nonrelativistic limit : $v \ll c$

$$T = m\gamma c^2 \approx m\left(1 + \frac{v^2}{2c^2}\right)c^2 = mc^2 + \frac{1}{2}mv^2$$

rest energy

nonrelativistic kin. energy

sometimes one defines

$$E_0 = mc^2 \qquad \text{(rest energy)}$$

rewrite force:

$$K^\mu = \gamma\left(\frac{1}{c}\frac{dT}{dt}, \vec{F}\right)^T = \gamma\left(\frac{1}{c}\frac{dT}{dt}, \frac{d\vec{p}}{dt}\right)$$

$$= \gamma \frac{d}{dt}\left(\frac{T}{c}, \vec{p}\right)^T = \frac{d}{d\tau}\left(\frac{T}{c}, \vec{p}\right)^T$$

identify $p^0 = m\gamma c = \frac{T}{c}$

rewrite equation for Lorentz scalar

$$\frac{T^2}{c^2} = (p^0)^2 = p^\mu p_\mu + \vec{p}^{\,2} = m^2c^2 + p^2$$

$$\Leftrightarrow T^2 = p^2c^2 + m^2c^4$$

$$\Rightarrow T = \pm\sqrt{p^2c^2 + m^2c^4}$$

(relativistic energy momentum relation)

Notes

a) two possible values (sign), also solutions with negative kinetic energy within special relativity
→ physical interpretation : antiparticles

b) 3-momentum reads

$$\vec{p} = m\gamma\vec{v}$$

sometimes one defines

$$m(v) = m\gamma(v) = \frac{m}{\sqrt{1 - \frac{v^2}{c^2}}}$$

relativistic "increase" of mass

here: mass is considered as Lorentz scalar

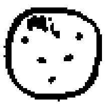
more correct: kinetic energy increases

$$T = m\gamma c^2 \geq mc^2 = E_0$$

Equivalence of mass and energy

Consider: explosive charge (bomb)

before explosion



nonrelativistic kinetic energy

$$T_{nonrel} = \frac{1}{2}\sum_i m_i v_i^2 = 0$$

after explosion

$$\tilde{T}_{nonrel} = \frac{1}{2}\sum_i m_i \tilde{v}_i^2 \neq 0$$

relativistic kinetic energy conservation

$$0 = T - \tilde{T} = \sum_i \left(m_i \gamma_i c^2 - m_i \tilde{\gamma}_i c^2\right) = \sum_i m_i c^2 \big(\underbrace{\ }_{\gamma_i = 0} 1 - \tilde{\gamma}_i\big)$$

$$\Rightarrow \tilde{\gamma}_i = 0 \quad \text{, i.e.} \quad \tilde{v}_i = 0 \qquad \text{(contradiction)}$$

What is wrong?

→ We used that $m_i = \text{const}$ for all particles.

Assuming $m_i \neq \tilde{m}_i$, we get

$$0 = T - \tilde{T} = \sum_i \left(m_i \gamma_i c^2 - \tilde{m}_i \tilde{\gamma}_i c^2\right) = \sum_i \Big[\tilde{m}_i c^2 (\tilde{\gamma}_i - 1) - (m_i c^2 - \tilde{m}$$

introducing $\Delta E_i = \Delta m_i c^2 = (m_i - \tilde{m}_i) c^2$

we can rewrite the equation as

$$\sum_i \Delta E_i = \sum_i \tilde{m}_i c^2 (\tilde{\gamma}_i - 1) \underset{\tilde{\gamma}_i = 1 + \frac{\tilde{v}_i^2}{2c^2}}{\approx} \frac{1}{2} \sum_i \tilde{m}_i \tilde{v}_i^2 > 0$$

$\rightarrow$ at least one "particle" has
$\Delta E_i > 0$ (loss of mass)

Where does the energy come from?

