Electromagnetic waves in metals

Consider : isotropic metal (conductor)

material equations $\vec{D} = \epsilon_0 \epsilon_r \vec{E}$

$$\vec{H} = \frac{1}{\mu_0 \mu_r} \vec{B}$$

Ohm's law $\vec{j}_{free} = \sigma \vec{E}$ (σ: conductivity)

$\rho = 0$ (no charges)

Maxwell's equations

$$\vec{\nabla} \cdot \vec{D} = \rho \qquad \vec{\nabla} \cdot \vec{B} = 0$$

$$\vec{\nabla} \times \vec{H} - \frac{\partial \vec{D}}{\partial t} = \vec{j} \qquad \vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$$

take curl of Faraday's law

$$\vec{\nabla} \times (\vec{\nabla} \times \vec{E}) + \underbrace{\vec{\nabla} \times \frac{\partial \vec{B}}{\partial t}}_{\mu_0 \mu_r \frac{\partial}{\partial t} \big(\underbrace{\vec{\nabla} \times \vec{H}}_{\vec{j}_{free} + \frac{\partial \vec{D}}{\partial t} = \sigma \vec{E} + \epsilon_0 \epsilon_r \frac{\partial \vec{E}}{\partial t}} \big)} = 0$$

$$\Rightarrow \vec{\nabla} (\underbrace{\vec{\nabla} \cdot \vec{E}}_{0}) - \Delta \vec{E} + \mu_0 \mu_r \frac{\partial}{\partial t} \left(\sigma \vec{E} + \epsilon_0 \epsilon_r \frac{\partial \vec{E}}{\partial t} \right) = 0$$

$$\Delta \vec{E} - \epsilon_0 \epsilon_r \mu_0 \mu_r \frac{\partial^2 \vec{E}}{\partial t^2} - \mu_0 \mu_r \sigma \frac{\partial \vec{E}}{\partial t} = 0$$

(Telegrapher's equation)

similar from Ampère's law

$$\Delta \vec{B} - \epsilon_0 \epsilon_r \mu_0 \mu_r \frac{\partial^2 \vec{B}}{\partial t^2} - \mu_0 \mu_r \sigma \frac{\partial \vec{B}}{\partial t} = 0$$

Ansatz :

$$\vec{E}(t, \vec{r}) = \vec{E}_0 \, e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

$$\vec{B}(t, \vec{r}) = \mathcal{B}_0 \, e^{i(\vec{k} \cdot \vec{r} - \omega t)}$$

insert in Telegrapher's equation

$$\left(-\vec{\kappa}^2 + \epsilon_0\epsilon_r\mu_0\mu_r\,\omega^2 + i\omega\mu_0\mu_r\,\sigma\right)\vec{E}_0\, e^{i(\vec{\kappa}\cdot\vec{r}-\omega t)} = 0$$

$$\vec{\kappa}^2 = \epsilon\mu\omega^2 + i\omega\mu\sigma \qquad \vec{k}^2 = \epsilon\mu\,\omega^2 = \frac{\omega^2}{u^2}$$ (wave vector of E.M. wave in insulator)

$$= \vec{k}^2\left(1 + \frac{i\sigma}{\epsilon\omega}\right)$$

$\vec{\kappa}$ is a complex vector and can be parametrized as

$$\vec{\kappa} = \hat{k}\,\kappa \qquad \hat{k} = \frac{\vec{k}}{k}$$ (direction)

$$\kappa = k\sqrt{1+\frac{i\sigma}{\epsilon\omega}} = k\sqrt{1+i\frac{2}{(k\delta)^2}} = \sqrt{k^2+\frac{2i}{\delta^2}}$$ (magnitude)

$$= \alpha + i\beta$$ (real and imaginary part) $\delta = \sqrt{\frac{2}{\mu\sigma\omega}}$ length scale: "skin depth"

$$\alpha = k\sqrt{\frac{\sqrt{1+\frac{\sigma^2}{\epsilon^2\omega^2}}+1}{2}} \qquad \beta = k\sqrt{\frac{\sqrt{1+\frac{\sigma^2}{\epsilon^2\omega^2}}-1}{2}}$$

rewrite Ansatz:

$$\vec{E}(t,\vec{r}) = \vec{E}_0\, e^{-\beta\hat{k}\cdot\vec{r}}\, e^{i(\alpha\hat{k}\cdot\vec{r}-\omega t)}$$

