}_i$ can be integrated by parts

$$\int_{t_1}^{t_2} dt\ p_i'\, \delta\dot{q}_i' = \int_{t_1}^{t_2} dt\ p_i' \frac{d}{dt}\delta q_i'$$
$$= p_i'\, \delta q_i' \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} dt\ \dot{p}_i'\, \delta q_i'$$

$$0 = \delta S = \sum_{i=1}^{S} \left(p_i' + \frac{\partial F_1}{\partial q_i'}\right) \delta q_i' \Big|_{t_1}^{t_2}$$
$$+ \int_{t_1}^{t_2} dt \left[\sum_{i=1}^{S} \left(\dot{q}_i' - \frac{\partial H'}{\partial p_i'}\right) \delta p_i' - \left(\dot{p}_i' + \frac{\partial H'}{\partial q_i'}\right) \delta q_i'\right]$$

Note: the variation of the new q_i' does <u>not</u> vanish at the endpoint times, but we have $p_i' = -\frac{\partial F_1}{\partial q_i'}$ from the transformation!

variation in the new variables p_i' and q_i' is independent, so we get the canonical equations

$$\dot{q}_i' = \frac{\partial H'}{\partial p_i'} \qquad \dot{p}_i' = -\frac{\partial H'}{\partial q_i'}$$

in the new variables, i.e. the transformation is canonical.

other forms of the generating function

$$\begin{tikzcd}
F_1(\vec{q},\vec{q}\,',t) \arrow[d, "LT"'] \arrow[dd, bend right=60, "LT"'] \\
F_2(\vec{q},\vec{p}\,',t) \\
F_3(\vec{p},\vec{q}\,',t) \arrow[d, "LT"'] \\
F_4(\vec{p},\vec{p}\,',t)
\end{tikzcd}$$

$$\begin{array}{lll}
F_1(\vec{q},\vec{q}\,',t) & p_i = \dfrac{\partial F_1}{\partial q_i} & p_i' = -\dfrac{\partial F_1}{\partial q_i'} \\
F_2(\vec{q},\vec{p}\,',t) & p_i = \dfrac{\partial F_2}{\partial q_i} & q_i' = \dfrac{\partial F_2}{\partial p_i'} \\
F_3(\vec{p},\vec{q}\,',t) & p_i' = -\dfrac{\partial F_3}{\partial q_i'} & q_i = -\dfrac{\partial F_3}{\partial p_i} \\
F_4(\vec{p},\vec{p}\,',t) & q_i = -\dfrac{\partial F_4}{\partial p_i} & q_i' = \dfrac{\partial F_4}{\partial p_i'}
\end{array}$$

in all cases $\quad H' = H + \dfrac{\partial F_i}{\partial t}$

partial derivatives of F_i : transformation to conjugated variable

Examples:

1) interchange of coordinate and momentum

$$F_1(\vec{q},\vec{q}\,') = -\sum_{j=1}^{s} q_j q_j'$$

calculate the transformations

$$p_i = \frac{\partial F_1}{\partial q_i} = -q_i' \qquad\qquad p_i' = -\frac{\partial F_1}{\partial q_i'} = +q_i$$

(we have already discussed this canonical transformation with $H'(\vec{q}\,',\vec{p}\,',t) = H(\vec{q},\vec{p},t)$)

2) Harmonic oscillator

we have discussed the Hamilton function

$$H(q,p) = \frac{p^2}{2m} + \frac{1}{2}K q^2$$

consider the generating function

$$F_1(q,q') = \frac{1}{2} m\,\omega_0\, q^2 \cot(q')$$

calculate the transformation

$$p = \frac{\partial F_1}{\partial q} = m\omega_0 q \cot(q')$$

$$p' = -\frac{\partial F_1}{\partial q'} = \frac{m}{2}\omega_0 q^2 \frac{1}{\sin^2(q')} \quad \Rightarrow \quad q^2 = \frac{2p'}{m\omega_0}\sin^2(q')$$

