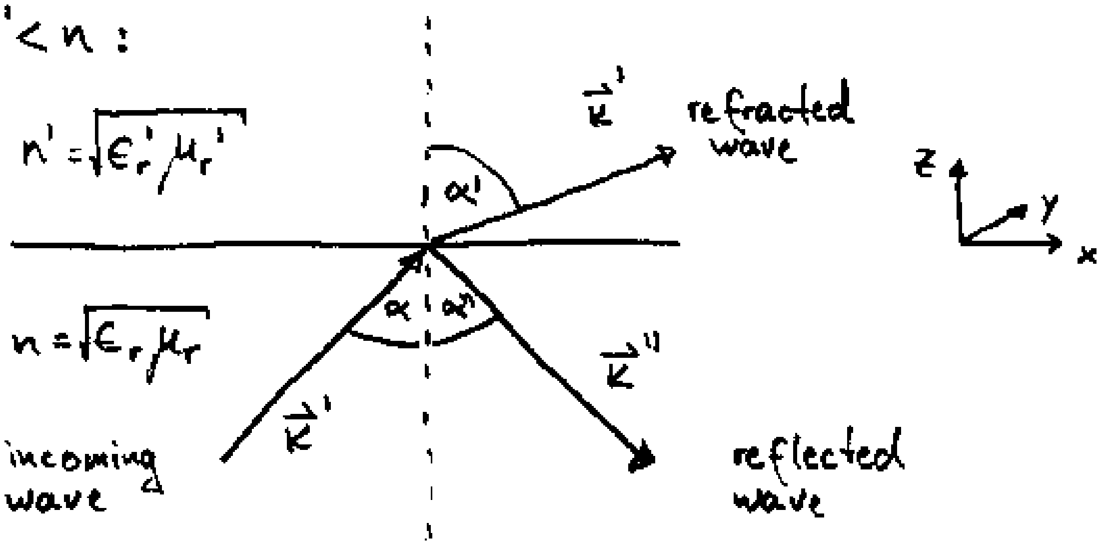
Plane wave: Reflection and refraction at insulating interfaces

drawing: $n' < n$:



α: incoming angle
α': refracted angle
α'': reflected angle

kinematics (usually discussed in school already): define direction of $\vec{k}'$ and $\vec{k}''$

a) reflection law: $\alpha = \alpha''$
b) refraction law (Snell's law)

$$\frac{\sin\alpha}{\sin\alpha'} = \frac{n'}{n} = n_r$$

(can be derived from Maxwell's equations)

However: EM waves are not only characterized by $\vec{k}$, but also by intensity, phase, polarization

→ here: derive dynamic properties from M.E.

Note: frequency fixed externally, thus we have

$$|\vec{k}| = |\vec{k}''| = \frac{\omega}{v} \qquad v = \frac{c}{n}$$

$$|\vec{k}'| = k' = \frac{\omega}{v'} \qquad v' = \frac{c}{n'}$$

Reminder: plane waves fulfill

incoming wave $\vec{E} = \vec{E}_0 \, e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ $\quad \vec{E}_0 \cdot \vec{k} = 0$, $\vec{B} = \frac{1}{\omega} \vec{k} \times \vec{E}$

refracted wave $\vec{E}' = \vec{E}_0' \, e^{i(\vec{k}'\cdot\vec{r}-\omega t)}$ $\quad \vec{E}_0' \cdot \vec{k}' = 0$, $\vec{B}' = \frac{1}{\omega} \vec{k}' \times \vec{E}'$

reflected wave $\vec{E}'' = \vec{E}_0'' \, e^{i(\vec{k}''\cdot\vec{r}-\omega t)}$ $\quad \vec{E}_0'' \cdot \vec{k}'' = 0$, $\vec{B}'' = \frac{1}{\omega} \vec{k}'' \times \vec{E}''$

wanted: $\vec{E}_0'$, $\vec{k}'$, $\vec{E}_0''$, $\vec{k}''$ as function of $\vec{E}_0$, $\vec{k}$

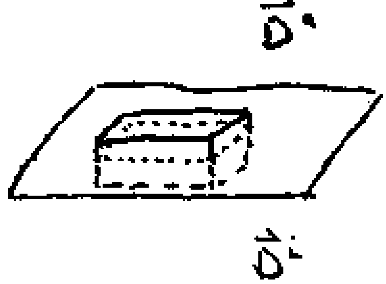
Use: Boundary conditions for the electromagnetic fields at interfaces

