7) Electrodynamics II

reminder: Maxwell's equations, potentials

$\vec{\nabla} \cdot \vec{B} = 0$ (no magnetic monopoles)

$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$ (Faraday's law of induction)

$\vec{\nabla} \cdot \vec{D} = \varrho$ (Coulomb's law)

$\vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{j}$ (Ampère's law + Maxwell's displacement)

material equations (linear media) $\vec{B} = \mu_0 \mu_r \vec{H}$, $\vec{D} = \epsilon_0 \epsilon_r \vec{E}$

continuity equation $\vec{\nabla} \cdot \vec{D} + \frac{\partial \varrho}{\partial t} = 0$

"scalar" potential $\varphi(\vec{r}, t)$, vector potential $\vec{A}(\vec{r}, t)$ $\Big\}$ $\vec{B} = \vec{\nabla} \times \vec{A}$, $\vec{E} = -\vec{\nabla}\varphi - \frac{\partial \vec{A}}{\partial t}$

(homogeneous wave equations trivially fulfilled)

2nd order D.E.:

$$\Box \varphi(\vec{r}, t) = -\frac{\varrho(\vec{r}, t)}{\epsilon_0 \epsilon_r}$$

$$\Box \vec{A}(\vec{r}, t) = -\mu_0 \mu_r \vec{j}(\vec{r}, t)$$

$\vec{\nabla} \cdot \vec{A} + \frac{1}{\mu_r \epsilon_r c^2} \frac{\partial \varphi}{\partial t} = 0$ (Lorenz gauge)

Question: Covariant formulation? (in vacuum) $\epsilon_r = 0$, $\mu_r = 0$

4-current

$$j^\mu = (c\varrho, \vec{j})^T$$

transforms as 4-vector: $j'^\mu = \Lambda^\mu{}_\nu j^\nu$

Consider : resting charge in Σ

$$j^\mu = (c\rho_0, \vec{0}) \qquad \rho_0 : \text{charge density in resting frame}$$

in relatively moving system Σ' :

$$j'^\mu = (c\rho, \rho\vec{v}) \qquad \rho : \text{charge density as measured in } \Sigma'$$

length contraction $\rho = \frac{\Delta q}{\Delta V'} = \frac{\Delta q}{\Delta x' \Delta y' \Delta z'} = \frac{\Delta q}{\Delta x \Delta y \Delta z}\gamma$

$$= \rho_0 \gamma$$

$$\Rightarrow j'^\mu = \rho_0 \gamma (c, \vec{v}) = \rho_0 u'^\mu \qquad u'^\mu : \text{4-velocity}$$

Covariant equations:

(1) $$\partial_\mu j^\mu = \frac{1}{\partial(ct)} j^0 + \frac{\partial}{\partial x^i} j^i \qquad \frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{j} = 0$$

$$= \frac{1}{c}\frac{\partial}{\partial t}(\rho c) + \frac{\partial}{\partial x^i} j^i = 0$$

(2) reminder $\Box = \partial_\nu \partial^\nu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \Delta$

$$-\mu_0 j^\mu = \partial_\nu \partial^\nu A^\mu$$

$$-\mu_0 j^0 = -\mu_0 c\rho = -\underbrace{\mu_0 \epsilon_0}_{\frac{1}{c^2}} c \frac{\rho}{\epsilon_0} = -\frac{1}{c}\frac{\rho}{\epsilon_0} = \frac{1}{c}\Box\varphi$$

$$-\mu_0 j^i = \Box A^i$$

read off 4-vector potential

$$A^\mu = \left(\frac{\varphi}{c}, \vec{A}\right)$$

φ : transforms as 0-component !

Lorenz-gauge : $\partial_\mu A^\mu = 0$

Construct further covariant quantities

field strength tensor: $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

tensor of 2^{nd} rank, antisymmetric : $F^{\mu\nu} = -F^{\nu\mu}$

$\rightarrow$ has 6 independent nonzero elements

explicitely write out :

$$F^{10} = \partial^1 A^0 - \partial^0 A^1$$
$$= -\frac{\partial}{\partial x}\left(\frac{\varphi}{c}\right) - \frac{\partial}{\partial (ct)} A^x = \frac{1}{c}\left(-\frac{\partial \varphi}{\partial x} - \frac{\partial A^x}{\partial t}\right)$$
$$= \frac{E^y}{c}$$