$$(1) \quad q(q', p') = \sqrt{\frac{2p'}{m\omega_0}}\sin(q')$$

$$(2) \quad p = m\omega_0\sqrt{\frac{2p'}{m\omega_0}}\sin(q')\frac{\cos(q')}{\sin(q')} = \sqrt{2m\omega_0 p'}\cos(q')$$

now, transform the Hamilton function

$$H' = H(p,q) + \frac{\partial F_1}{\partial t}$$

$$= \frac{\left[\sqrt{2m\omega_0 p'}\cos(q')\right]^2}{2m} + \frac{1}{2}k\left[\sqrt{\frac{2p'}{m\omega_0}}\sin(q')\right]^2 + 0$$

$$= \omega_0 p'\cos^2(q') + \omega_0 p'\sin^2(q') = \omega_0 p'$$

canonical equations:

$$\dot{p}' = -\frac{\partial H'}{\partial q'} = 0 \qquad p' = \text{const} = \alpha$$

$$\dot{q}' = \frac{\partial H'}{\partial p'} = \omega_0 \qquad q' = \omega_0 t + \beta$$

(solutions almost trivial)

interpretation by use of transformations (1,2)

$$q = \sqrt{\frac{2\alpha}{m\omega_0}}\sin(\omega_0 t + \beta)$$

$$p = \sqrt{2m\omega_0\alpha}\cos(\omega_0 t + \beta)$$

known solutions of harmonic oscillator.

4) Hamilton Jacobi Theory

Motivation: Find a canonical transformation which greatly simplifies the equations of motion.

Strategies

1) choose transformation that maps to a $H'(\vec{q},\vec{p},t)$ for which the equations of motion have been solved already.

2) choose transformation such that all coordinates q_i' are cyclic. Then: $H' = H'(\vec{p}\,')$

$$\frac{\partial H'}{\partial q_i'} = 0 = -\dot{p}_i' \qquad p_i' = \alpha_i \; (= \text{const})$$

for time independent H', we have

$$\left.\frac{\partial H'}{\partial p_i'}\right|_{p_j' = \alpha_j} = \text{const} = \dot{q}_i' = \omega_i$$

$$q_i' = \omega_i t + \beta_i \qquad \text{(compare harmonic oscillator)}$$

$2s$ parameters (α_i, β_i) need to be fixed by initial conditions

3) choose transformation such that

$$H = \text{const} \qquad H = H(\cancel{q}, \cancel{p}, \cancel{t})$$

$$0 = \frac{\partial H'}{\partial q_i'} = -\dot{p}_i' \qquad p_i' = \alpha_i = \text{const}$$

$$0 = \frac{\partial H'}{\partial p_i'} = \dot{q}_i' \qquad q_i' = \beta_i = \text{const}$$

(α_i, β_i) fixed by initial cond.

The dynamics of the system is then in the transformation :

$$q_i = q_i(\vec{q}\,', \vec{p}\,', t) = q_i(\vec{\beta}, \vec{\alpha}, t)$$
$$p_i = p_i(\vec{q}\,', \vec{p}\,', t) = p_i(\vec{\beta}, \vec{\alpha}, t)$$

How to obtain desired transformation ?

We want $\quad H' = \text{const} = 0$

(*) $H' = H + \frac{\partial F_2}{\partial t} = 0 \qquad F_2 = F_2(\vec{q}, \vec{p}\,', t)$ (generating function)

we have :

$$\left.\begin{aligned} p_i &= \frac{\partial F_2}{\partial q_i} \\ q_i' &= \frac{\partial F_2}{\partial p_i'} \end{aligned}\right\} \text{canonical transformation}$$

Then (*) is a differential equation for F_2 :

$$H\left(q_1, \dots, q_s, \frac{\partial F_2}{\partial q_1}, \dots, \frac{\partial F_2}{\partial q_s}, t\right) + \frac{\partial F_2}{\partial t} = 0$$

(Hamilton - Jacobi differential equation)

Notes :

1) HJD is a first order, nonlinear D.E. in $2s+1$ variables ; the solution has $2s+1$ parameters $(\vec{\gamma}, \vec{\delta}, \eta)$
   → one parameter is a trivial constant η : if F_2 is a solution, also $F_2 + \eta$ is a solution.