Maxwell's equations yield:

$$\vec{E}_0\cdot\hat{k} = 0$$

$$\vec{B}(t,\vec{r}) = \frac{\vec{\kappa}}{\omega}\times\vec{E}(t,\vec{r}) = \kappa\,\frac{\hat{k}}{\omega}\times\vec{E}(t,\vec{r})$$

observations

1) finite conductivity yields exponential decay of the wave : $e^{-\beta\hat{k}\cdot\vec{r}}$
2) fields are not in phase
   κ: complex

Special case: "good" conductor : $\vec{j} = \sigma \vec{E}$

large σ such that $\frac{\sigma}{\epsilon\omega} \gg 1$

$$\frac{\sigma}{\epsilon\omega} = \frac{\sigma\omega\mu}{\epsilon\mu\omega^2} = \frac{2}{(k\delta)^2} \qquad \delta = \sqrt{\frac{2}{\mu\sigma\omega}}$$

$$k = \sqrt{\epsilon\mu}\,\omega$$

other interpretation:

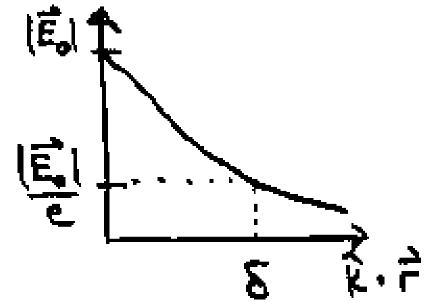
$$k\delta = \sqrt{\epsilon\mu}\,\omega\delta \ll 1 \quad \text{(good conductor)}$$

we calculate $k = \alpha + i\beta = (1+i)\frac{1}{\delta}$

$$\left.\begin{aligned} \alpha &\approx k\sqrt{\frac{\sigma}{2\epsilon\omega}} = \frac{1}{\delta} \\ \beta &\approx k\sqrt{\frac{\sigma}{2\epsilon\omega}} = \frac{1}{\delta} \end{aligned}\right\}$$

exponential decay: $e^{-\frac{\hat{k}\cdot\vec{r}}{\delta}}$

(skin effect)



field energies $\quad w = w_e + w_m$

$$= \frac{1}{2}\left(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H}\right)$$

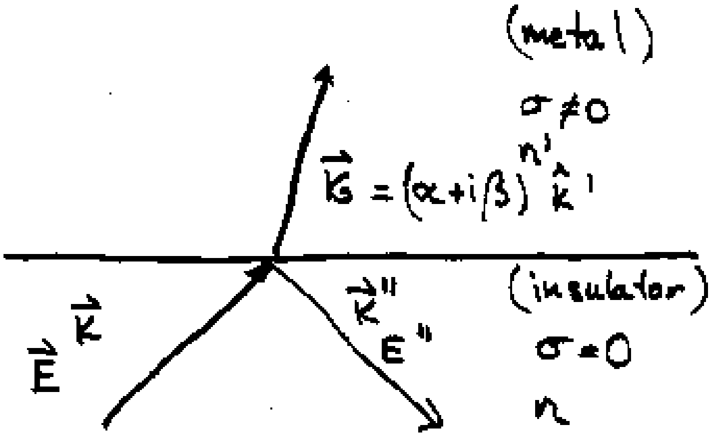
$$w_e = \frac{1}{2}\epsilon|\vec{E}|^2 \quad \text{(electric contribution)}$$

