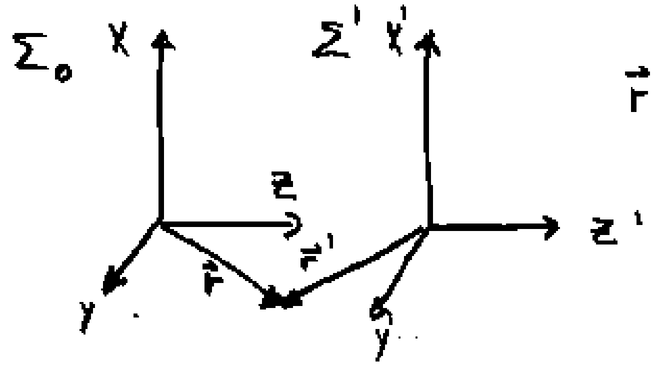
6) Special relativity

So far: Newton mechanics
absolute space and time: "preferred frame"
inertial system: moves relative to the preferred frame with constant velocity

Galilei transformation

$$x = x', \quad y = y', \quad z = z' - vt, \quad t = t'$$



$$\vec{r}\,' = \vec{r} - \vec{v}t$$

$$\ddot{\vec{r}} = \frac{d^2}{dt^2}\vec{r} = \frac{d^2}{dt^2}\left(\vec{r}\,' + \vec{v}t\right) = \frac{d^2}{dt^2}\vec{r}\,' = \frac{d^2}{dt'^2}\vec{r}\,' = \ddot{\vec{r}}\,'$$

Newton equations of motion identical:

$$\vec{F} = m\ddot{\vec{r}} = m\ddot{\vec{r}}\,' = \vec{F}\,'$$

Assumption: Light propagates with speed c in Σ_0 ("preferred frame")

Σ_0: $\vec{v} = \dot{\vec{r}} = c\vec{e}_r$ (spherical coordinates)

Σ' (use Galilei transformation)

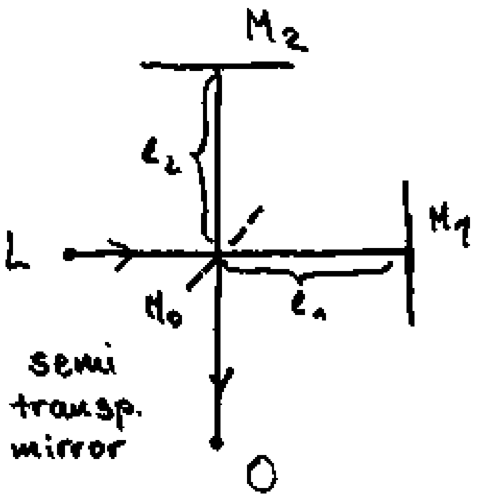
$$\dot{\vec{r}}\,' = \dot{\vec{r}} - \vec{v} = c\vec{e}_r - \vec{v} \qquad \vec{v} = (0,0,v)^T$$

$$|\dot{\vec{r}}\,'| = \sqrt{c^2\sin^2\vartheta + (c\cos\vartheta - v)^2} \qquad \vec{e}_r = \begin{pmatrix}\cos\varphi\sin\vartheta\\ \sin\varphi\sin\vartheta\\ \cos\vartheta\end{pmatrix}$$

$$= c\sqrt{1 - \frac{2v}{c}\cos\vartheta + \frac{v^2}{c^2}} \neq c$$

no spherical wave in Σ'

The Michelson-Morley experiment



$\vec{v}$ assumed velocity relative to preferred frame

interference experiment
fix lengths ℓ_1, ℓ_2 such that light (waves) show constructive interference:

$$\delta = m\lambda \qquad m \in \mathbb{Z}$$

λ: wavelength of light

calculate distance in Σ_0:

$$\delta = c\left(t_{02} + t_{20} - (t_{01} + t_{10})\right) \quad (*)$$

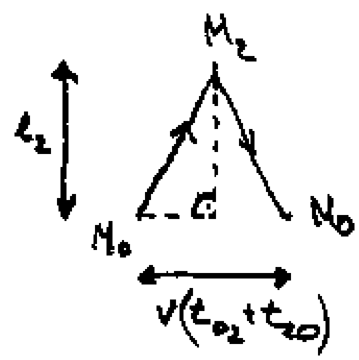
t_{ij}: time from mirror i to mirror j.