2) The general solution is a set of generating functions F_2, no guarantee that it exists

3) The "dynamics" of the Hamilton function $H' = 0$ is just a point in phase space $\vec{\pi}' = (\vec{q}', \vec{p}') = (\vec{\beta}, \vec{\alpha})$
Solution of the original problem by inversion of the transformation $\vec{\pi} = (\vec{q}(\vec{q}', \vec{p}', t), \vec{p}(\vec{q}', \vec{p}', t$

4) The generating function is the action:

$$F_2 = S = \int_{t_1}^{t_2} L(\vec{q}, \dot{\vec{q}}, t)\, dt + \text{const}$$

to see this, we recall: $F_2 = F_2(\vec{q}, \vec{p}', t)$

$$p_i = \frac{\partial F_2}{\partial q_i}, \quad q_i' = \frac{\partial F_2}{\partial p_i'}$$

$$H' = H + \frac{\partial F_2}{\partial t} = 0$$

$$\frac{dF_2}{dt} = \sum_{i=1}^{s} \left(\frac{\partial F_2}{\partial q_i} \dot{q}_i + \frac{\partial F_2}{\partial p_i'} \dot{p}_i' \right) + \overbrace{\frac{\partial F_2}{\partial t}}^{H' - H}$$

$$= \sum_{i=1}^{s} \left(p_i \dot{q}_i + q_i' \dot{p}_i' \right) + H' - H$$

$\dot{p}_i' = 0$
$H' = 0$

$$= \sum_{i=1}^{s} p_i \dot{q}_i - H = L$$

(Definition of the Hamilton fct.)

integration over t from $[t_1 \dots t_2]$ yields

$$F_2(t_2) - F_2(t_1) = \int_{t_1}^{t_2} \frac{dF_2}{dt} dt = \int_{t_1}^{t_2} L\, dt$$

need to know $\vec{q}(t)$ to evaluate r.h.s

$\to$ this does <u>not</u> help to calculate F_2

Some special cases:

1) Hamilton function is conserved : $\frac{\partial H}{\partial t} = 0$

$$H\left(\vec{q}, \frac{\partial S}{\partial q_1}, \dots, \frac{\partial S}{\partial q_s}\right) + \frac{\partial S}{\partial t} = 0 \qquad (\text{use}: F_2 = S)$$

Ansatz: (separation of time dependence)

$$S(\vec{q}, \vec{P}', t) = W(\vec{q}, \vec{P}') - E(\vec{P}', t) \cdot t$$

then:

$$H\left(\vec{q}, \frac{\partial W}{\partial q_1}, \dots, \frac{\partial W}{\partial q_s}\right) = E(\vec{P}')$$

total energy (if H is total energy)

$W(\vec{q}, \vec{P}')$: Hamilton's characteristic function

2) separable problems :

$$H(\vec{q}, \vec{p}) = \sum_{i=1}^{s} H_i(q_i, p_i) \Rightarrow \sum_{i=1}^{s} H_i\left(q_i, \frac{\partial W_i}{\partial q_i}\right) = E(\vec{P}')$$

ansatz $\quad W(\vec{q}, \vec{P}') = \sum_i W_i(q_i, \vec{P}')$

each function obeys: $\quad H_i\left(q_i, \frac{\partial W_i}{\partial q_i}\right) = C_i$

s-dimensional harmonic oscillator:

$$H(\vec{q}, \vec{p}) = \sum_{i=1}^{s} \left(\frac{p_i^2}{2m} + \frac{1}{2} K_i q_i^2\right)$$

$$\Rightarrow \frac{1}{2m}\left(\frac{\partial W_i}{\partial q_i}\right)^2 + \frac{1}{2} K_i q_i^2 = C_i$$