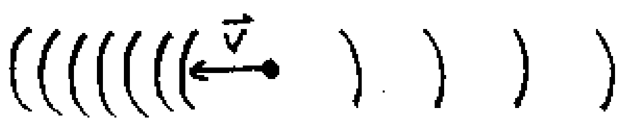
$\rightarrow$ binding energy that form particles
a) chemical binding energy
b) nuclear binding energy

$\rightarrow$ particle / antiparticle reactions

$$e^+ + e^- \rightarrow 2\gamma \leftarrow \text{"photons"}$$

Doppler effect and aberration

classical



frequency shift of sound waves (in air)

relativistic: light

momentum $p^\mu = \left(\frac{T}{c}, \vec{p}\right)$, $p^\mu p_\mu = mc^2 \underset{m=0}{=} 0$

relativistic kinetic energy

$$T = \sqrt{p^2 c^2 + \underset{0}{\underbrace{m^2}} c^4} = pc$$

photon : $T = \hbar\omega = \frac{h\nu}{c} = \hbar k = \frac{h}{\lambda}$

$\hbar$: Planck constant

$\frac{\omega}{2\pi} = \nu$: frequency

$k = \frac{\omega}{c}$ wave number

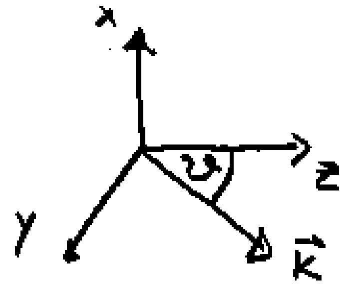
$\lambda = \frac{c}{\nu}$ wavelength

conveniently define a 4-vector

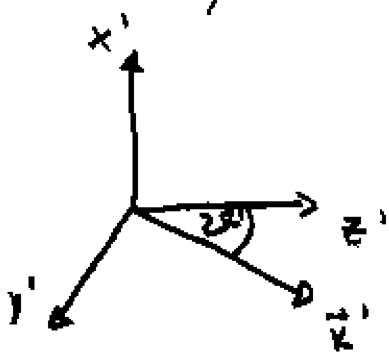
$k^\mu = \frac{p^\mu}{\hbar} = \left(\frac{\omega}{c}, \vec{k}\right)$, $k^\mu k_\mu = 0$

transforms as 4-vector : $k'^\mu = \Lambda^\mu{}_\nu k^\nu$

consider : light source in reference system Σ



observer in relatively moving reference system Σ'



Lorentz transformation

$$\left.\begin{aligned} \frac{\omega'}{c} &= \gamma\left(\frac{\omega}{c} - \beta k^z\right) \\ k'^x &= k^x \\ k'^y &= k^y \\ k'^z &= \gamma\left(k^z - \beta\frac{\omega}{c}\right) \end{aligned}\right\}$$

looking for :

$\nu'(\nu, \vartheta)$

$\nu'(\nu, \vartheta')$

$\vartheta'(\nu, \vartheta)$

convenient choice of coordinate systems

$$k^z = k\cos\vartheta \qquad k'^z = k'\cos\vartheta'$$
$$k^y = 0 \qquad k'^y = 0$$
$$k^x = k\sin\vartheta \qquad k'^x = k'\sin\vartheta'$$

from $k^\mu k_\mu = 0 = \left(\frac{\omega}{c}\right)^2 - \vec{k}^2$ follows $k = \frac{\omega}{c}$

$$k^z = \frac{\omega}{c}\cos\vartheta \qquad k'^z = \frac{\omega'}{c}\cos\vartheta'$$

use zero component of transformation

$$\frac{\omega'}{c} = \gamma\left(\frac{\omega}{c} - \beta\frac{\omega}{c}\cos\vartheta\right)$$

and third component

$$\frac{\omega'}{c}\cos\vartheta' = \gamma\left(\frac{\omega}{c}\cos\vartheta - \beta\frac{\omega}{c}\right)$$

multiply with $\frac{c}{2\pi}$

$$(1)\quad \nu' = \gamma\nu\,(1-\beta\cos\vartheta)$$
$$(2)\quad \nu'\cos\vartheta' = \gamma\nu\,(\cos\vartheta - \beta)$$

2 equations
4 parameters $\nu, \nu', \vartheta, \vartheta'$

$$(2)\quad \cos\vartheta' = \frac{\nu}{\nu'}\gamma(\cos\vartheta - \beta) \overset{(1)}{=} \frac{\nu\gamma(\cos\vartheta-\beta)}{\gamma\nu\,(1-\beta\cos\vartheta)}$$
$$= \frac{\cos\vartheta - \beta}{1-\beta\cos\vartheta} \to \vartheta'(\nu,\vartheta)$$