$$F^{20} = \partial^2 A^0 - \partial^0 A^2 = \frac{1}{c}\left(-\frac{\partial \varphi}{\partial y} - \frac{\partial A^y}{\partial t}\right) = \frac{E^y}{c}$$
$$F^{30} = \partial^3 A^0 - \partial^0 A^3 = \frac{E^z}{c}$$

$$F^{21} = \partial^2 A^1 - \partial^1 A^2$$
$$= -\frac{\partial}{\partial y} A^x + \frac{\partial}{\partial x} A^y = B^z$$

$$F^{31} = \partial^3 A^1 - \partial^1 A^3$$
$$= -\frac{\partial}{\partial z} A^x + \frac{\partial}{\partial x} A^z = -B^y$$

$$F^{23} = \partial^2 A^3 - \partial^3 A^2$$
$$= -\frac{\partial}{\partial y} A^z + \frac{\partial}{\partial z} A^y = -B^x$$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -\frac{E^x}{c} & -\frac{E^y}{c} & -\frac{E^z}{c} \\ \frac{E^x}{c} & 0 & -B^z & B^y \\ \frac{E^y}{c} & B^z & 0 & -B^x \\ \frac{E^z}{c} & -B^y & B^x & 0 \end{pmatrix}$$

transformation behavior of the electromagnetic fields:

$$F'^{\mu\nu} = \Lambda^{\mu}{}_{\alpha}\,\Lambda^{\nu}{}_{\beta}\,F^{\alpha\beta}$$

$$= \Lambda^{\mu}{}_{\alpha}\,F^{\alpha\beta}\,(\Lambda^{T})_{\beta}{}^{\nu} \quad \text{(matrix multiplication)}$$

$$(\dots) = \begin{pmatrix} 0 & -\gamma\left(\frac{E^x}{c}-\beta B^y\right) & -\gamma\left(\frac{E^y}{c}+\beta B^x\right) & -\frac{E^z}{c} \\ \gamma\left(\frac{E^x}{c}-\beta B^y\right) & 0 & -B^z & \gamma\left(B^y-\beta\frac{E^x}{c}\right) \\ \gamma\left(\frac{E^y}{c}+\beta B^x\right) & B^z & 0 & -\gamma\left(B^x+\beta\frac{E^y}{c}\right) \\ \frac{E^z}{c} & -\gamma\left(B^y-\beta\frac{E^x}{c}\right) & \gamma\left(B^x+\beta\frac{E^y}{c}\right) & 0 \end{pmatrix}$$

read off transformation for each component

$$E'^x = \gamma\left(\frac{E^x}{c}-\beta B^y\right) \qquad B'^x = \gamma\left(B^x+\beta\frac{E^y}{c}\right)$$

$$E'^y = \gamma\left(\frac{E^y}{c}+\beta B^x\right) \qquad B'^y = \gamma\left(B^y-\beta\frac{E^x}{c}\right)$$

$$E'^z = E^z \qquad B'^z = B^z$$

Notes:
- components longitudinal to boost direction are unchanged (in contrast to transformation of 4-vectors)
- Lorentz transformation mixes components of electric field and magnetic induction

$$\vec{E}' = \vec{E}'_{\parallel} + \vec{E}'_{\perp} \qquad \vec{E}'_{\parallel} = \vec{E}_{\parallel}$$

$$\vec{E}'_{\perp} = \gamma\left(\vec{E}_{\perp} + c\,(\vec{\beta}\times\vec{B})\right) \qquad \vec{\beta} = \frac{\vec{v}}{c}$$

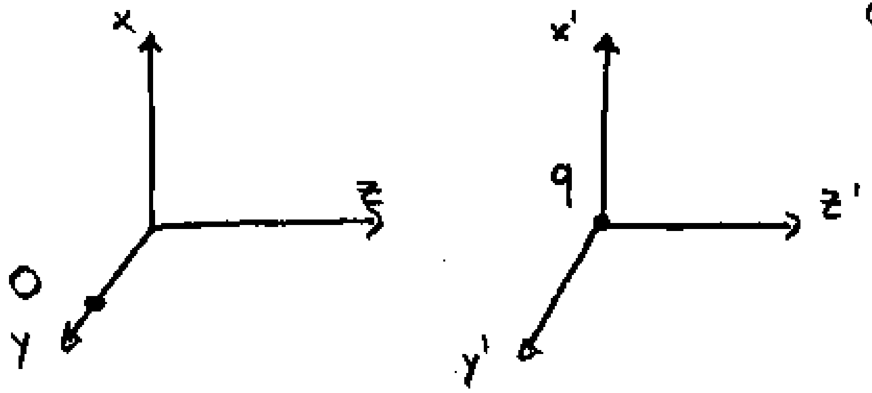
$$\vec{B}' = \vec{B}'_{\parallel} + \vec{B}'_{\perp} \qquad \vec{B}'_{\parallel} = \vec{B}_{\parallel}$$

$$\vec{B}'_{\perp} = \gamma\left(\vec{B}_{\perp} - \frac{1}{c}(\vec{\beta}\times\vec{E})\right)$$