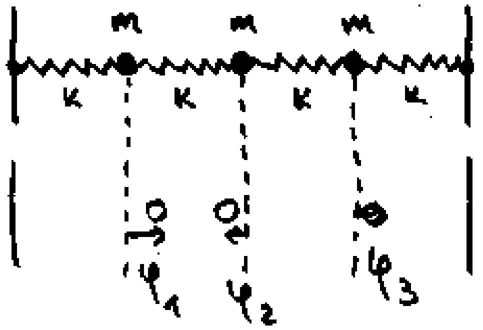
s (ordinary) nonlinear differential equations, solve by integration

5) Lagrange mechanics for (continous) fields

so far countable number of degrees of fredom $\vec{q}(t) = (q_1, \dots, q_s)$ (s degrees of fredom)

now: fields $\varphi(t, \vec{r})$ represents <u>one</u> degree of freedom

Example: 1D chain of masses, connected by springs



equilibrium position: masses separated by a

configuration fixed by $\vec{q}(t) = (\varphi_1, \varphi_2, \dots, \varphi_s)$

Lagrange function $L = T - V$

kinetic energy $T = \frac{1}{2} m \sum_{i=1}^{s} \dot{\varphi}_i^2$

potential $V = \frac{1}{2} k \sum_{i=1}^{s} (\varphi_{i+1} - \varphi_i)^2$

check by calculating the force

$$F_j = -\frac{\partial V}{\partial \varphi_j} = -k(\varphi_j - \varphi_{j-1}) + k(\varphi_{j+1} - \varphi_j)$$

(correct if setting $\varphi_0 = \varphi_{s+1} = 0$ for fixed walls)

$$L = T - V = \frac{1}{2} \sum_{i=1}^{s} \left(m \dot{\varphi}_i^2 - k(\varphi_{i+1} - \varphi_i)^2\right)$$

$$= \frac{1}{2} \sum_{i=1}^{s} a\left[\mu \dot{\varphi}_i^2 - \sigma \left(\frac{\varphi_{i+1} - \varphi_i}{a}\right)^2\right]$$

$\mu = \frac{m}{a}$ (mass per length) $\quad \sigma = ka$

Euler-Lagrange equations of motion

$$0 = \frac{d}{dt}\frac{\partial L}{\partial \dot\varphi_i} - \frac{\partial L}{\partial \varphi}$$

$$0 = m\,\ddot\varphi_i - k\,(\varphi_{i-1} - 2\varphi_i + \varphi_{i+1})$$

in vector notation

$$M\ddot{\vec q} + K\vec q = 0 \qquad M = \delta_{ij}\,m \quad \text{(mass matrix)}$$

$$K = k\begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & 2 & -1 & \\ & & -1 & 2 & -1 \\ & & & & \ddots \end{pmatrix}$$

"force" matrix

the force matrix is
a real symmetric matrix
→ exists an orthogonal matrix U such that

$$U^T K U = \mathrm{diag}(\lambda_1, \dots, \lambda_s) = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_s \end{pmatrix} = D$$

λ_i : eigenvalues of K

define: transformed coordinates

$$\vec q\,' = U^T \vec q \quad \Leftrightarrow \quad U\vec q\,' = \underbrace{UU^T}_{1\,\text{(orthogonal)}}\vec q = \vec q$$

equations of motion:

$$MU\ddot{\vec q}\,' + KU\vec q\,' = 0 \qquad \text{(multiply with } U^T\text{)}$$

$$\underbrace{U^T M U}_{M}\ddot{\vec q}\,' + \underbrace{U^T K U}_{D}\vec q\,' = 0$$

decoupled equations of motion

$$m\,\ddot q_i' + \lambda_i\, q_i' = 0 \qquad \omega_i^2 = \frac{\lambda_i}{m} \quad \text{(eigen frequencies)}$$

solutions $\quad q_i'(t) = A_i \cos(\omega_i t) + B_i \sin(\omega_i t)$

(A_i and B_i fixed by initial conditions)

positions of masses via transformation $\mathcal{U}$.

