$\Lambda(v) = \Lambda(v_2)\,\Lambda(v_1)$ (two Lorentz boosts)

$$\begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix} = \begin{pmatrix} \gamma_2 & -\beta_2\gamma_2 \\ -\beta_2\gamma_2 & \gamma_2 \end{pmatrix} \begin{pmatrix} \gamma_1 & -\beta_1\gamma_1 \\ -\beta_1\gamma_1 & \gamma_1 \end{pmatrix} = \gamma_1\gamma_2 \begin{pmatrix} 1+\beta_1\beta_2 & -\beta_1-\beta_2 \\ -\beta_1-\beta_2 & 1+\beta_1\beta_2 \end{pmatrix}$$

$$= \gamma_1\gamma_2(1+\beta_1\beta_2) \begin{pmatrix} 1 & -\frac{\beta_1+\beta_2}{1+\beta_1\beta_2} \\ -\frac{\beta_1+\beta_2}{1+\beta_1\beta_2} & 1 \end{pmatrix}$$

read off:

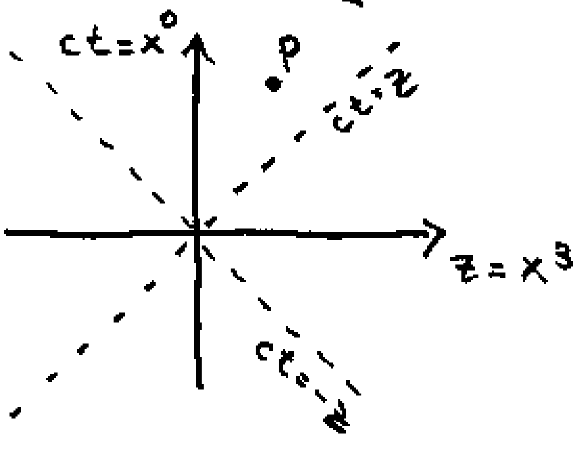
$$\beta = \frac{\beta_1+\beta_2}{1+\beta_1\beta_2}$$

$$v = \frac{v_1+v_2}{1+\frac{v_1 v_2}{c^2}}$$ (relativistic addition of velocities)

Note: Lorentz boost leaves c invariant

$$\frac{v_1+c}{1+\frac{v_1 c}{c^2}} = \frac{v_1+c}{1+\frac{v_1}{c}} = c\,\frac{v_1+c}{v_1+c} = c$$

Minkowski diagrams (spacetime diagrams)



special line: light signal

$c^2t^2 - z^2 = 0$

$\pm z = ct$

moving reference frame (Lorentz boost in z dir

Σ' at $t=t'=0$, same origins $z=z'=0$

coordinate axis of Σ':

a) $0 = z' = \gamma(z - vt)$

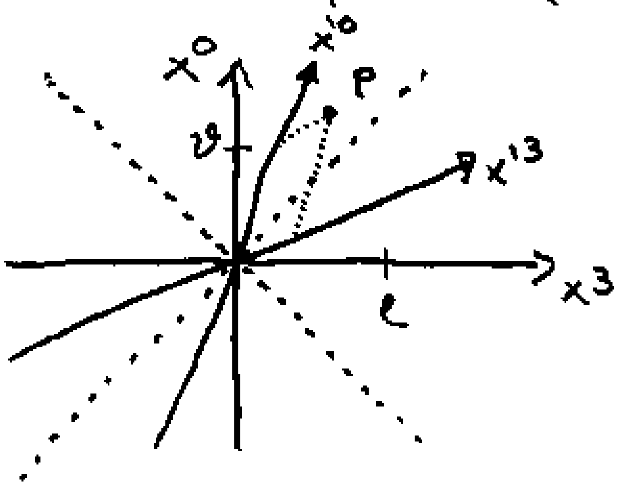
$\Rightarrow \quad z = vt$

$\Leftrightarrow \quad ct = \frac{c}{v} t = \frac{z}{\beta}$ (condition for ct' axis)

b) $0 = ct' = c\gamma\left(t - \frac{v}{c^2} z\right)$

$\Rightarrow \quad t = \frac{v}{c^2} z$

$\Leftrightarrow \quad ct = \frac{v}{c} z = \beta z$ (condition for z' axis)