Now: calculate times in Σ' by assuming addition of velocities $\vec{c}\,' = \vec{c} \pm \vec{v}$

$$\left.\begin{aligned} t_{01} &= \frac{\ell_1}{c-v} = \frac{\ell_1}{c}\,\frac{1}{1-\frac{v}{c}} \\ t_{10} &= \frac{\ell_1}{c+v} = \frac{\ell_1}{c}\,\frac{1}{1+\frac{v}{c}} \end{aligned}\right\}$$ mirror moves relative to velocity of light

calculate times in Σ_0 by considering the path traveled

$$c\,t_{02} = c\,t_{20} = \sqrt{\ell_2^2 + v^2\left(\frac{t_{02}+t_{20}}{2}\right)^2} \overset{t_{02}=t_{20}}{=} \sqrt{\ell_2^2 + v^2 t_{02}^2}$$



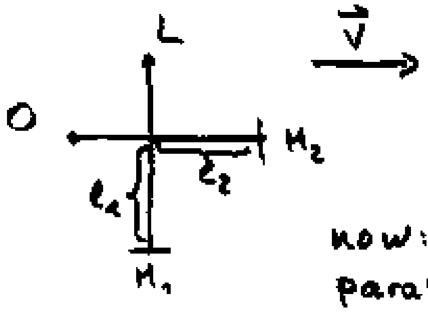
$$\Rightarrow \quad t_{02}^2 = \frac{\ell_2^2}{c^2 - v^2}\,, \qquad t_{02} = \frac{\ell_2}{c}\,\frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

insert into (*)

$$\delta = c\left[2t_{02} - (t_{01} + t_{10})\right] = \frac{2\ell_2}{\sqrt{1-\frac{v^2}{c^2}}} - \ell_1\left(\frac{1}{1+\frac{v}{c}} + \frac{1}{1-\frac{v}{c}}\right)$$

$$= \frac{2\ell_2}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{2\ell_1}{1-\frac{v^2}{c^2}}$$

now: turn apparatus:



$$\delta' = \frac{2\ell_2}{1-\frac{v^2}{c^2}} - \frac{2\ell_1}{\sqrt{1-\frac{v^2}{c^2}}}$$

now: $\vec{\ell}_2$ parallel to $\vec{v}$

calculate difference:

$$\delta' - \delta = 2\ell_2\left(\frac{1}{1-\frac{v^2}{c^2}} - \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\right) - 2\ell_1\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - \frac{1}{1-\frac{v^2}{c^2}}\right)$$

$$= \frac{2(\ell_1+\ell_2)}{\sqrt{1-\frac{v^2}{c^2}}}\left(\frac{1}{\sqrt{1-\frac{v^2}{c^2}}} - 1\right)$$

$\frac{1}{\sqrt{1-x^2}} \approx 1 + \frac{x^2}{2}$

$$\approx 2(\ell_1+\ell_2)\left(1+\frac{v^2}{2c^2}\right)\left(1+\frac{v^2}{2c^2} - 1\right) \approx (\ell_1+\ell_2)\frac{v^2}{c^2}$$

if $\delta' - \delta$ is of the order of λ, a change in the measured amplitude should be visible (interference)

Michelson	1881	$\ell_i \approx 1.2\,m$:	$v < 20\,\frac{km}{s}$
Michelson-Morley	1887	$\ell_i \approx 11\,m$	$v < 6\,\frac{km}{s}$
⋮			
Trimmer	1973		$v < 2.5\,\frac{cm}{s}$

Conclusion: Light propagates with the same speed in all inertial systems.

Then: $\delta = 2\ell_2 - 2\ell_1$ (independent of $\vec{v}$)

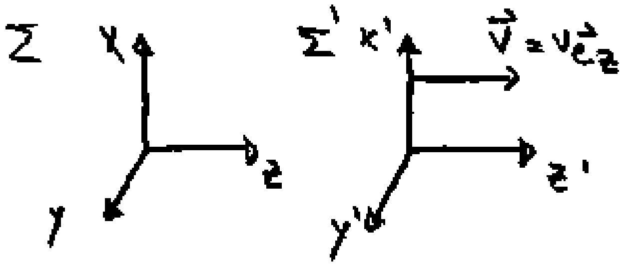
Einstein's postulates

1) All laws of physics take the same form in all inertal frames of reference.
2) Light always propagates in empty space with velocity c (independ of inertial frame of reference).

