The energy-momentum tensor

reminder : 4-momentum $p^\mu = m u^\mu = m\gamma (c, \vec{v})^T$
$= (\frac{E}{c}, \vec{p})$

generalization of Newton's equation

$$K^\mu = \frac{dp^\mu}{d\tau} = \gamma \left(\frac{\vec{F}\cdot\vec{v}}{c}, \vec{F} \right)$$

particle in the electromagnetic field

Lorentz force $\vec{F} = q(\vec{E} + \vec{v}\times\vec{B})$

$$\vec{K} = \gamma\vec{F} = \gamma q (\vec{E} + \vec{v}\times\vec{B})$$

we expect : $K^\mu = q F^{\mu\nu} u_\nu$

explicit calculation

$$K^0 = \gamma\frac{\vec{F}\cdot\vec{v}}{c} = \gamma q \frac{(\vec{E}+\vec{v}\times\vec{B})\cdot\vec{v}}{c} = q\,\frac{\vec{E}}{c}\cdot\gamma\vec{v}$$

$u^\mu = \gamma(c, \vec{v})$

$$= q\,\frac{E^i}{c}\cdot u^i = q F^{i0} u^i = q F^{0i} u_i$$

$$= q F^{0\mu} u_\mu$$

$F^{\mu\nu} = -F^{\nu\mu}$
$u^i = -u_i$

$$K^1 = q\gamma\left(\frac{E^x}{c}c + v^x B^z - v^z B^y\right)$$

$$= q\left(F^{10}u^0 + u^2 F^{21} - u^3 F^{13}\right)$$

$$= q\left(F^{10}u_0 + F^{12}u_2 + F^{13}u_3 + F^{11}u_1\right) = q F^{1\mu}u_\mu$$

similar other components

$$K^2 = q F^{2\mu} u_\mu$$
$$K^3 = q F^{3\mu} u_\mu$$

Energy-Momentum tensor $T^{\mu\nu} = \frac{1}{\mu_0}\left(\frac{1}{4} g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} - F^{\mu\alpha} F^{\nu}{}_{\alpha}\right)$

it can be shown (exercises)

$$\partial_\nu T^{\mu\nu} = - F^{\mu\alpha} j_\alpha \qquad \text{(Energy-Momentum theorem)}$$

interpretation of both sides:

right side: 4-force density
we have seen $K^\mu = F^{\mu\alpha} \underbrace{q u_\alpha}$

current density of point charge with charge density $\rho = q\, \delta^{(3)}(\vec{r}-\vec{r}_q)$

$$\Rightarrow F^{\mu\alpha} j_\alpha = k^\mu \quad \text{(force density)}$$

left side : derivative of (E^2, B^2, EB)

calculation of elements of $T^{\mu\nu}$:

$$T^{00} = \frac{1}{\mu_0}\left(\frac{1}{4} F^{\alpha\beta} F_{\alpha\beta} - F^{0\alpha} F^{0}{}_{\alpha}\right)$$

$$= \frac{1}{\mu_0}\left[-\frac{1}{2}\left(\frac{\vec{E}^2}{c^2} - \vec{B}^2\right) - F^{0i} F^{0}{}_{i}\right] \quad F^{0}{}_{i} = -F^{0i}$$

$$= \frac{1}{\mu_0}\left[-\frac{E^2}{2c^2} + \frac{B^2}{2} + \frac{E^i}{c}\frac{E^i}{c}\right]$$

$$= \frac{1}{2\mu_0}\left(\frac{\vec{E}^2}{c^2} + B^2\right) = \frac{1}{2}\left(\epsilon_0 \vec{E}^2 + \frac{1}{\mu_0}\vec{B}^2\right) = \frac{1}{2}\left(\vec{E}\cdot\vec{D} + \vec{H}\cdot\vec{B}\right)$$