$$w_m = \frac{1}{2\mu}|\vec{B}|^2 \quad \text{(magnetic contribution)}$$

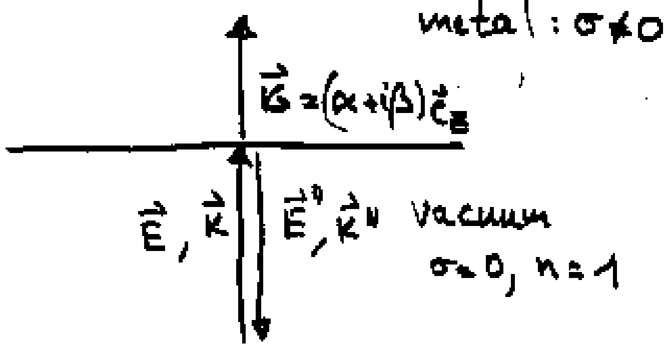
$$\vec{B} = \frac{\vec{k}\times\vec{E}}{\omega} \longrightarrow = \frac{1}{2\mu}\epsilon\mu\frac{|k|^2}{k^2}\vec{E}^2 = \frac{|k|^2}{k^2}w_e$$

$$\approx \frac{2}{k^2\delta^2} = \left|\frac{\sigma}{\epsilon\omega}\right| w_e \gg w_e$$

good conductor $\qquad\qquad$ good conductor metallic reflection



similar approach
as for interfaces
between insulators
→ more complicated
due to complex wavevector $\vec{k}$



here: only special case

1) linear polarized wave (polarization in plane)

2) non-magnetic, good conductor $\mu = \mu_0$, $\frac{\sigma}{\epsilon\omega} \gg 1$

Then, we obtain for the ratio of reflected to incoming intensity

$$R = \frac{|\vec{E}''|^2}{|\vec{E}|^2} \approx 1 - 2\sqrt{\frac{2\epsilon_0\omega}{\sigma}} = 1 - 2\frac{\omega}{c}\delta + O(\delta^2)$$

$R \approx 1$: good metals shine

Note: The Fresnel formula $\mu_r = \mu_r'$

$$\frac{E_0''}{E_0} = \frac{n_r\cos\alpha - \sqrt{1 - n_r^{-2}\sin^2\alpha}}{n_r\cos\alpha + \sqrt{1 - n_r^{-2}\sin^2\alpha}}$$

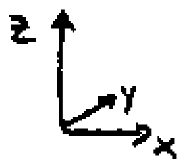
can be used with a complex optical density

$$n(\omega) = \sqrt{\epsilon_r\mu_r}\sqrt{1 + \frac{i\sigma}{\epsilon\omega}}$$

which corresponds to a complex dielectric constant $\epsilon(\omega) = \epsilon + i\frac{\sigma}{\omega}$

Fields on the surface and inside a conductor

Consider following interface



vacuum	$\vec{E}(t,\vec{r})$, $\vec{B}(t,\vec{r})$
ϵ_0, μ_0	$\vec{D} = \epsilon_0 \vec{E}$
$\sigma = 0$	$\vec{H} = \frac{1}{\mu_0}\vec{B}$
metal	$\vec{E}_m(t,\vec{r})$, $\vec{B}_m(t,\vec{r})$
$\epsilon, \mu, \sigma \neq 0$	$\vec{D}_m = \epsilon_0 \epsilon_r \vec{E}_m$
$\vec{j}_{free} = \sigma \vec{E}_m$	$\vec{H}_m = \frac{1}{\mu_0 \mu_r} \vec{B}$

given : fields in vacuum
want : fields at interface and in metal

notation : fields at interface

$$\left.\begin{aligned} \vec{E}^a(t,\vec{r}) &= \lim_{z\to 0^+} \vec{E}(t,\vec{r}) \\ \vec{E}^i(t,\vec{r}) &= \lim_{z\to 0^-} \vec{E}_m(t,\vec{r}) \end{aligned}\right\} \text{similar for } \vec{B}, \vec{D}, \vec{H}$$