Lorentz transformation

a) Minkowski space

consider: two inertial systems



$t = 0 = t'$
systems have same origin

generation of (spherical) light wave at $t = 0$

Σ: velocity of light (wavefronts): $\vec{v} = \dot{\vec{r}} = c\,\vec{e}_r$

position by integration: $\vec{r} = c\,t\,\vec{e}_r$

calculate scalar product $\vec{r}\cdot\vec{r} = x^2+y^2+z^2 = c^2t^2$

Σ': same considerations $\vec{r}\,'\cdot\vec{r}\,' = x'^2+y'^2+z'^2 = c^2t'^2$

especially, we can equate

$$c^2t^2 - (x^2+y^2+z^2) = 0 = c^2t'^2 - (x'^2+y'^2+z'^2)$$

under Lorentz transformation, the quantity $c^2t^2 - (x^2+y^2+z^2)$ is invariant.

reminder: rotations in 3 dimensions

$$\vec{r}' = D\vec{r} = \sum_{i,j=1}^{3} D^{ij} x^j \vec{e}_i \qquad D^T = D^{-1}$$

(orthogonal transformation)

$$\vec{r}' \cdot \vec{r}' = \vec{r}'^T \vec{r}'$$

$$= \vec{r}^T D^T D \vec{r} = \vec{r}^T \vec{r} = \vec{r} \cdot \vec{r}$$

(scalar product invariant)

now: define contravariant vector

$$x^\mu = (x^0, x^1, x^2, x^3) = (ct, \vec{r})^T$$

and metric tensor

$$g_{\mu\nu} = \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix}$$

invariant quantity

$$\sum_{\mu,\nu} x^\mu g_{\mu\nu} x^\nu = c^2t^2 - x^2 - y^2 - z^2$$

we can also introduce $x_\mu = \sum_\nu g_{\mu\nu} x^\nu = (ct, -\vec{r})$

(covariant 4-vector)

invariant quantity

$$\sum_\mu x^\mu x_\mu = \sum_\mu x'^\mu x'_\mu \qquad \text{(4-scalar product)}$$

other notation $\quad X \cdot X = X' \cdot X$

Einstein sum convention: implicit sum over covariant and covariant indices

$$X \cdot X = \sum_\mu x^\mu x_\mu = x^\mu x_\mu$$

b) Lorentz transformation

look for transformation (matrix) leaving 4-scalar product invariant

$$x'^\mu = \sum_{\nu=1}^{3} \Lambda^\mu{}_\nu x^\nu \qquad \Lambda^\mu{}_\nu \in \mathbb{L}$$

(Lorentz group)

subgroup of Lorentz transformations:

orthogonal transformations in 3 dimensions: $O(3)$
representations: orthogonal 3x3 matrices D
with $D^T = D^{-1}$, leave $\vec{r}\cdot\vec{r} = \vec{r}\,'\cdot\vec{r}\,'$ invariant
here: consider only those transformations with $\det D = +1$
special orthogonal transformations $SO(3) \subset \mathbb{L}$

$$\Lambda_D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & & & \\ 0 & & D & \\ 0 & & & \end{pmatrix}$$

component-wise transformation:

$$x'^0 = ct' = x^0 = ct$$

$$x'^i = d^i{}_j x^j \qquad i = 1..3$$

then: $c^2 t'^2 - \vec{r}\,'^2 = c^2 t^2 - \vec{r}^{\,2}$

Lorentz-boosts (no subgroup of Lorentz group)

explicit calculation of boost in z direction

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \Lambda^0{}_0 & 0 & 0 & \Lambda^0{}_3 \\ 0 & & & 0 \\ 0 & & M & 0 \\ \Lambda^3{}_0 & 0 & 0 & \Lambda^3{}_3 \end{pmatrix} \qquad M = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$$

symmetries $\quad x'^0 = x'^0(x^0, x^3)$ (no point in x^1-x^2 plane is special)