Note: $\lambda_i \in \mathbb{R}$ (because $K = K^T$)
but for general case (s coupled oscillators) not necessarily positive

1) $\lambda_i > 0$ stable direction, oscillations
2) $\lambda_i < 0$ instable direction $\gamma_i = i\omega_i$
$q_i'(t) = A_i \cosh(\gamma_i t) + B_i \sinh(\gamma_i t)$
3) $\lambda_i = 0$ $q_i'(t) = A_i + B_i t$

Remarks: 1) For s coupled oscillators, the Lagrange function can be written as:

$$L(\vec{q}, \dot{\vec{q}}) = \frac{1}{2}\left(\dot{\vec{q}}^T M \dot{\vec{q}} - \vec{q}^T K \vec{q}\right)$$

M: generalized mass matrix
K: force matrix
linear terms in $\vec{q}$ do not occur if $\vec{q} = 0$ is the equilibrium position

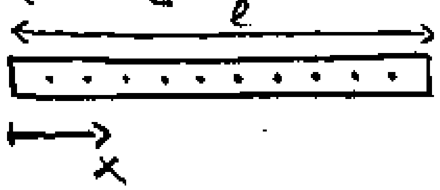
2) If the mass matrix is not diagonal, one needs another transformation $\vec{q}' = \mathcal{U}_M^T \vec{q}$ where $\mathcal{U}_M$ is the othogonal matrix that diagonalizes M:

$$\mathcal{U}_M^T M \mathcal{U} = \mathrm{diag}(\tilde{m}_1, \dots, \tilde{m}_s)$$

Limit: $s \to \infty$, $a \to 0$: $\mu = \text{const}$, $\kappa = \text{const}$

$$L = \frac{1}{2} \sum_{i=1}^{s} a \Big[\underbrace{\mu}_{\text{mass per length}} \dot{\varphi}_i^{\,2} - \underbrace{\kappa}_{\text{elastic modulus}} \left(\frac{\varphi_{i+1} - \varphi_i}{a} \right)^2 \Big]$$

description of longitudinal elastic waves of a stick



$\varphi_i(t) \to \varphi(x,t)$

position of mass i — position of infinitest. mass element at x

$$\lim_{a\to 0} \frac{\varphi_{i+1}(t) - \varphi_i(t)}{a} = \lim_{a\to 0} \frac{\varphi(t, x+a) - \varphi(t,x)}{a} = \frac{\partial \varphi(t,x)}{\partial x}$$

$$\lim_{a\to 0} \frac{\varphi_{i+1}(t) - 2\varphi_i(t) + \varphi_{i+1}(t)}{a^2} = \lim_{a\to 0} \frac{\varphi(t,x+a) - 2\varphi(t,x) + \varphi(t,x-a)}{a} = \frac{\partial^2 \varphi(t,x)}{\partial x^2}$$

formally, the time derivative becomes a partial derivative

$$\dot{\varphi}_i \longrightarrow \frac{\partial \varphi(x,t)}{\partial t}$$

The lagrange function becomes

$$L = \lim_{a\to 0} \sum_{i=1}^{s} a\, \mathcal{L}_i = \int_0^{\ell} dx\, \mathcal{L}$$

$$\mathcal{L}_i = \frac{1}{2} \mu \dot{\varphi}_i^{\,2} - \frac{\kappa}{2} \left(\frac{\varphi_{i+1} - \varphi_i}{a} \right)^2$$

$$\mathcal{L} = \frac{1}{2} \mu \left(\frac{\partial \varphi}{\partial t} \right)^2 - \frac{\kappa}{2} \left(\frac{\partial \varphi}{\partial x} \right)^2 \quad \text{(Lagrange density)}$$

equations of motion :

$$0 = \mu \frac{\partial^2 \varphi}{\partial t^2} - K \frac{\partial^2 \varphi}{\partial x^2} \qquad \text{partial D.E.}$$

(wave equation for longitudinal sound waves in a homogeneous stick)