conditions for fields on interface :

$$D_\perp^a - D_\perp^i = \sigma_{free} \quad \longleftarrow \text{surface charge density}$$

$$B_\perp^a - B_\perp^i = 0$$

$$\vec{E}_\parallel^a - \vec{E}_\parallel^i = 0$$

$$\vec{n} \times (\vec{H}_\parallel^a - \vec{H}_\parallel^i) = \vec{j}_{free,\sigma} \quad \longleftarrow \text{surface current density} \quad \vec{j}_{free} = \vec{j}_{free,\sigma}\,\delta(z)$$

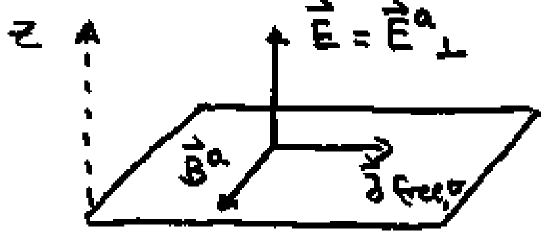
$\vec{n} = \vec{e}_z$ (surface normal)

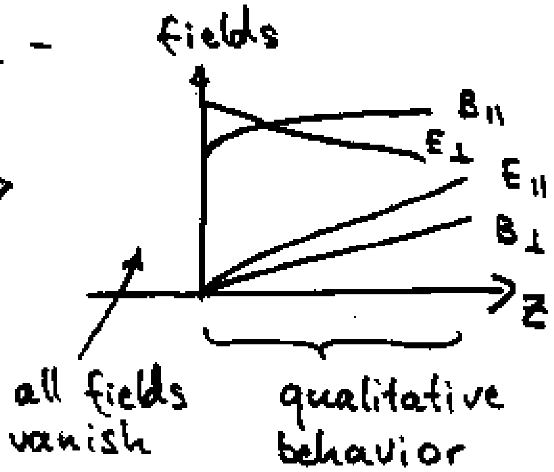
a) ideal conductor $\sigma \to \infty$

(charges extremely mobile, such that no fields inside the metal $\vec{E}_m = 0$, $\vec{B}_m = 0$, $\vec{D}_m = 0$, $\vec{H}_m = 0$

conditions read :

$$D_\perp^a = \sigma_{free}, \quad B_\perp^a = 0, \quad \vec{E}_\parallel^a = 0, \quad \vec{n} \times \vec{H}_\parallel^a = \vec{j}_{free,\sigma}$$

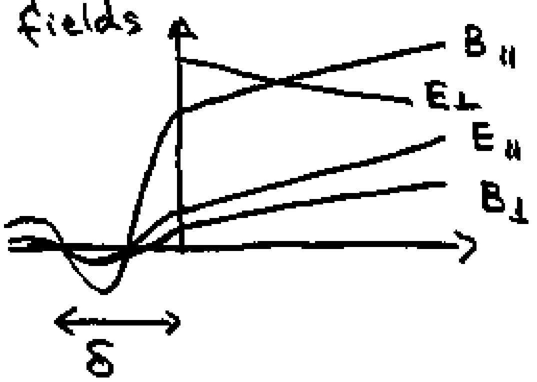




b) good conductor, but not ideal

$0 < \frac{\epsilon\omega}{\sigma} = \frac{1}{2}(k\delta)^2 \ll 1$

qualitative behavior:



$E_\parallel, B_\perp, B_\parallel$ penetrate into conductor and decay as $e^{-\frac{|z|}{\delta}}$

detailed analysis of fields inside conductor

<u>small parameter</u> : $k\delta = \sqrt{\epsilon\mu}\,\omega\delta = \sqrt{\frac{2\epsilon\omega}{\sigma}}$

1) zeroth order $\quad \sigma \to \infty$, order $(k\delta)^0$

outside (e.m. waves in vacuum):