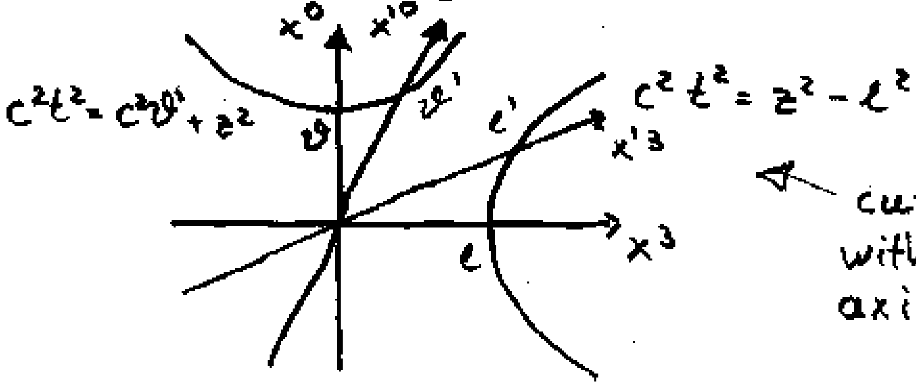
scaling of axis using invariant (distance)

$$s^2 = x^\mu x_\mu = c^2 t^2 - (x^2 + y^2 + z^2)$$

for "scale" on x^3 axis, we have $s^2 = -\ell^2$

thus all points with $c^2t^2 - z^2 = -\ell^2$
are invariant: $(ct)^2 = z^2 - \ell^2$ (hyperbola in Minkowski diagram)

for "scale" ϑ on x^0 axis, we have $s^2 = c^2\vartheta^2$
all points with $c^2t^2 - z^2 = c^2\vartheta^2$
are invariant $c^2t^2 = c^2\vartheta^2 + z^2$ (hyperbola)



cuts of hyperbolas with coordinate axis fix scale

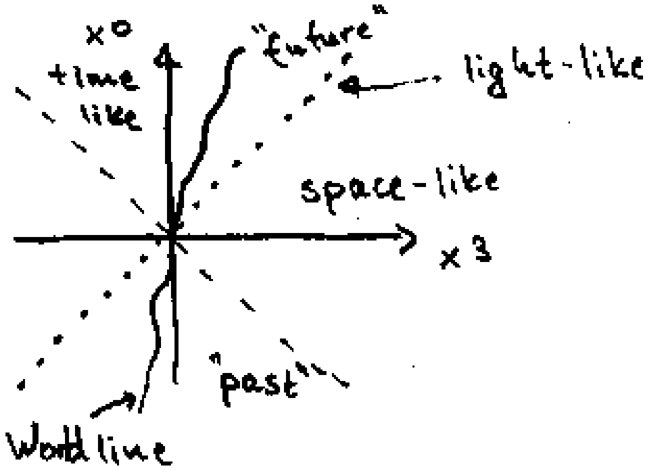
types of 4-vectors, worldline

$$s^2 = x^\mu x_\mu = c^2t^2 - (x^2 + y^2 + z^2)$$

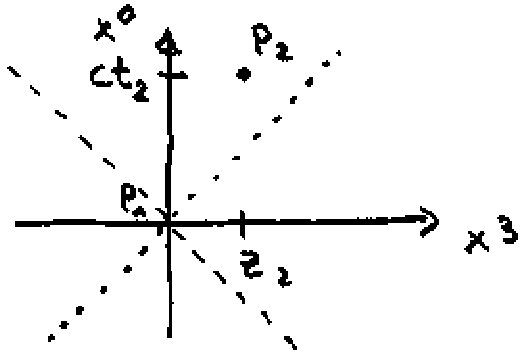
time-like 4 vector $\quad s^2 > 0 \ , \ |ct| > |\vec{r}|$

light-like 4 vector $\quad s^2 = 0 \ , \ |ct| = |\vec{r}|$

space-like 4 vector $\quad s^2 < 0 \ , \ |ct| < |\vec{r}|$



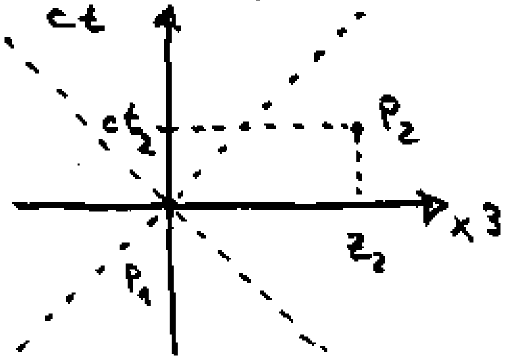
events with time-like distance $\quad s^2 > 0$