Lagrange formalism for continuous fields

Lagrange function (generalization to multiple dimension

$$L = \int_V dV \, \mathcal{L}\left(\varphi, \frac{\partial \varphi}{\partial t}, \vec{\nabla}\varphi, t, \vec{r}\right)$$

note the correspondences :

$\varphi \rightarrow \varphi_i$ generalized coordinates

$\frac{\partial \varphi}{\partial t}, \vec{\nabla}\varphi \rightarrow$ generalized velocities

$t, \vec{r} \rightarrow$ additional explicit dependences

Hamilton's principle

$$\delta S = \delta \int_\Omega dt \, dV \, \mathcal{L} = 0$$

$\Omega \leftarrow$ space-time volume

variation of the degrees of fredom $\left(\varphi, \frac{\partial \varphi}{\partial t}, \vec{\nabla}\varphi\right)$

no variation of the field on the surface of Ω : $\delta \varphi(t, \vec{r})\big|_{\partial \Omega} = 0$

(compare : no variation of q_i at t_1 and t_2)

use chain rule to obtain for variation

$$0 = \int_\Omega dt\, dV \left[\frac{\partial \mathcal{L}}{\partial \varphi} \delta\varphi + \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\varphi}{\partial t}\right)} \underbrace{\underbrace{\delta \frac{\partial \varphi}{\partial t}}_{\frac{\partial}{\partial t}(\delta\varphi)}}_{-\frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\varphi}{\partial t}\right)} \delta\varphi} + \sum_{i=1}^{d} \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\varphi}{\partial x_i}\right)} \underbrace{\underbrace{\delta\left(\frac{\partial \varphi}{\partial x_i}\right)}_{\frac{\partial}{\partial x_i}(\delta\varphi)}}_{-\frac{\partial}{\partial x_i} \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\varphi}{\partial x_i}\right)} \delta\varphi} \right]$$

integration by parts over t
"surface terms vanish

integration by parts

$$0 = \int_\Omega dt\, dV \underbrace{\left[\frac{\partial \mathcal{L}}{\partial \varphi} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\varphi}{\partial t}\right)} - \sum_{i=1}^{d} \frac{\partial}{\partial x_i} \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\varphi}{\partial x_i}\right)} \right]}_{=0} \delta\varphi(\vec{r}, t)$$

variation on each t $\vec{r}$ independent

Euler Lagrange equations for continous functions

Example: lagrangian for stick

$$\mathcal{L} = \frac{1}{2} \mu \left(\frac{\partial \varphi}{\partial t}\right)^2 - \frac{\kappa}{2} \left(\frac{\partial \varphi}{\partial x}\right)^2$$

$$\frac{\partial \mathcal{L}}{\partial \varphi} = 0 \quad , \quad \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\varphi}{\partial t}\right)} = \mu \frac{\partial \varphi}{\partial t} \quad , \quad \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\varphi}{\partial x}\right)} = -\kappa \frac{\partial \varphi}{\partial x}$$

$$0 = -\frac{\partial}{\partial t}\left(\mu \frac{\partial \varphi}{\partial t}\right) - \frac{\partial}{\partial x}\left(-\kappa \frac{\partial \varphi}{\partial x}\right) = \underbrace{-\mu \frac{\partial^2 \varphi}{\partial t^2} + \kappa \frac{\partial^2 \varphi}{\partial x^2}}_{\text{(see above)}} \quad \checkmark$$

Note: for multiple fields $\varphi \to \varphi_\alpha$ $\alpha = 1 \dots n$, we have

$$\frac{\partial \mathcal{L}}{\partial \varphi_\alpha} - \frac{\partial}{\partial t} \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\varphi_\alpha}{\partial t}\right)} - \sum_{i=1}^{d} \frac{\partial}{\partial x_i} \frac{\partial \mathcal{L}}{\partial\left(\frac{\partial\varphi_\alpha}{\partial x_i}\right)} = 0$$

Euler-Lagrange equ. for each field $\varphi_\alpha(t, \vec{r})$