$E_\parallel^a = 0, \quad B_\perp^a = 0 \qquad |E_\perp^a| = c\,|B_\parallel^a|$

inside $\quad \vec{E}^i = 0, \quad B_\perp^i = 0$

no surface currents in this approximation:

$\vec{j}_{free} = \sigma \vec{E}_m$

it follows that $\vec{B}_\parallel^a = \vec{B}_\parallel^i$ (field $\vec{B}_\parallel$ continous

$\vec{B}_m = \vec{B}_\parallel^a \; e^{-|z|/\delta} \; e^{i(z/\delta - \omega t)}$ (discussion of skin effect)

2) first order contribution from $k\delta$

inside $\vec{E}_m^{(1)} = -\frac{\omega}{k} \hat{k} \times \vec{B}_m^{(0)} \approx -\frac{\omega\delta}{1+i} \hat{k} \times \vec{B}_m^{(0)}$

$k = (1+i)\frac{1}{\delta}$

field parallel to interface : $\hat{k} = \vec{e}_z$, $\vec{B}_m^{(0)} = |\vec{B}_m^{0}| \vec{e}_{\parallel}$

outside : $\vec{E}_{\parallel}^{a} = \vec{E}_m^{(1)}(z=0)$ (parallel component continous at interface)

magnitude of field components

$$\frac{|\vec{E}_{\parallel}^{a}|}{|\vec{E}_{\perp}^{a}|} = \text{const} \cdot (k\delta)^{1} \ll 1$$

use $\frac{\partial \vec{B}}{\partial t} = -\vec{\nabla} \times \vec{E}$ to find the same for $\vec{B}$:

$$\frac{|\vec{B}_{\perp}^{a}|}{|\vec{B}_{\parallel}^{a}|} = \text{const}' \cdot (k\delta)^{1} \ll 1$$

3) 2nd order contribution $(k\delta)^2$
(only relevant for quantities that are zero so far)

→ small electric field perpendicular to surface
reconsider boundary condition

$$D_{\perp}^{a} - D_{\perp}^{i} = \sigma_{free}$$

$$\Rightarrow \epsilon_0 E_{\perp}^{a} = \epsilon_0 \epsilon_r E_{\perp}^{i} + \sigma_{free}$$

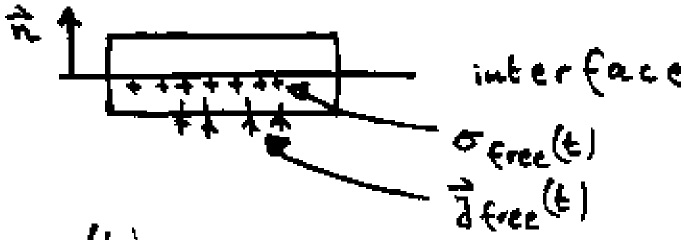
also charge density oscillates

$$\sigma_{free}(t) = \sigma_{free}^{0} e^{-i\omega t}$$

$$\frac{\partial \sigma_{free}(t)}{\partial t} = -i\omega \, \sigma_{free}(t)$$

continuity equation $\vec{\nabla} \cdot \vec{j}_{free} = -\frac{\partial \rho}{\partial t}$

integrate over Gaußian volume



$$\vec{n} \cdot \vec{j}_{free} = -i\omega\, \sigma_{free}(t)$$

use Ohm's law $\vec{j}_{free} = \sigma \vec{E}^i$

$$\vec{n} \cdot \vec{j}_{free} = \sigma\, \vec{n} \cdot \vec{E}^i = \sigma E_\perp^i = -i\omega\, \sigma_{free}$$