TP 2: derivation for static fields

tools: small volumia / surfaces across interface + Gauß theorem / Stokes theorem

derivation for perpendicular components still valid:

$$\left.\begin{array}{l}\vec{\nabla}\cdot\vec{D} = \rho \\ \vec{\nabla}\cdot\vec{B} = 0\end{array}\right\} \text{ no time dependence!}$$

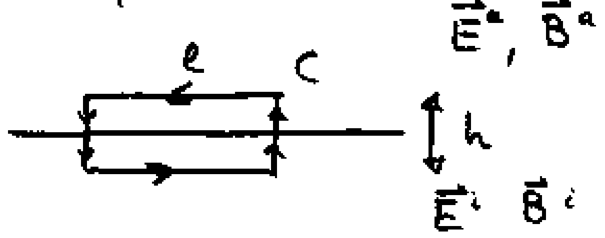


$$D_\perp^a - D_\perp^i = \sigma = 0$$ (no surface charges)

$$B_\perp^a - B_\perp^i = 0$$

components for parallel components

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$



$$\vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{J}$$

$$\int \left(\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} \right) \cdot d\vec{a} = 0$$

$$\oint \vec{E} \cdot d\vec{r} + \int \frac{\partial \vec{B}}{\partial t} \cdot d\vec{a} = 0$$ evaluate line integral and take $h \to 0$

$$E_\parallel \cdot l - E_\parallel' \cdot l = 0$$

$$\Rightarrow \vec{E}_\parallel^a = \vec{E}_\parallel^i$$

similar for: $\vec{\nabla} \times \vec{H} - \frac{\partial D}{\partial t} = \vec{J} \Rightarrow$

$$\Rightarrow \vec{H}_\parallel^a - \vec{H}_\parallel^i = \vec{J}_s = 0$$ (no surface current)

Boundary conditions at $z = 0$ have to be fulfilled at all times t, thus we have:

$$e^{i(\vec{k} \cdot \vec{r} - \omega t)} = e^{i(\vec{k}' \cdot \vec{r} - \omega t)} = e^{i(\vec{k}'' \cdot \vec{r} - \omega t)}$$

$$\Rightarrow \vec{k} \cdot \vec{r} \big|_{z=0} = \vec{k}' \cdot \vec{r} \big|_{z=0} = \vec{k}'' \cdot \vec{r} \big|_{z=0}$$

or explicitely $\quad k_x = k_x' = k_x'' \quad$ for $\vec{r} = \vec{e}_x$

$\qquad\qquad\qquad\quad k_y = k_y' = k_y'' \quad$ for $\vec{r} = \vec{e}_y$

($\vec{k}, \vec{k}', \vec{k}''$ are linear dependent, lie in one plane) choice of coordinate system : $k_y = k_y' = k_y'' = 0$

read off from geometry

$k \sin\alpha = k' \sin\alpha' = k'' \sin\alpha''$

we have additionally : $k'' = k$, $\frac{k'}{k} = \frac{n'}{n}$

and get $\sin\alpha = \sin\alpha'' \Rightarrow \alpha = \alpha''$ (reflection law)

$$\frac{\sin\alpha}{\sin\alpha'} = \frac{k'}{k} = \frac{n'}{n} \quad \text{(refraction law)}$$

write down wavevectors explicitely :

$$\vec{k} = k\begin{pmatrix}\sin\alpha\\0\\\cos\alpha\end{pmatrix} \qquad \vec{k}' = k'\begin{pmatrix}\sin\alpha'\\0\\\cos\alpha'\end{pmatrix} = k\begin{pmatrix}\sin\alpha\\0\\\sqrt{n_r^2-\sin^2\alpha}\end{pmatrix}$$

$k' = n_r k$

$n_r \sin\alpha' = \sin\alpha$

$$\vec{k}'' = k\begin{pmatrix}\sin\alpha\\0\\-k\cos\alpha\end{pmatrix}$$

use boundary conditions

note : $\vec{D}^a = \vec{D} + \vec{D}''$, $\vec{D}^i = \vec{D}'$ (...)

$$D_\perp^a = D_\perp^i \Rightarrow \epsilon_r \left(\vec{E}_0 + \vec{E}_0''\right)\cdot\vec{e}_z = \epsilon_r' \vec{E}_0'\cdot\vec{e}_z \qquad (1)$$

$$B_\perp^a = B_\perp^i \Rightarrow \left(\vec{k}\times\vec{E}_0 + \vec{k}''\times\vec{E}_0''\right)\cdot\vec{e}_z = \left(\vec{k}'\times\vec{E}_0'\right)\cdot\vec{e}_z \qquad (2)$$

$$\vec{E}_\parallel^a = E_\parallel^i \Rightarrow \left(\vec{E}_0 + \vec{E}_0''\right)\times\vec{e}_z = \vec{E}_0'\times\vec{e}_z \qquad (3)$$

$$\vec{H}_\parallel^a = \vec{H}_\parallel^i \Rightarrow \frac{1}{\mu_r}\left(\vec{k}\times\vec{E}_0 + \vec{k}''\times\vec{E}_0''\right)\times\vec{e}_z = \frac{1}{\mu_r'}\left(\vec{k}'\times\vec{E}_0'\right)\times\vec{e}_z \qquad (4)$$

Without loss of generality: ← why?!
Consider linear polarized plane wave only.