(energy density of the electromagnetic field)

$$T^{0i} = -\frac{1}{\mu_0} F^{0\alpha} F^{i}{}_{\alpha} = -\frac{1}{\mu_0} F^{0j} F^{i}{}_{j} = \frac{1}{\mu_0} F^{0j} F^{ij}$$

$$T^{01} = -\frac{1}{\mu_0}\frac{E^x}{c} F^{11} - \frac{1}{\mu_0}\frac{E^y}{c} F^{12} - \frac{1}{\mu_0}\frac{E^z}{c} F^{13}$$

$$= \frac{1}{\mu_0}\frac{E^y}{c} B^z - \frac{1}{\mu_0}\frac{E^z}{c} B^y = \frac{1}{\mu_0 c}\left[\vec{E}\times\vec{B}\right]^x$$

$$= \frac{1}{c}\left[\vec{E}\times\vec{H}\right]^x = \frac{S^x}{c} \qquad \vec{S} = \vec{E}\times\vec{H}$$

(Poynting-vector)

$$T^{ij} = (\dots) = \frac{1}{\mu_0}\left[\delta^{ij}\ \frac{1}{2}\left(\frac{\vec{E}^2}{c^2}+B^2\right) - \frac{E^i E^j}{c^2} - B^i B^j\right]$$

$$= -T^{ij}_{(M)}$$

(Maxwell's stress tensor)

$$\partial_\nu T^{\mu\nu} = -F^{\mu\alpha} j_\alpha \quad \Big\}\ \text{Poynting theorem + Momentum conservation}$$

Poynting theorem (compare TP 2)

$$\vec{j}\cdot\vec{E} = -\frac{\partial w_{field}}{\partial t} - \vec{\nabla}\cdot\vec{S}$$

mechanical work (change of kin. energy)

change in field energy

radiated energy (note: only $\vec{\nabla}\cdot(\vec{E}\times\vec{H})$ enters)

$$\vec{f} = \vec{\nabla}\cdot\overleftrightarrow{T}_M - \frac{1}{c^2}\frac{d}{dt}\vec{S}$$

total force by integration (compare TP2)

$$\vec{F} = \int_V dV\,\vec{f} = \oint_{S(V)} \overleftrightarrow{T}\cdot d\vec{f} - \frac{1}{c^2}\frac{d}{dt}\int \vec{S}\,dV$$

(contribution of fields on surface) (radiation force)

Summary : relativistic formulation of electrodynamics

4-current density $j^\mu = \gamma\,(c\varrho, \vec{v})$

continuity equation : $\partial_\mu j^\mu = 0$

Lorenz gauge $\partial_\mu A^\mu = 0$ of 4-potential $A^\mu = \left(\frac{\varphi}{c}, \vec{A}\right)$

$\Rightarrow \quad \partial_\nu \partial^\nu A^\mu = -\mu_0 j^\mu$ (wave equations for A^μ)

construction of tensors:

$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \qquad \tilde{F}^{\mu\nu} : \ \frac{\vec{E}}{c} \to \vec{B}, \ \vec{B} \to -\frac{\vec{E}}{c}$

$T^{\mu\nu} = \frac{1}{\mu_0}\left(\frac{1}{4}\, g^{\mu\nu} F^{\alpha\beta} F_{\alpha\beta} + F^{\mu\alpha} F_\alpha{}^\nu\right)$

→ transformation of fields, em. energy density, Poynting vector, stress tensor

invariants $\quad F_{\mu\nu} F^{\mu\nu} = 2\left(\vec{B}^2 - \frac{\vec{E}^2}{c^2}\right)$

$\qquad F_{\mu\nu} \tilde{F}^{\mu\nu} = -\frac{4}{c}\, \vec{E} \cdot \vec{B}$

Maxwell equations

$\partial_\alpha F^{\alpha\beta} = \mu_0 j^\beta \qquad \partial_\alpha \tilde{F}^{\alpha\beta} = 0$