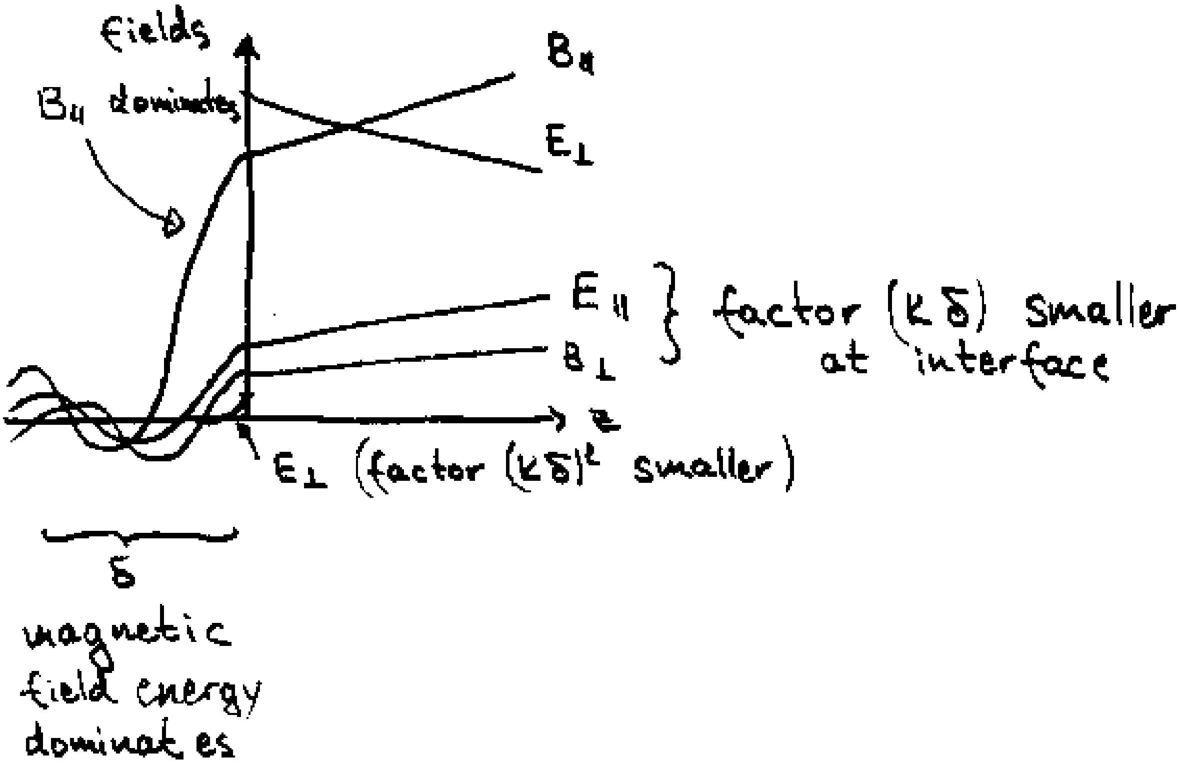
$$\Leftrightarrow \quad \sigma_{free} = \frac{\sigma}{-i\omega} E_\perp^i = i \frac{\sigma}{\omega} E_\perp^i$$

insert: $\epsilon_0 E_\perp^a = \epsilon_0 \epsilon_r E_\perp^i + i \frac{\sigma}{\omega} E_\perp^i = \epsilon_0 \epsilon_r \underbrace{\left(1 + i \frac{\sigma}{\epsilon\omega}\right)}_{\text{metallic screening}} E_\perp^i$

calculate ratio

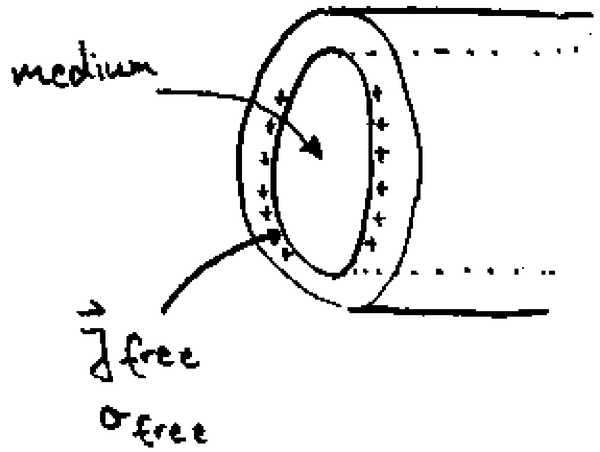
$$\frac{E_\perp^i}{E_\perp^a} = \frac{\epsilon_0}{\epsilon_0 \epsilon_r \left(1 + i \frac{\sigma}{\epsilon\omega}\right)} \approx -i \frac{\epsilon_0 \omega}{\sigma} = \text{const.} (k\delta)^2$$