$x'^1 = x'^1(x^1, x^2)$
$x'^2 = x'^2(x^1, x^2)$ (no point in $x^{1'}$ $x^{2'}$ plane is special)

no rotation : $x'^1 = x'^1(x^1)$
$x'^2 = x'^2(x^2)$

calculate 4-scalar product:

$$x'^\mu x'_\mu = x'^\mu g_{\mu\nu} x'^\nu = \Lambda^\mu{}_\alpha x^\alpha g_{\mu\nu} \Lambda^\nu{}_\beta x^\beta$$
$$= x^\alpha \Lambda^\mu{}_\alpha g_{\mu\nu} \Lambda^\nu{}_\beta x^\beta \underset{\text{invariance}}{=} x^\alpha g_{\alpha\beta} x^\beta$$

we have therefore:

$$g_{\alpha\beta} = \Lambda^\mu{}_\alpha g_{\mu\nu} \Lambda^\nu{}_\beta = (\Lambda^T)_\alpha{}^\mu g_{\mu\nu} \Lambda^\nu{}_\beta$$

$$\begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} = \begin{pmatrix} \Lambda^0{}_0 & 0 & 0 & \Lambda^3{}_0 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ \Lambda^0{}_3 & 0 & 0 & \Lambda^3{}_3 \end{pmatrix} \begin{pmatrix} 1 & & & \\ & -1 & & \\ & & -1 & \\ & & & -1 \end{pmatrix} \begin{pmatrix} \Lambda^0{}_0 & 0 & 0 & \Lambda^0{}_3 \\ 0 & a & 0 & 0 \\ 0 & 0 & a & 0 \\ \Lambda^3{}_0 & 0 & 0 & \Lambda^3{}_3 \end{pmatrix}$$

$$= \begin{pmatrix} (\Lambda^0{}_0)^2 - (\Lambda^3{}_0)^2 & 0 & 0 & \Lambda^0{}_0 \Lambda^0{}_3 - \Lambda^3{}_0 \Lambda^3{}_3 \\ 0 & -a^2 & 0 & 0 \\ 0 & 0 & -a^2 & 0 \\ \Lambda^0{}_3 \Lambda^0{}_0 - \Lambda^3{}_3 \Lambda^3{}_0 & 0 & 0 & (\Lambda^0{}_3)^2 - (\Lambda^3{}_3)^2 \end{pmatrix}$$

comparison component-wise

$-1 = -a^2 \qquad a = \pm 1 \qquad$ choose: $a = 1$

($a = -1$ would be additional mirror operation)

(1) $\quad 1 = (\Lambda^0{}_0)^2 - (\Lambda^3{}_0)^2$
(2) $\quad -1 = (\Lambda^0{}_3)^2 - (\Lambda^3{}_3)^2$
(3) $\quad 0 = \Lambda^0{}_0 \Lambda^0{}_3 - \Lambda^3{}_0 \Lambda^3{}_3$

} 3 equations for 4 unknown

Ansatz with one parameter

$$\Lambda^0{}_0 = \Lambda^3{}_3 = \cosh\chi$$
$$\Lambda^0{}_3 = \Lambda^3{}_0 = -\sinh\chi$$

(1) $1 = \cosh^2\chi - \sinh^2\chi$ ✓

(2) $-1 = \sinh^2\chi - \cosh^2\chi$ ✓

(3) $0 = \cosh\chi(-\sinh\chi) - (-\sinh\chi)\cosh\chi$ ✓

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \cosh\chi & 0 & 0 & -\sinh\chi \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\chi & 0 & 0 & \cosh\chi \end{pmatrix} \qquad \chi: \text{rapidity}$$

Interpretation of $\chi(v)$:

origin of Σ' as observed in Σ:

$$x^3 = vt = \frac{v}{c}x^0 \qquad (4)$$

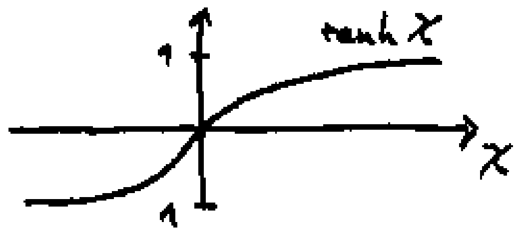
origin of Σ' as observed in Σ':

$$0 = x'^3 \overset{LT}{=} -\sinh\chi\, x^0 + \cosh\chi\, x^3 \overset{(4)}{=} x^0\left(-\sinh\chi + \frac{v}{c}\cosh\chi\right)$$

$$\Rightarrow \frac{v}{c}\cosh\chi = \sinh\chi$$

$$\frac{v}{c} = \frac{\sinh\chi}{\cosh\chi} = \tanh\chi \qquad \text{define } \beta = \frac{v}{c}$$

$$\cosh\chi = \frac{1}{\sqrt{1-\tanh^2\chi}}$$
$$= \frac{1}{\sqrt{1-\beta^2}} = \gamma$$