Energy-momentum conservation

$\partial_\nu T^{\mu\nu} = -F^{\mu\alpha} j_\alpha$

so far : derivation of D.E., no discussion of solutions

TP 2 : Electrostatics, Magnetostatics

Now: solutions of wave-equations

$$\Box A^\mu = \mu_0 j^\mu$$

So far already discussed: homogeneous wave eqn

$$\Box A^\mu = 0$$

formally, we can also allow waves in media (fixed ref. system)

$$\Box \to \Box_u = \frac{1}{u^2}\frac{\partial^2}{\partial t^2} - \Delta$$

$$u = \frac{c}{n} \quad , \quad n = \sqrt{\epsilon_r \mu_r}$$

Maxwell eqn yield the same eqn:

$$\Box_u \vec{E} = 0$$

$$\Box_u \vec{B} = 0$$

we have seen $\psi = \psi(\varphi(\vec{r},t))$

with the phase $\varphi(\vec{r},t) = \omega t - \vec{k}\cdot\vec{r}$ $(= k^\mu x_\mu)$

solves $\Box_u \psi = \frac{\partial^2 \psi}{\partial \varphi^2}\left(\frac{\omega^2}{u^2} - \vec{k}^2\right) = 0$

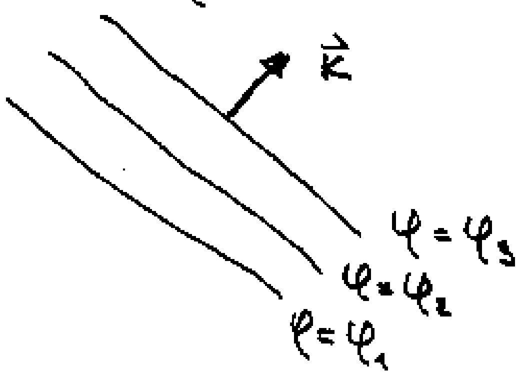
if $\omega = \pm u k = \omega_0$ $\qquad$ u: phase velocity

for fixed $\vec{k}$, the solution reads

$$\psi = f_1(\omega_0 t - \vec{k}\cdot\vec{r}) + f_2(-\omega_0 t - \vec{k}\cdot\vec{r})$$

wave propagating in $-\vec{k}$ direction

$\varphi(\vec{r},t)$ fixes planes where f_1 and f_2 have the same values



Additional constraints for EM waves in free space

$$\left.\begin{array}{l} \Box_u \vec{E} = 0 \\ \Box_u \vec{B} = 0 \\ \vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \\ \vec{\nabla}\times\vec{B} - \frac{1}{u^2}\frac{\partial \vec{E}}{\partial t} = 0 \\ \vec{\nabla}\cdot\vec{E} = 0 \\ \vec{\nabla}\cdot\vec{B} = 0 \end{array}\right\} \quad \begin{array}{ll} \vec{E} = \vec{E}_0\, e^{-i(\omega_0 t - \vec{k}\cdot\vec{r})} & \\ \vec{B} = \vec{B}_0\, e^{-i(\omega_0 t - \vec{k}\cdot\vec{r})} & \\ \vec{k}\times\vec{E}_0 = \omega_0 \vec{B}_0 & (1) \\ \vec{k}\times\vec{B}_0 = -\frac{\omega_0}{u^2}\vec{E}_0 & (1') \\ \vec{k}\perp\vec{E}_0 & (2) \\ \vec{k}\perp\vec{B}_0 & (2') \end{array}$$