Two cases:

1) $\vec{E}_0$ is perpendicular to plane defined by $\vec{k}$ and $\vec{e}_z$ (plane of incidence).

$\vec{E}_0 = \begin{pmatrix} 0 \\ E_0 \\ 0 \end{pmatrix}$ also $\vec{E}_0'$ and $\vec{E}_0''$ are of that form:

$\vec{E}_0' = E_0' \vec{e}_y$, $\vec{E}_0'' = E_0'' \vec{e}_y$

the conditions then read:

(3) : $E_0 + E_0'' = E_0'$

(4) : $\frac{1}{\mu_r} k (E_0 \cos\alpha - E_0'' \cos\alpha) \vec{e}_y = \frac{1}{\mu_r'} n_r k E_0' \cos\alpha' \vec{e}_y$

$\Rightarrow \frac{1}{\mu_r} (E_0 - E_0'') \cos\alpha = \frac{1}{\mu_r'} n_r E_0' \cos\alpha'$

(1), (2) : trivially fulfilled

use $n_r \cos\alpha' = \sqrt{n_r^2 - \sin^2\alpha}$ and substitute $E_0' = E_0 + E_0''$ to get:

$$\frac{E_0''}{E_0} = \frac{\cos\alpha - \frac{\mu_r}{\mu_r'} \sqrt{n_r^2 - \sin^2\alpha}}{\cos\alpha + \frac{\mu_r}{\mu_r'} \sqrt{n_r^2 - \sin^2\alpha}}$$

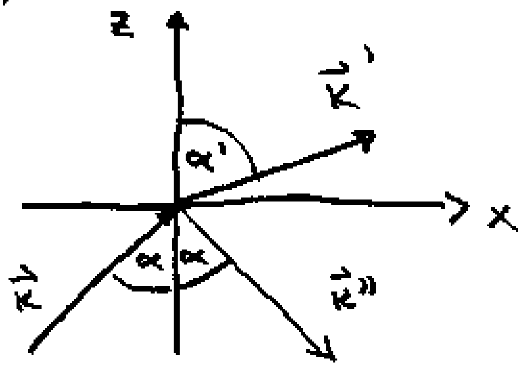
(amplitude of reflected wave for polarization perpendicular to plane of inciden

use:

$$\frac{E_0'}{E_0} = 1 + \frac{E_0''}{E_0} = \frac{2\cos\alpha}{\cos\alpha + \frac{\mu_r}{\mu_r'} \sqrt{n_r^2 - \sin^2\alpha}}$$

(amplitude of refracted wave)

2) $\vec{E}_0$ lies in plane of incidence:



$$\vec{E}_0 = \begin{pmatrix} -E_0 \cos\alpha \\ 0 \\ E_0 \sin\alpha \end{pmatrix}$$

$$\vec{E}_0' = \begin{pmatrix} -E_0' \cos\alpha' \\ 0 \\ E_0 \sin\alpha' \end{pmatrix} \qquad \vec{E}_0'' = \begin{pmatrix} E_0'' \cos\alpha \\ 0 \\ E_0'' \sin\alpha \end{pmatrix}$$

explicit calculation yields

$$(3) \qquad \cos\alpha\,(E_0 - E_0'') = \cos\alpha'\, E_0'$$

$$(4) \qquad \frac{1}{\mu_r}(E_0 + E_0'') = \frac{1}{\mu_r'}\, n_r\, E_0'$$

after canceling common prefactors and using $\sin^2\alpha + \cos^2\alpha = 1$

solving the linear system of equations yields:

$$\frac{E_0''}{E_0} = \frac{\frac{\mu_r}{\mu_r'}\, n_r \cos\alpha - \sqrt{1 - n_r^{-2}\sin^2\alpha}}{\frac{\mu_r}{\mu_r'}\, n_r \cos\alpha + \sqrt{1 - n_r^{-2}\sin^2\alpha}}$$