Lorentz gamma

$\gamma \geq 1$

small χ: $\tanh\chi \approx \chi = \frac{v}{c}$

$$\sinh\chi = \cosh\chi\tanh\chi = \gamma\beta$$

rewrite Lorentz transformation

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix}$$

further properties

1) $\det \Lambda = \det \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} = \gamma^2 - \beta^2\gamma^2 = \frac{1}{1-\beta^2}(1-\beta^2) = 1$

but $\Lambda^{-1} \neq \Lambda^T$ (Λ not orthogonal)

2) inverse for $v \to -v$ $\quad \Lambda(v)\ \Lambda(-v) = 1$

$$\Lambda^\mu{}_\nu(-v) = \begin{pmatrix} \gamma & 0 & 0 & \beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \beta\gamma & 0 & 0 & \gamma \end{pmatrix} = \left[\Lambda^\mu{}_\nu(v)\right]^{-1}$$

3) Back to Galilei transformation for $\frac{v}{c} = \beta \ll 1$

(I) $\quad x'^0 = ct' = \gamma ct - \beta\gamma z = \gamma c\left(t - \frac{v}{c^2} z\right)$

$x'^1 = x^1$

$x'^2 = x^2$

(IV) $\quad x'^3 = -\beta\gamma ct + \gamma x^3 = \gamma(z - vt)$

expansion for $\frac{v}{c} \ll 1$:

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = 1 + \frac{\beta^2}{2} + O(\beta^4)$$

(I) $\quad t' = \gamma\left(t - \frac{v}{c^2} z\right) \approx \left(1 + \frac{\beta^2}{2}\right)\left(t - \frac{\beta}{c} z\right) \approx t$

(IV) $\quad z' = \gamma(z - vt) \approx \left(1 + \frac{\beta^2}{2}\right)(z - vt) \approx z - vt$ ✓

4) Time dilatation

two events at the same place: x_1^μ, x_2^μ

$\Delta t = t_2 - t_1$ in reference frame Σ

$$\Delta t' = t_2' - t_1' \quad \text{in reference frame } \Sigma'$$

$$= \frac{1}{c}\left(x_2'^{0} - x_1'^{0}\right) \overset{LT}{=} \frac{1}{c}\gamma\left(x_2^0 - x_1^0 - \beta\underbrace{(z_2 - z_1)}_{=0 \text{ (same position)}}\right)$$

$$= \gamma\left(t_2 - t_1\right) = \gamma\Delta t$$

$$\Rightarrow \Delta t' = \frac{1}{\sqrt{1-\beta^2}}\Delta t \geq \Delta t$$

5) Length contraction (in boost direction)

measure a length in Σ: $\ell = z_2 - z_1$
(at a fixed time t)

now measure length in Σ' at fixed time $t_1' = t_2'$:

$$t_2' = \gamma\left(t_2 - \frac{v}{c^2}z_2\right) = t_1' = \gamma\left(t_1 - \frac{v}{c^2}z_1\right)$$

$$\Rightarrow t_2 - t_1 = \frac{v}{c^2}\left(z_2 - z_1\right) = \frac{v}{c^2}\ell$$

(corresponds to different times in Σ)

now transform spatial coordinates

$$\ell' = z_2' - z_1' = \gamma\left[z_2 - z_1 - v\left(t_2 - t_1\right)\right]$$

$$= \gamma\left[\ell - v\frac{v}{c^2}\ell\right] = \gamma\ell\left(1 - \frac{v^2}{c^2}\right)$$

$$= \frac{\ell}{\gamma} < \ell$$

in Σ': length appears shorter

6) Addition of velocities

Galilei: $v = v_1 + v_2$ (apply two Galilei transformations with v_i from an inertial system where mass m is at rest)