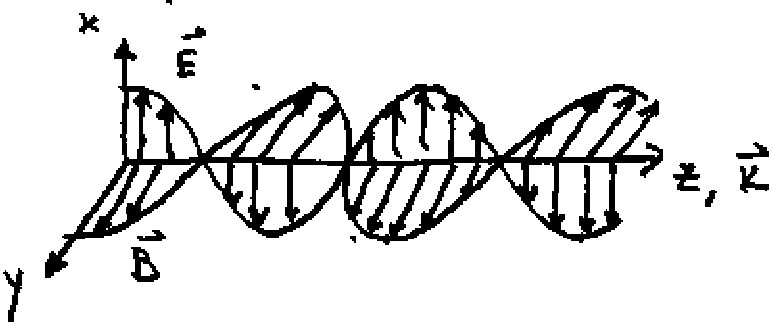
only propagation in $\vec{k}$ here

$\vec{E}_0, \vec{B}_0, \vec{k}$: right handed triad of orthogonal vectors

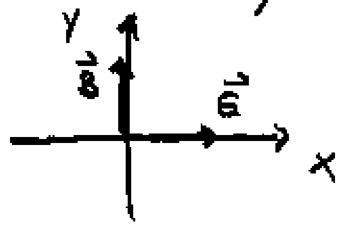
Polarization : two sets of 4 equations for 6 unknown quantities $\vec{B}_0, \vec{E}_0$
$\to$ two free parameters

convenient choice $\vec{k} = k\,\vec{e}_z$ (coordinate system)

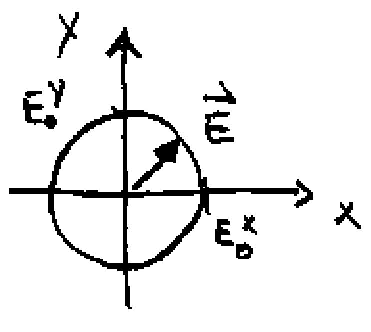
a) linear polarization



$\vec{E}$ is in $\pm x$ direction
$\vec{B}$ is in $\pm y$ direction

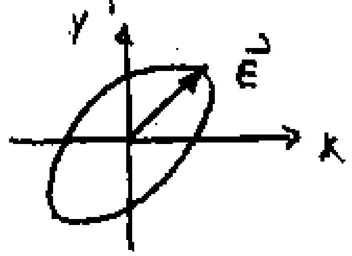


b) circular polarization
special case $|E_0^x| = |E_0^y|$



c) elliptical polarization $|E_0^x| = |E_0^y|$

d)



general case : $\vec{E}$ evolves on rotated ellipse (superposition of two circular polarized waves with opposite rotation direction)

Wave packets

wave equation $\Box_u \psi = 0$

linear D.E. : if ψ_1 and ψ_2 are solutions,
also $\tilde{\psi} = \psi_1 + \psi_2$ are solutions

arbitrary linear combination of plane waves

$$F_\pm(t,z) = \int_{-\infty}^{\infty} a(k)\, f_\pm(\pm\omega_0 t - kz)$$

weight function

$f_\pm(\pm\omega_0 t - kz) = e^{i(\pm\omega_0 t - kz)}$
in $\vec{k} = \pm k\,\vec{e}_z$ propagating plane wave

then :

$$\Box_u F_\pm(t,z) = \int_{-\infty}^{\infty} a(k)\, \underbrace{\Box_u f_\pm(\pm\omega_0 t - kz)}_{=0}\, dk = 0$$

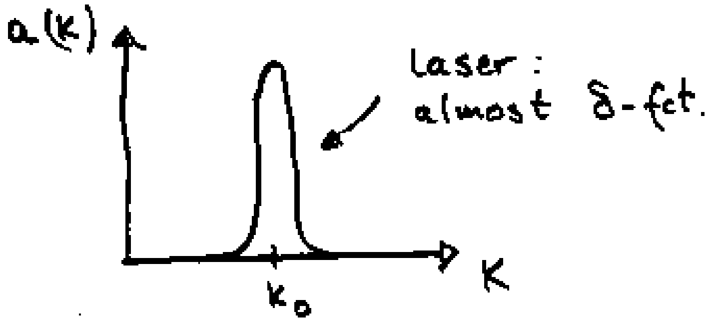
generalization to 3 dimensions

$$F_\pm(t,\vec{r}) = \int_{-\infty}^{\infty} a(\vec{k})\, f_\pm(\pm\omega_0 t - \vec{k}\cdot\vec{r})\, d^3\vec{k}$$