(aberration)

solve for $\cos\vartheta$:

$$\cos\vartheta = \frac{\beta + \cos\vartheta'}{\beta\cos\vartheta' + 1}$$

$$(1)\quad \nu' = \gamma\nu\left(1 - \beta\,\frac{\beta+\cos\vartheta'}{\beta\cos\vartheta'+1}\right) = \nu\gamma\,\frac{1-\beta^2}{1+\cos\vartheta'\beta}$$

$\gamma = \frac{1}{\sqrt{1-\beta^2}}$

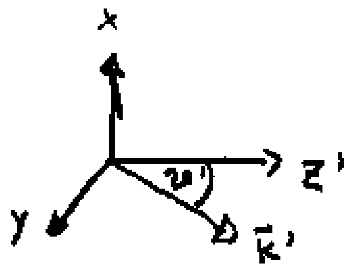
$$= \nu\,\frac{\sqrt{1-\beta^2}}{1+\cos\vartheta'\beta} \to \nu'(\nu,\vartheta')$$

(Doppler effect)

special cases

1) $\cos\vartheta' = -1$ $\qquad \vartheta' = \pi$

$$\frac{\nu}{\nu'} = \frac{\sqrt{1-\beta^2}}{1-\beta} = \sqrt{\frac{1+\beta}{1-\beta}} \geq 1$$

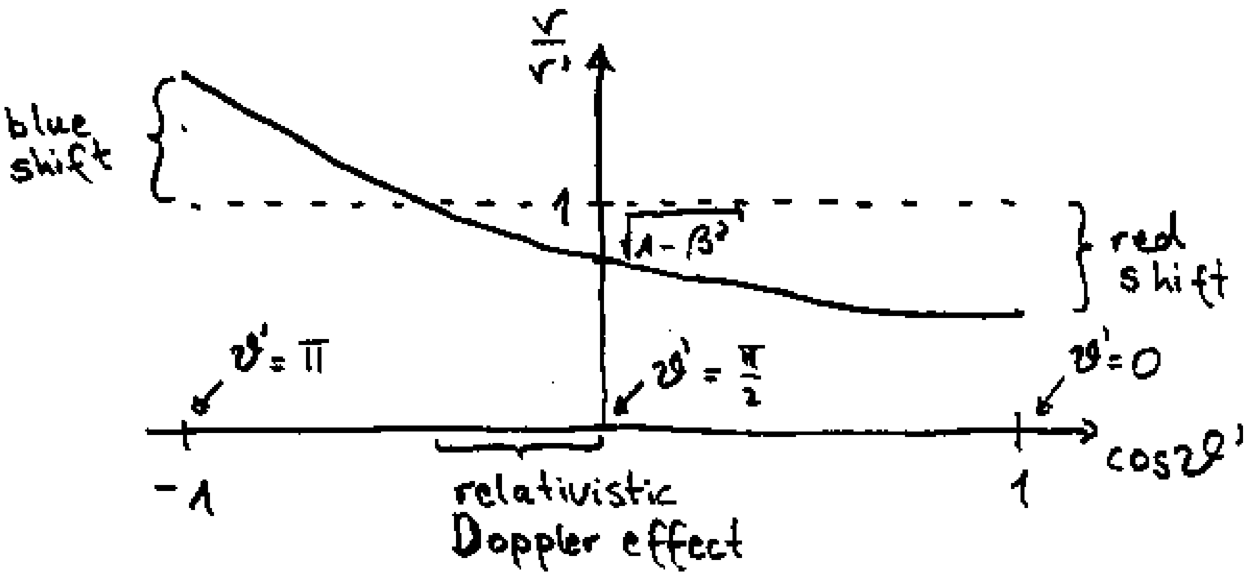


2) $\cos\vartheta' = 1$ $\qquad \vartheta' = 0$

$$\frac{\nu}{\nu'} = \frac{\sqrt{1-\beta^2}}{1+\beta} = \sqrt{\frac{1-\beta}{1+\beta}} \leq 1$$

3) $\cos\vartheta' = 0$ $\qquad \vartheta' = \frac{\pi}{2}$

$$\frac{\nu}{\nu'} = \sqrt{1-\beta^2} \leq 1$$



Aberration $\qquad \cos\vartheta' = \dfrac{\cos\vartheta - \beta}{1-\beta\cos\vartheta}$

1) $\cos\vartheta = -1$, $\cos\vartheta' = -1$

2) $\cos\vartheta = 1$, $\cos\vartheta' = 1$

3) $\cos\vartheta = 0$, $\cos\vartheta' = -\beta$