Example: Field of a moving point charge

two reference frames

Σ : system of observer Σ' : reference frame where charge is at rest



$t = t' = 0$
origin of Σ and Σ' identical

position of observer

$\vec{r}_0 = (0, y_0, 0)$ $\vec{r}_0{}' = (0, y_0, -vt')$ (moving observer)

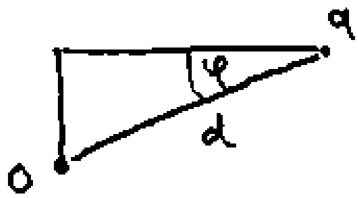
Lorentz transformation for zero component:

$$t' = \gamma\left(t - \frac{v}{c^2}\underset{\overset{\shortparallel}{0}}{z}\right) = \gamma t$$

parametrize:

$$d = |\vec{r}_0{}'| = \sqrt{y_0^2 + v^2 t'^2} = \sqrt{y_0^2 + v^2 \gamma^2 t^2}$$

$y_0 = d \sin\varphi$
$vt' = d \cos\varphi$



fields in Σ': point charge at rest

$$\left.\begin{aligned} \vec{B}' &= 0 \\ \vec{E}' &= \frac{q}{4\pi\epsilon_0} \frac{\vec{r}'}{r'^3} \end{aligned}\right\}$$ evaluate at $\vec{r}_0{}'$:

$$\vec{E}' = \frac{q}{4\pi\epsilon_0} \frac{(0, y_0, -vt')}{\sqrt{y_0^2 + v^2 t'^2}^{\,3}}$$

$$= E'^y \vec{e}_y + E'^z \vec{e}_z$$

transformation of fields:

$$E^x = \gamma\left(E'^x + vB'^y\right) = 0$$
$$E^y = \gamma\left(E'^y - vB'^x\right) = \gamma E'^y$$
$$E^z = E'^z$$

$$B^x = \gamma\left(B'^x - \frac{v}{c^2}E'^y\right) = -\gamma\frac{v}{c^2}E'^y$$
$$B^y = \gamma\left(B'^y + \frac{v}{c^2}E'^x\right) = 0$$
$$B^z = B'^z = 0$$

$$\sqrt{y_0^2+v^2\gamma^2t^2} = d\gamma\sqrt{\frac{1}{\gamma^2}\sin^2\varphi+\cos^2\varphi} = d\gamma\sqrt{1-\beta^2\sin^2\varphi}$$

more explicitely:

$$E^y = \gamma E'^y = \gamma\frac{q}{4\pi\epsilon_0}\frac{y_0}{\sqrt{y_0^2+v^2\gamma^2t^2}^3} = \frac{\gamma q}{4\pi\epsilon_0}\frac{d\sin\varphi}{d^3\gamma^3\sqrt{1-\beta^2\sin^2\varphi}^3}$$

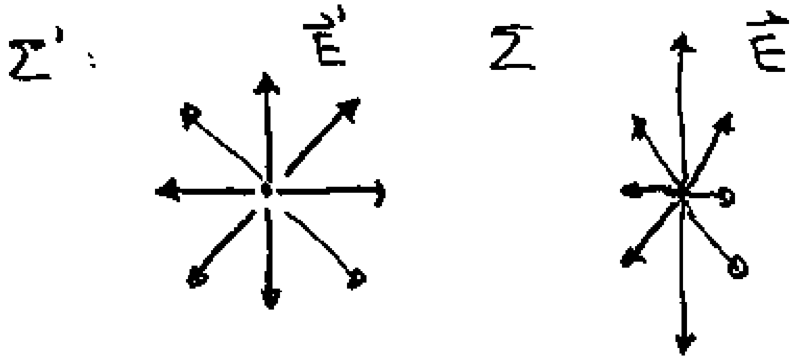
$$E^z = -\gamma\frac{q}{4\pi\epsilon_0}\frac{v\gamma t}{\sqrt{y_0^2+v^2\gamma^2t^2}} = -\frac{\gamma q}{4\pi\epsilon_0}\frac{d\cos\varphi}{d^3\gamma^3\sqrt{1-\beta^2\sin^2\varphi}^3}$$