exists reference frame in which P_1 and P_2 is at the same place: boost with

$$\beta = \frac{z_2}{ct_2}$$

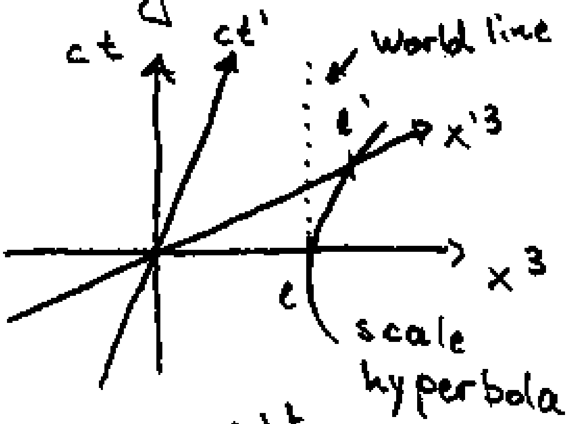
events with space-like distance



Lorentz boost with
$\beta = \frac{ct_2}{z_2}$
(events at the same time)

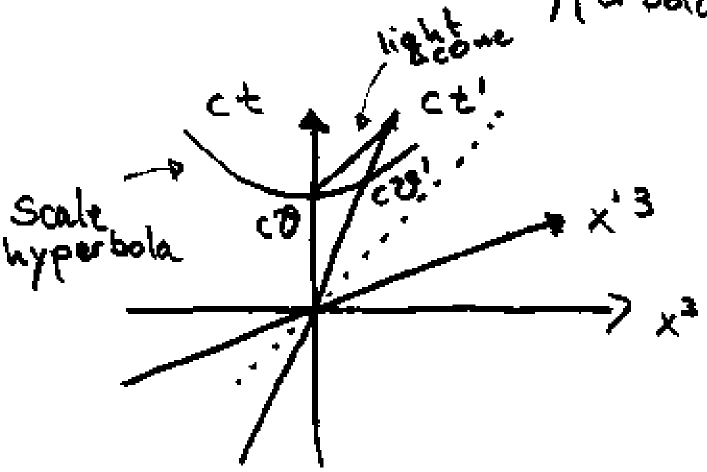
Length contraction and time dilatation in Minkowski diagrams

1)



→ rod appears shorter on scale of x'^3 axis

2)



light cone of event at $x^3 = 0$ reaches ct' axes at time $> c\vartheta'$

Covariant formulation of classical mechanics

goal: write physical laws such that these are ~~for~~ invariant under Lorentz transformat

note: $\vec{F} = \dot{\vec{p}}$ is only Galilei invariant

Reminder: tensors of nth rank

$n = 0$ scalars invariant

example: $\vec{r}\cdot\vec{r} = r^2$ in $\mathbb{R}^n$

$ds^2 = dx^\mu dx_\mu$ in Minkowski space

$n = 1$ vectors (tensors of 1st rank)

example $\vec{r} \to \vec{r}\,' = D\vec{r} = \sum_{i,j=1}^{3} d^{ij} x^j \vec{e}_i$

$x^\mu \to x'^\mu = \Lambda^\mu{}_\nu x^\nu \quad (*)$

$n = 2$ tensors

example inertial tensor

$$J^{lm} = \int d^3\vec{r}\,\left(\delta^{lm}\vec{r}^{\,2} - x^l x^m\right)\rho(\vec{r})$$

$$J^{lm} \to J'^{lm} = \left(D J D^T\right)^{lm}$$

$$F^{\mu\nu} \to F'^{\mu\nu} = \Lambda^\mu{}_\alpha \Lambda^\nu{}_\beta F^{\alpha\beta}$$

Lorentz transformation as partial derivative:

$$\frac{\partial x'^\mu}{\partial x^\nu} = \Lambda^\mu{}_\nu \qquad \text{(use } *)$$

alternatively: $(\Lambda^{-1})^\nu{}_\mu x'^\mu = \overbrace{(\Lambda^{-1})^\nu{}_\mu \Lambda^\mu{}_{\nu'}}^{\delta^\nu{}_{\nu'}} x^{\nu'}$