recapitulate:



Waveguides (and cavities)

consider: cylinder of ideal metal, filled with dielectric ($\epsilon = \epsilon_0 \epsilon_r$, $\mu = \mu_0 \mu_r$)



induced currents and surface charges: screening of metallic casing

boundary conditions

$B_\perp^a = 0 \quad \Leftrightarrow \quad \vec{n} \cdot \vec{B}^a = 0$

$\vec{E}_\parallel^a = 0 \quad \Leftrightarrow \quad \vec{n} \times \vec{E}^a = 0$

perpendicular to inner surface

Solutions of Maxwell equ. with oscillating time dependence

$$\vec{E}(t, \vec{r}) = \vec{E}_\omega(\vec{r})\, e^{-i\omega t}$$
$$\vec{B}(t, \vec{r}) = \vec{B}_\omega(\vec{r})\, e^{-i\omega t}$$

Maxwell's equations then read

$$\vec{\nabla} \cdot \vec{E}_\omega = 0 \qquad \vec{\nabla} \cdot \vec{B}_\omega = 0$$
$$\vec{\nabla} \times \vec{B}_\omega = -i\omega \mu \epsilon_0 \vec{E}_\omega \qquad \vec{\nabla} \times \vec{E}_\omega = i\omega \vec{B}_\omega$$

rewrite fields in transverse and longitudinal components

$$\vec{E}_\omega(\vec{r}) = E_z(\vec{r})\, \vec{e}_z + \vec{E}_t(\vec{r})$$
$$\vec{B}_\omega(\vec{r}) = B_z(\vec{r})\, \vec{e}_z + \vec{B}_t(\vec{r})$$

define transverse part of Nabla operator:

$$\vec{\nabla}_t = \vec{e}_x \frac{\partial}{\partial x} + \vec{e}_y \frac{\partial}{\partial y}$$

formally rewrite M.E.:

$$\vec\nabla_t \cdot \vec E_t = -\partial_z E_z \qquad\qquad \vec\nabla_t \cdot \vec B_t = -\partial_z B_z$$

$$\vec e_z \cdot (\vec\nabla_t \times \vec B_t) = -i\omega\mu\epsilon E_z \qquad\qquad \vec e_z \cdot (\vec\nabla_t \times \vec E_t) = i\omega B_z$$

$$\partial_z \vec E_t + i\omega \vec e_z \times \vec B_t = \vec\nabla_t E_z \qquad\qquad \partial_z \vec B_t - i\omega\mu\epsilon\, \vec e_z \times \vec E_t = \vec\nabla_t B_z$$

two observations : a) E_z and B_z on r.h.s.
(inhomogeneous D.E. for $\vec E_t, \vec B_t$)
b) symmetry $\omega \leftrightarrow -\omega\mu\epsilon$
$\vec E \leftrightarrow \vec B$

construct solution in two steps

a) take ∂_z derivative of last equations

b) make ansatz for z-dependence

$$\left.\begin{aligned} \vec E_t &= \vec E_t'(x,y)\, e^{ikz} \\ E_z &= E_z'(x,y)\, e^{ikz} \end{aligned}\right\} \text{ same for } \vec B$$

k ← parameter to be fixed later

$$\partial_z \left(\partial_z \vec E_t + i\omega \vec e_z \times \vec B_t \right) = \partial_z \vec\nabla_t E_z$$