$$\vec{E} = \frac{q}{4\pi\epsilon_0}\frac{1}{d^2\gamma^2}\frac{(0,\sin\varphi,-\cos\varphi)}{\sqrt{1-\beta^2\sin^2\varphi}^3}$$

note: in Σ': isotropic field $|\vec{E}'| = \frac{q}{4\pi\epsilon_0}\frac{1}{d^2}$

in Σ: anisotropic field

$$|\vec{E}| = \frac{q}{4\pi\epsilon_0}\frac{1}{d^2\gamma^2}\frac{1}{\sqrt{1-\beta^2\sin^2\varphi}^3}$$



(time dependence due to position of charge)

limiting special cases

$$\frac{|E(0)|}{|E'|} = \frac{E(\pi)}{|E'|} \overset{\sin\varphi=0}{=} \frac{1}{\gamma^2} < 1 \qquad \frac{E(\frac{\pi}{2})}{E'} = \frac{E(-\frac{\pi}{2})}{E'} \overset{\sin^2\varphi=1}{=} \gamma > 1$$

magnetic induction

$$\vec{B} = B^x \vec{e}_x = -\gamma \frac{v}{c^2} E'^y \vec{e}_x = \gamma \frac{\vec{v} \times \vec{E}'}{c^2}$$

consider $\vec{v} = v\vec{e}_z$: $\vec{v} \times \vec{E}' = -v\, E'^y \vec{e}_x$

substitute back $\vec{E}' = \frac{q}{4\pi\epsilon_0} \frac{\vec{r}'}{r'^3}$

$$\vec{B} = \frac{\gamma q}{4\pi\epsilon_0} \epsilon_0 \mu_0 \frac{\vec{v} \times \vec{r}'}{r^3} = \frac{\gamma q \mu_0}{4\pi} \frac{\vec{v} \times \vec{r}'}{r^3}$$

(Biot - Savart law for $v \ll c$, $\gamma \approx 1$)

Lorentz invariants in electrodynamics

q : charge

define : dual field tensor

$$\tilde{F}^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} F_{\alpha\beta} = \frac{1}{2} \epsilon^{\mu\nu\alpha\beta} g_{\alpha\eta} g_{\beta\varsigma} F^{\eta\varsigma}$$

$\epsilon^{\mu\nu\alpha\beta}$: antisymmetric tensor of rank 4

$$\epsilon^{\mu\nu\alpha\beta} = \begin{cases} +1 & (\mu\nu\alpha\beta) \text{ even permutation of } 0123 \\ -1 & (\mu\nu\alpha\beta) \text{ odd permutation of } 0123 \\ 0 & \text{otherwise} \end{cases}$$

antisymmetric $\tilde{F}^{\mu\nu} = -\tilde{F}^{\nu\mu}$ (6 independent components)

explicit calculation yields:

$$\tilde{F}^{\mu\nu} = \begin{pmatrix} 0 & -B^x & -B^y & -B^z \\ B^x & 0 & \frac{E^z}{c} & -\frac{E^y}{c} \\ B^y & -\frac{E^z}{c} & 0 & \frac{E^x}{c} \\ B^z & \frac{E^y}{c} & -\frac{E^x}{c} & 0 \end{pmatrix}$$

$$F^{\mu\nu} \to \tilde{F}^{\mu\nu} \; : \quad \frac{\vec{E}}{c} \to +\vec{B}, \quad \vec{B} \to -\frac{\vec{E}}{c}$$

Invariants by contraction

$$F_{\mu\nu} F^{\mu\nu} = F_{0i} F^{0i} + F_{i0} F^{i0} + F_{ij} F^{ij}$$

$$g^{\mu\nv} \quad = -F^{0i} F^{0i} - F^{i0} F^{i0} + F^{ij} F^{ij}$$

$$= -2 \frac{E^i E^i}{c^2} + 2 B^i B^i = 2 \left(\vec{B}^2 - \frac{\vec{E}^2}{c^2} \right)$$

Note : for a pure $\vec{B}$ field in a reference frame, we have $F_{\mu\nu} F^{\mu\nu} > 0$, so this cannot be transformed to a pure $\vec{E}$ - field which has $F_{\mu\nu} F^{\mu\nu} < 0$

$$F_{\mu\nu} \tilde{F}^{\mu\nu} = -F^{0i} \tilde{F}^{0i} - F^{i0} \tilde{F}^{i0} + F^{ij} \tilde{F}^{ij}$$

$$= -2 \frac{E_i}{c} B^i - 2 B^i \frac{E^i}{c} = -\frac{4}{c} \vec{E} \cdot \vec{B}$$

third invariant $\tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} = -2 \left(\vec{B}^2 - \frac{\vec{E}^2}{c^2} \right)$