=> also this yields $\Box_u F_\pm(t,\vec{r}) = 0$

linearly polarized "monochromatic" light

$$\vec{E} = F_{+}(t,\vec{r})\,\vec{e}_x$$



Note: in general, we have

$$\omega_0 = u \cdot k = \frac{c}{n} \cdot k \qquad n = \frac{1}{\sqrt{\epsilon_r \mu_r}} = \frac{1}{\sqrt{\epsilon_r(k)\,\mu_r(k)}}$$

$$= u(k)\; k$$

"dispersion": waves with different k travel with different velocities in media

consider: Taylor expansion for monochromatic light:

$$\omega_0(k) = \omega_0(k_0) + \underbrace{\left.\frac{d\omega_0(k)}{dk}\right|_{k=k_0}}_{\text{group velocity}}(k-k_0) \qquad v_g = \frac{d\,\omega_0(k)}{d\,k}$$

rewrite plane wave part:

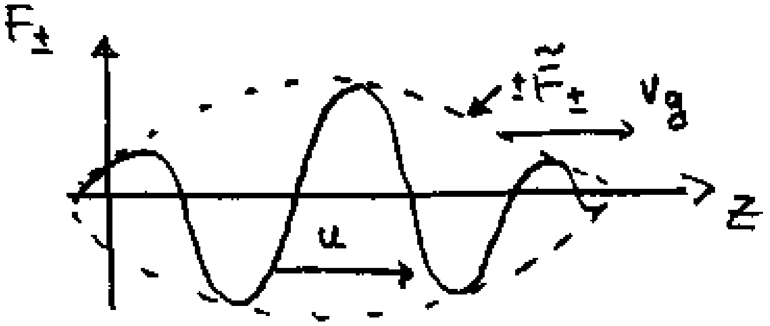
$$f_\pm = e^{-i\left(\pm\omega_0 t \pm (k-k_0)v_g t - kz\right)}$$
$$= e^{-i(\pm\omega_0 t \pm k_0 z)}\; e^{i(k-k_0)(z \mp v_g t)}$$

$$F_\pm(t,z) = e^{-i(\pm\omega_0 t \pm k_0 z)} \int_{-\infty}^{\infty} a(k)\, e^{i(k-k_0)(z\mp v_g t)}\, dk$$

$$= e^{-i(\pm\omega_0 t \pm k_0 z)} \int_{-\infty}^{\infty} a(k_0+q)\, e^{iq(z\mp v_g t)}\, dq$$

$$= e^{-i(\pm\omega_0 t \pm k_0 z)} \cdot \tilde{F}_\pm(k_0,\, z \mp v_g t)$$

plane wave with wavenumber k_0 — amplitude depending on z and t.



- enveloppe function moves with velocity v_g
- phase fronts of $F_\pm$ move with u

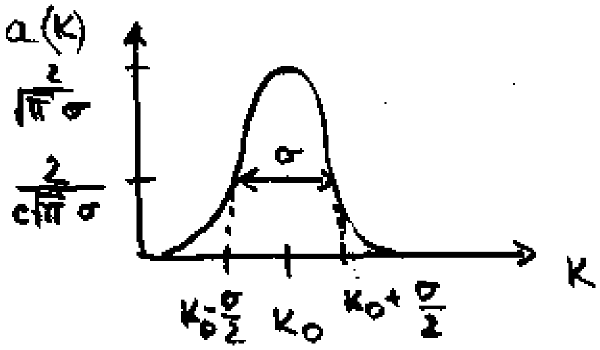
in general $u \neq v_g$

$$u = \frac{\omega(k)}{k}$$

$$v_g = \frac{d\omega(k)}{dk}$$

Example : Gauß wavepacket

given weight function: $a(k) = \frac{2}{\sqrt{\pi}\,\sigma} \cdot e^{-\frac{(k-k_0)^2}{\sigma^2/4}}$