$\Rightarrow \dfrac{\partial x^\nu}{\partial x'^\mu} = (\Lambda^{-1})^\nu{}_\mu$

contravariant vector

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}\, x^{\nu}$$

covariant vector

$$x'_{\mu} = (\Lambda^{-1})^{\nu}{}_{\mu}\, x_{\nu} = x_{\nu} (\Lambda^{-1})^{\nu}{}_{\mu}$$

(transforms as coordinate axis)

check correctness by Lorentz invariance:

$$x'_{\mu} x'^{\mu} = x_{\nu} (\Lambda^{-1})^{\nu}{}_{\mu'}\, \Lambda^{\mu'}{}_{\nu'}\, x^{\nu'} = x_{\nu}\, g^{\nu}{}_{\nu'} x^{\nu'} = x_{\nu} x^{\nu}$$

tensors of 2^{nd} rank

contravariant: $F^{\mu\nu}$ — each index transforms as contravariant vector

mixed $F^{\mu}{}_{\nu}$

covariant $F_{\mu\nu}$

Tensor contraction:

$$F^{\mu}{}_{\nu} \to F^{\mu}{}_{\mu} = \sum_{\mu=0}^{3} F^{\mu}{}_{\mu}$$

transformation: $F^{\mu}{}_{\mu} \to F'^{\mu}{}_{\mu} = \Lambda^{\mu}{}_{\alpha} F^{\alpha}{}_{\beta}\, (\Lambda^{-1})^{\beta}{}_{\mu}$

$$= (\Lambda^{-1})^{\beta}{}_{\mu}\, \Lambda^{\mu}{}_{\alpha}\, F^{\alpha}{}_{\beta}$$

$$= g^{\beta}{}_{\alpha} F^{\alpha}{}_{\beta} = F^{\beta}{}_{\beta}$$

$\to$ transforms as scalar

$A^{\mu}{}_{\nu\alpha\beta}$: tensor of 4th rank

$A^{\mu}{}_{\mu\alpha\beta}$ $\to$ transforms as covariant tensor of 2nd rank

Definitions of differential operators

4-gradient

a) $\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \vec{\nabla}\right)$ covariant vector

b) $\partial^\mu = \frac{\partial}{\partial x_\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, -\vec{\nabla}\right)^T$ contravariant vector

4-divergence

$$\partial^\mu a_\mu = \partial_\mu a^\mu = \frac{1}{c}\frac{\partial a^0}{\partial t} + \vec{\nabla}\vec{a} \qquad \text{(scalar)}$$

D'Alembert operator (box)

$$\Box = \partial_\mu \partial^\mu = \partial^\mu \partial_\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta \qquad \text{(scalar)}$$

differentials

$$dx^\mu = (c\,dt, d\vec{r})^T \qquad \text{(4-vector)}$$

$$ds^2 = d(x_\mu x^\mu) = c(dt)^2 - (d\vec{r})^2 \qquad \text{scalar}$$

proper time

$$d\tau^2 = \frac{1}{c^2} ds^2 \qquad \text{defined for time-like } ds^2 > 0$$

$$d\tau = \frac{1}{c}\sqrt{ds^2}$$

consider: object described

1) in a reference system where it rests

$d\vec{r} = 0$, $(d\vec{r})^2 = 0$

$ds^2 = c^2(dt)^2$

proper time $d\tau = \frac{1}{c}\sqrt{ds^2} = \frac{1}{c}\sqrt{c^2(dt)^2} = dt$

2) in an arbitrary reference frame

$$c^2(d\tau)^2 = c^2(dt^2) - (d\vec{r})^2 = c^2(dt)^2\left[1 - \frac{(d\vec{r})^2}{c^2(dt)^2}\right]$$

$$= c^2(dt)^2\left[1 - \frac{1}{c^2}\left(\frac{d\vec{r}}{dt}\right)^2\right]$$

$$\frac{d\vec{r}}{dt} = \vec{v} \longrightarrow \quad = c^2(dt)^2\left[1 - \frac{v^2}{c^2}\right] = c^2(dt)^2\left[1-\beta^2\right]$$

$$= c^2(dt)^2\,\frac{1}{\gamma^2}$$

$$\Rightarrow \quad d\tau = \frac{dt}{\gamma} \quad \Leftrightarrow \quad dt = \gamma\, d\tau$$