(check by replacement rule)

Maxwell's equations in covariant form:

need : first order D.E of fields $\vec{E}, \vec{B}$

$(\vec{E}, \vec{B}) \sim F^{\mu\nu}$

inhomogeneous M.E. $\vec{\nabla} \cdot \vec{D} = \rho$

multiply with $c\mu_0$: $\mu_0 c \rho = \mu_0 c \epsilon_0 \vec{\nabla} \cdot \vec{E}$

$$\mu_0 j^0 = \frac{\partial}{\partial x} \frac{E^x}{c} + \frac{\partial}{\partial y} \frac{E^y}{c} + \frac{\partial}{\partial z} \frac{E^z}{c}$$

$$= \partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30}$$

inhomogeneous M.E. $\vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{j}$

multiply with μ_0, use $\vec{D} = \epsilon_0 \vec{E}$, $\vec{H} = \frac{1}{\mu_0} \vec{B}$

$$\vec{\nabla} \times \vec{B} - \frac{1}{c^2} \frac{\partial \vec{E}}{\partial t} = \mu_0 \vec{j}$$

$$\mu_0 j^1 = \frac{\partial}{\partial y} B^z - \frac{\partial}{\partial z} B^y - \frac{1}{c^2} \frac{\partial E^x}{\partial t} \qquad \partial_1 F^{11} = 0$$

$$= \partial_2 F^{21} + \partial_3 F^{31} + \partial_0 F^{01} = \partial_\alpha F^{\alpha 1}$$

$$\mu_0 j^2 = \frac{\partial}{\partial z} B^x - \frac{\partial}{\partial x} B^z - \frac{1}{c^2} \frac{\partial E^y}{\partial t}$$

$$= \partial_3 F^{32} + \partial_1 F^{12} + \partial_0 F^{02} = \partial_\alpha F^{\alpha 2}$$

$$\mu_0 j^3 = (\ldots) = \partial_\alpha F^{\alpha 3}$$

all together: $\partial_\alpha F^{\alpha\beta} = \mu_0 j^\beta$

homogeneous M.E.:

$$\vec{\nabla} \cdot \vec{B} = 0 = \frac{\partial}{\partial x} B^x + \frac{\partial}{\partial y} B^y + \frac{\partial}{\partial z} B^z$$

$$= \partial_1 F^{32} + \partial_2 F^{13} + \partial_3 F^{21}$$

$\partial_i = -\partial^i$, $F^{ij} = -F^{ji}$

$$\stackrel{\uparrow}{=} \partial^1 F^{23} + \partial^2 F^{31} + \partial^3 F^{12}$$

permutation (1 2 3)

similar for components of: $\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$

$$0 = \frac{1}{c}\frac{\partial}{\partial y} E^z - \frac{1}{c}\frac{\partial}{\partial z} E^y + \frac{1}{c}\frac{\partial B^x}{\partial t} = \partial_2 F^{30} + \partial_3 F^{02} - \partial_0 F^{23}$$

$$= -\left(\partial^2 F^{30} + \partial^3 F^{02} + \partial^0 F^{23}\right)$$

permutation (2 3 0)

(similar for the other two components)

all together: $\partial^\alpha F^{\beta\gamma} + \partial^\beta F^{\gamma\alpha} + \partial^\gamma F^{\alpha\beta} = 0$

(Jacobi identity)

α, β, γ arbitrary $(0, 1, 2, 3)$

(all together $4^3 = 64$ equations)

alternative formulation:

$$\partial_\alpha \tilde{F}^{\alpha\beta} = 0 \qquad \beta = 0, 1, 2, 3$$

(4 equations)

Covariant Lagrange formulation of electrodynamics

reminder: $L = L(\vec{q}, \dot{\vec{q}}, t) \qquad \frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0$

$$L = \int_V \mathcal{L}\, dV \qquad \mathcal{L}: \text{ Lagrange density}$$

$$\mathcal{L} = \mathcal{L}\left(\varphi, \frac{\partial \varphi}{\partial t}, \vec{\nabla}\varphi, t, \vec{r}\right)$$

$$= \mathcal{L}\left(\varphi, \partial^\mu \varphi, x^\mu\right)$$

$$\frac{\partial \mathcal{L}}{\partial \varphi} - \frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial(\partial^\mu \varphi)}\right) = 0 \qquad \text{(equations of motion)}$$