(reflected wave, polarized in plane of incidence)

$$\frac{E_0'}{E_0} = \frac{\mu_r'}{n_r \mu_r}\left(1 + \frac{E_0''}{E_0}\right)$$

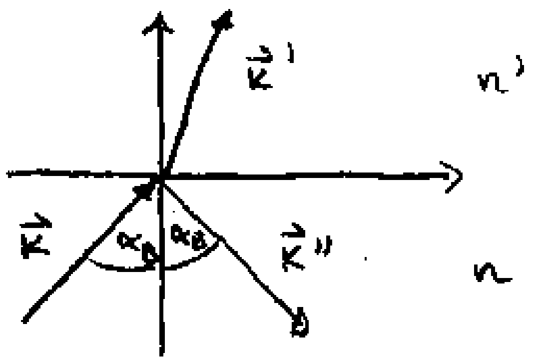
$$= \frac{2\cos\alpha}{\frac{\mu_r}{\mu_r'}\, n_r \cos\alpha + \sqrt{1 - n_r^{-2}\sin^2\alpha}}$$

(amplitude of refracted wave)

(Fresnel formulas)

Discussion

1) consider interface from optically light to optically dense medium



special angle α_B, when reflected wave has no polarization in the plane of incidence

Fesnel formula:

$$\frac{E_0''}{E_0} = \frac{\frac{\mu_r}{\mu_r'} n_r \cos\alpha_B - \sqrt{1 - n_r^{-2}\sin^2\alpha_B}}{\frac{\mu_r}{\mu_r'} n_r \cos\alpha_B + \sqrt{1 - n_r^{2}\sin^2\alpha_B}} \stackrel{!}{=} 0$$

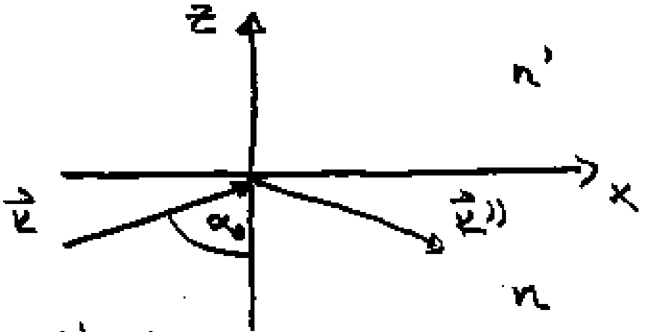
assume $\mu_r' = \mu_r$ (usually media are not susceptible to magnetic fields)

$$n_r \cos\alpha_B = \sqrt{1 - n_r^{-2}\sin^2\alpha_B}$$

equation fulfilled if

$$\tan\alpha_B = n_r = \frac{n'}{n} \quad \text{(Brewster angle)}$$

2) interface from optically denser medium to lighter medium : $n' < n$



Special case $\alpha' = \frac{\pi}{2}$

Snell's law:

$$\frac{\sin\alpha}{\sin\alpha'} = n_r = \frac{n'}{n} \qquad \sin\alpha' = 1$$

$\Rightarrow \sin\alpha_0 = n_r$ (critical angle)

for $\alpha > \alpha_0$ no "refracted wave" (total reflection)

What happens to refracted wave?

our ansatz was: $\vec{E}' = \vec{E}_0' \, e^{i(\vec{k}'\cdot\vec{r} - \omega t)}$

consider spatial dependence:

$$e^{i\vec{k}'\cdot\vec{r}} = e^{ik'(\sin\alpha' x + \cos\alpha' z)} \qquad \vec{k}' = k\begin{pmatrix}\sin\alpha' \\ 0 \\ \cos\alpha'\end{pmatrix}$$

$$\sin\alpha' = \frac{1}{n_r}\sin\alpha = \frac{\sin\alpha}{\sin\alpha_0} \qquad \text{(Snell's law + critical angle)}$$

$$\cos\alpha' = \sqrt{1-\sin^2\alpha'} = \sqrt{1-\frac{\sin^2\alpha}{\sin^2\alpha_0}}$$

$$= i\sqrt{\left(\frac{\sin\alpha}{\sin\alpha_0}\right)^2 - 1} \qquad \text{(imaginary)}$$

$$e^{i\vec{k}'\cdot\vec{r}} = e^{-k'\sqrt{\left(\frac{\sin\alpha}{\sin\alpha_0}\right)^2-1}\,z} \; e^{ik'\frac{\sin\alpha}{\sin\alpha_0}x}$$

propagation along x, exponential decay in z direction

(no energy flow into medium with n', $\left|\frac{E_0''}{E_0}\right| = 1$)

evaluate Fresnel formula $\rightarrow$