(time dilatation)

Physical quantities

1) 4-Velocity

$$u^\mu = \frac{dx^\mu}{d\tau}$$

is a contravariant vector since x^μ is one and τ is scalar

$$u^\mu = \frac{d}{d\tau}(x^\mu) = \frac{d}{d\tau}(ct, \vec{r})^T$$

$$= \gamma \frac{d}{dt}(ct, \vec{r}) = \gamma(c, \vec{v})$$

construct Lorentz scalar:

$$u^\mu u_\mu = \gamma^2(c^2 - \vec{v}^2) = \frac{1}{1-\beta^2}\left(c^2 - \frac{\beta^2}{c^2}\right)$$

$$= \frac{1}{1-\beta^2}(1-\beta^2)\,c^2 = c^2$$

2) 4-momentum

$p^\mu = m\,u^\mu$ (contravariant vector) $\quad$ m: mass, Lorentz scalar construct Lorentz scalar

$$p^\mu p_\mu = m^2 u^\mu u_\mu = m^2 c^2$$

a closer look at the components

$$p^\mu = (p^0, \vec{p})^T = m\gamma (c, \vec{v})^T \qquad p^0 = m\gamma c, \quad \vec{p} = m\gamma \vec{v}$$

for $v \ll c$, we have $\gamma = 1 + O\left(\frac{v^2}{c^2}\right)$

$$\vec{p} = m\gamma\vec{v} \approx m\vec{v} = \vec{p}_{\text{nonrel}}$$

3) relativistic generalization of Newton's equation

4 - force (Minkowski force)

$$K^\mu = \frac{d}{d\tau} p^\mu = m \frac{du^\mu}{d\tau}$$

a closer look at the components

$$K^\mu = (K^0, \vec{K}) = \left(\gamma \frac{dp^0}{dt}, \gamma \frac{d\vec{p}}{dt}\right)^T$$

$$K^0 = \gamma \frac{dp^0}{dt} = \gamma \frac{d}{dt}(m\gamma c) = m\gamma c \frac{d\gamma}{dt}$$

$$\vec{K} = m\gamma \frac{d(\gamma\vec{v})}{dt}$$

<u>define</u> a 3-vector force

$$\vec{F} = \frac{d\vec{p}}{dt} \qquad \text{such that} \quad \vec{K} = \gamma \vec{F}$$

and we get the nonrelativistic limit for $v \ll c$

$$\vec{F} = \frac{d\vec{p}}{dt} = \frac{d}{dt}\left(\gamma \vec{p}_{\text{nonrel}}\right) \approx \frac{d\vec{p}_{\text{nonrel}}}{dt} = \vec{F}_{\text{nonrel}}$$

(Newton's equation)

4) Relativistic energy momentum relation

Question: How to interpret K^0 ?

construct Lorentz scalar

$$K^\mu u_\mu = m \frac{du^\mu}{d\tau} u_\mu \overset{\text{product rule + metric tensor } g^{\mu\nu}}{=} \frac{m}{2} \frac{d}{d\tau}\left(u^\mu u_\mu\right)$$
$$= \frac{m}{2} \frac{d}{d\tau}\left(c^2\right) = 0$$

in other notation: $K \cdot U = 0$
4-force is "orthogonal" to 4-velocity

a closer look to components

$$0 = K^\mu u_\mu = K^0 \gamma c - \vec{K} \cdot (\gamma \vec{v})$$
$$= \gamma c \left(K^0 - \gamma \vec{F} \cdot \frac{\vec{v}}{c} \right)$$

read off: $K^0 = \gamma \dfrac{\vec{F} \cdot \vec{v}}{c}$

nonrelativistic mechanics $\vec{F} \cdot \vec{v} = -P$ (power)

we obtain:

$$-P = \vec{F} \cdot \vec{v} = \frac{c}{\gamma} K^0 \overset{K^0 = m\gamma c \frac{d\gamma}{dt}}{=} m c^2 \frac{d\gamma}{dt} = \frac{d}{dt}\left(m \gamma c^2\right)$$

more on nonrelativistic mechanics:

$P = -\dot{T}$ $\qquad (dW = -dT)$

we can read off the relativistic generalization of the kinetic energy

$$T = m \gamma c^2 \quad , \text{ then } \quad -P = \vec{F} \cdot \vec{v} = \frac{dT}{dt} = \dot{T}$$