Lagrange density of the electromagnetic field?

$$\mathcal{L} = \mathcal{L}(A^\mu, \partial^\nu A^\mu, x^\mu)$$

also dependence via $j^\nu(x^\mu)$

$$S = \int_{t_1}^{t_2} \tilde{L}(t)\,dt = \int_\Omega \mathcal{L}\, dt\, dV$$

(action)

(Lorentz invariant)

How does the "4-volume" transform

$$c\,dt\,dV \to c\,dt'\,dV' = c\,dt\,dV\; F^{(xx')}$$

Jacobean determinan

$$F^{(xx')} = \left|\det\left(\frac{\partial x'^\mu}{\partial x^\nu}\right)\right| = \left|\det\left(\Lambda^\mu{}_\nu\right)\right| = 1$$

(invariant)

Conclusion: also $\mathcal{L}$ needs to be Lorentz invariant

possible terms $A_\mu A^\mu$, $j_\mu A^\mu$, $F_{\mu\nu}F^{\mu\nu}$, $F_{\mu\nu}\tilde{F}^{\mu\nu}$, $A_\mu F^{\mu\nu} j_\nu$ (...)

correct Lagrange density has only two terms

$$\mathcal{L}_{ED} = -\frac{1}{4\mu_0} F_{\mu\nu}F^{\mu\nu} - j_\mu A^\mu$$

$$= \frac{1}{2}\left(\epsilon_0 \vec{E}^2 - \frac{1}{\mu_0}\vec{B}^2\right) - \rho\varphi + \vec{j}\cdot\vec{A}$$

(non covariant form)

equations of motion:

$$\frac{\partial \mathcal{L}_{ED}}{\partial (A_\nu)} - \partial_\mu \frac{\partial \mathcal{L}_{ED}}{\partial(\partial_\mu A_\nu)} = 0$$

calculation of derivatives:

$$\frac{\partial \mathcal{L}_{ED}}{\partial (A_\nu)} = \frac{\partial}{\partial (A_\nu)} \Big[- \frac{1}{4\mu_0} \underbrace{F_{\mu\nu'}}_{\partial_\mu A_{\nu'} - \partial_{\nu'} A_\mu} F^{\mu\nu'} - j_\mu A^\mu \Big]$$

no dependence on A_ν !

$$= - \frac{\partial}{\partial A_\nu} \left(j_\mu \, g^{\mu\beta} A_\beta \right) = - j_\mu \, g^{\mu\beta} \, \delta^\nu{}_\beta$$

$$= - j_\mu \, g^{\mu\nu} = - j^\nu$$

side calculation :

$$\frac{\partial F_{\alpha\beta}}{\partial (\partial_\mu A_\nu)} = \frac{\partial}{\partial (\partial_\mu A_\nu)} \left(\partial_\alpha A_\beta - \partial_\beta A_\alpha \right) = \delta^\mu_\alpha \delta^\nu_\beta - \delta^\mu_\beta \delta^\nu_\alpha$$

$$\frac{\partial \mathcal{L}_{ED}}{\partial (\partial_\mu A_\nu)} = - \frac{1}{4\mu_0} \frac{\partial}{\partial (\partial_\mu A_\nu)} \Big[F_{\alpha\beta} \underbrace{F^{\alpha\beta}}_{F_{\eta\varsigma} g^{\alpha\eta} g^{\beta\varsigma}} \Big]$$

product rule

$$\overset{\downarrow}{=} - \frac{1}{4\mu_0} \left(\frac{\partial F_{\alpha\beta}}{\partial (\partial_\mu A_\nu)} F_{\eta\varsigma} + F_{\alpha\beta} \frac{\partial F_{\eta\varsigma}}{\partial (\partial_\mu A_\nu)} \right) g^{\alpha\eta} g^{\beta\varsigma}$$

$$= - \frac{1}{4\mu_0} \left(F^{\mu\nu} - F^{\nu\mu} + F^{\mu\nu} - F^{\nu\mu} \right) = - \frac{F^{\mu\nu}}{\mu_0}$$

put everything together

$$- j^\nu - \partial_\mu \left(- \frac{F^{\mu\nu}}{\mu_0} \right) = 0$$

$$\Rightarrow \quad \partial_\mu F^{\mu\nu} = \mu_0 j^\nu$$

(inhomogeneous Maxwell equations)