$$-k^2 \vec E_t + i\omega \vec e_z \times \underbrace{(\partial_z \vec B_t)}_{i\omega\mu\epsilon\, \vec e_z \times \vec E_t + \vec\nabla_t B_z} = -ik \vec\nabla_t E_z$$

$$\Rightarrow \left(-k^2 + \omega^2\mu\epsilon\right) \vec E_t + i\omega\, \vec e_z \times \vec\nabla_t B_z = -ik \vec\nabla_t E_z$$

$$\vec E_t = \frac{i}{\mu\epsilon\omega^2 - k^2} \left(k \vec\nabla_t E_z - \omega\, \vec e_z \times \vec\nabla_t B_z \right)$$

$$\vec B_t = \frac{i}{\mu\epsilon\omega^2 - k^2} \left(k \vec\nabla_t B_z + \mu\epsilon\omega\, \vec e_z \times \vec\nabla_t E_z \right)$$

(symmetry)

$\vec E_t$ and $\vec B_t$ fixed if E_z and B_z known

Three different types of solutions

1) TEM waves (transverse electromagnetic)
degenerate case $E_z = 0$, $B_z = 0$
(homogeneous D.E. for $\vec{E}_t$ and $\vec{B}_t$)

$\vec{\nabla}_t \times \vec{E}_t = 0 \qquad \vec{\nabla}_t \cdot \vec{E}_t = 0$

$\vec{B}_t = \frac{\mu \epsilon \omega}{k} \, \vec{e}_z \times \vec{E}_t$ (compare plane wave)

strategy: solve first two D.E. to get $\vec{E}_t$
then construct $\vec{B}_t$

→ electrostatic boundary value problem
in 2 dimensions: $\vec{\nabla}_t \times \vec{E}_t = 0$
$\vec{E}_t$ is potential field $\vec{E}_t = -\vec{\nabla}_t \psi$
then: $\vec{\nabla}_t \cdot \vec{E}_t = -\vec{\nabla}_t \cdot (\vec{\nabla}_t \psi) = -\Delta_t \psi = 0$

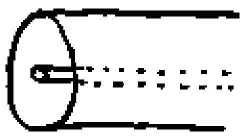
2D Laplace equation:
$$\Delta_t \psi(x,y) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \psi(x,y) = 0$$

boundary: ideal conductor, i.e.
$\psi(x,y)\big|_{surface} = \text{const}$ (same potential)

Theorem TP 2 Laplace equation for constant boundary value has only one solution $\psi = \text{const}$ (everywhere) $\Rightarrow \vec{E}_t = 0$

no solutions of TEM waves in cylindrical waveguide but: TEM waves in cylindrical geometry with 2 or more surfaces exist.

Example: coaxial cable



independent of size / geometry, the waves are not damped and have the dispersion

$$\omega = \pm u k = \pm \frac{1}{\sqrt{\epsilon\mu}} k$$

(as plane wave in infinite media)

2) TM waves (transverse magnetic)

Reminder: fields fulfill the wave equation

$$\Box_u \vec{E} = 0 \qquad \Box_u = \frac{1}{u^2}\partial_t^2 - \Delta$$
$$= \frac{1}{u^2}\partial_t^2 - \Delta_t - \partial_z^2$$

this also holds for E_z!

$$E_z = E_z'(x,y)\, e^{-i\omega t}\, e^{ikz} \qquad \text{(our ansatz)}$$

then $(\Delta_t + \gamma^2)\, E_z' = 0 \qquad \gamma^2 = \mu\epsilon\omega^2 - k^2$

(2d Helmholtz equation)

boundary conditions $E_z|_{surface} = 0$

boundary value problem has only discrete solutions for certain values γ_n^2, corresponding functions $E_z'^{(n)}(x,y)$ are called "modes": $\gamma_n^2 = \mu\epsilon\omega^2 - k_n^2$