Properties of weight function

$$\int_{-\infty}^{\infty} a(k)\,dk = 1$$

substitute $u = \frac{2(k-k_0)}{\sigma}$

$$\lim_{\sigma\to 0} a_\sigma(k) = \delta(k-k_0)$$

for $\sigma \to 0$, we have

$$F_\pm(t,z) = e^{-i(\pm\omega_0 t \pm k_0 z)} \cdot \int_{-\infty}^{\infty} \delta(k-k_0)\, e^{i(k-k_0)(z\mp v_g t)} d$$

$$= e^{-i(\pm\omega_0 t \pm k_0 z)}$$

(plane wave; discussed earlier)

general case:

$$\tilde{F}_\pm = \int_{-\infty}^{\infty} a(k_0+q)\, e^{iq(z\mp v_g t)}\, dq$$

side calculation:

$$a(K_0+q) = \frac{2}{\pi\sigma}\, e^{-\frac{q^2}{\sigma^2/4}}$$

total exponent

$$-\frac{q^2}{\sigma^2/4} + i\, q\left(z \pm v_g t\right) = -\frac{4}{\sigma^2}\left[q^2 - 2i\, q\left(z \pm v_g t\right)\frac{\sigma^2}{8}\right]$$

$$= -\frac{4}{\sigma^2}\left[q - i\left(z \pm v_g t\right)\frac{\sigma^2}{8}\right]^2 - \frac{\sigma^2}{16}\left(z \pm v_g t\right)^2$$

completion of square

independent of q

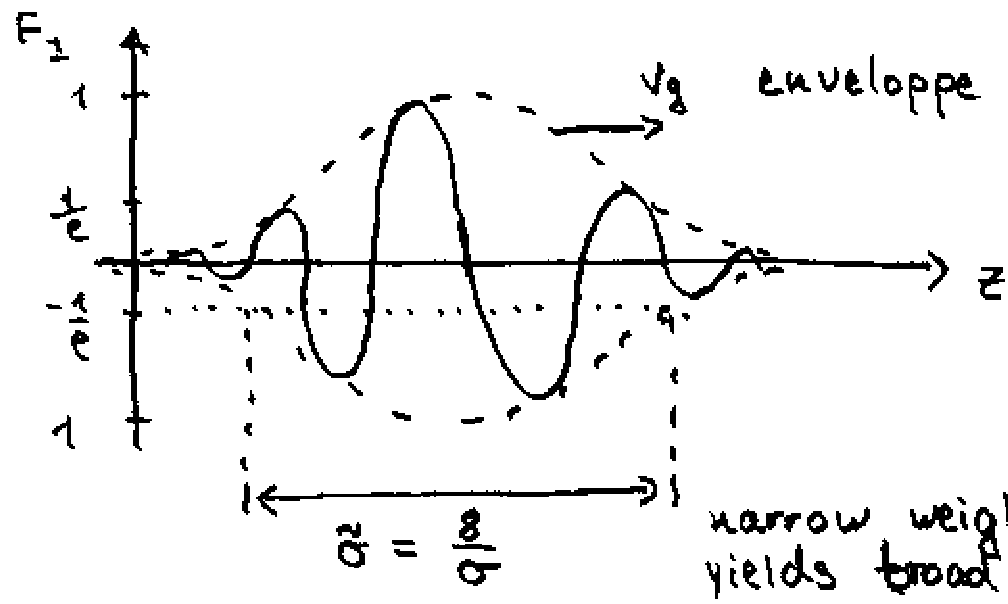
then, we obtain

$$\tilde{F}_\pm\left(K_0, z \pm v_g t\right) = e^{-\frac{\sigma^2}{16}\left(z \pm v_g t\right)^2} \underbrace{\int_{-\infty}^{\infty} \frac{2}{\sigma\sqrt{\pi}}\, e^{-\frac{\left(q - i\left(z \pm v_g t\right)\frac{\sigma^2}{8}\right)^2}{\sigma^2/4}}\, dq}_{\text{shifted Gauß integral now in complex plane, but still yields } 1}$$

$$\tilde{F}_\pm\left(K_0, z \pm v_g t\right) = e^{-\frac{\sigma^2}{16}\left(z \pm v_g t\right)^2}$$

$$= e^{-\frac{\left(z \pm v_g t\right)^2}{\tilde{\sigma}^2/4}}$$

Gauß function with $\tilde{\sigma} = \frac{8}{\sigma}$



narrow weight function yields broad enveloppe function