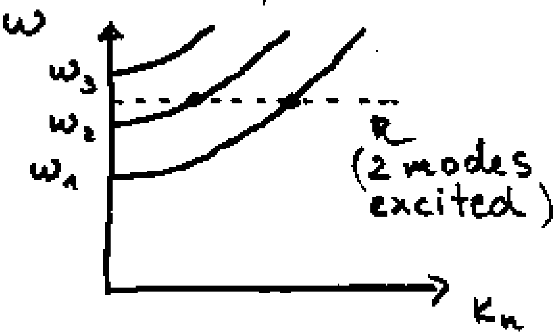
we want undamped solutions:

$$k_n^2 = \mu\epsilon\omega^2 - \gamma_n^2 > 0 \qquad (\text{i.e. } k_n \in \mathbb{R})$$

for each n, there is a minimum frequency:

$$\omega_n = \frac{\gamma_n}{\sqrt{\mu\epsilon}} \qquad \text{for } \omega \geq \omega_n : \quad \omega(k_n) = \sqrt{\frac{k_n^2 + \gamma_n^2}{\mu\epsilon}}$$



phase velocity:

$$v = \frac{\omega(k_n)}{k_n} = \frac{1}{\sqrt{\mu\epsilon}} \underbrace{\sqrt{1 + \frac{\gamma_n^2}{k_n^2}}}_{>1}$$

$\Rightarrow v > u$

diverges for $\omega \to \omega_n$ ($k_n \to 0$)

other fields from:

$B_z = 0$ (transverse magnetic)

$$\left.\begin{aligned} \vec{E}_t &= \frac{i k_n}{\gamma_n^2} \vec{\nabla}_t E_z^{(n)}(x,y) \\ \vec{B}_t &= \frac{\omega}{k_n} \epsilon\mu\, \vec{e}_z \times \vec{E}_t \end{aligned}\right\} \text{special case of equs. above}$$

3) TE (transverse electric, $E_z = 0$)

then consider wave equation for B_z:

$$(\Delta_t + \gamma^2) B_z' = 0 \qquad \gamma^2 = \mu\epsilon\omega^2 - k^2$$

(2d Helmholtz equation)

but: $\left.\dfrac{\partial B_z'}{\partial n}\right|_{\text{surface}} = 0$ (different boundary condition)

$\rightarrow$ discrete solutions for certain values $\gamma_n'^2$, but qualitatively similar dispersions.

Mathematical background: Fourier transformation and δ-function

Definition (Fourier transformation):

given function $f(x) : \mathbb{R} \to \mathbb{R}$

$$\tilde{f}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{-ikx}\, dx$$

is the Fourier transformed of $f(x)$

Definition (Convolution):

given $f(x)$ and $g(x)$:

$$F(x) = \int_{-\infty}^{\infty} f(y)\, g(x-y)\, dy \qquad \text{(Convolution of } f \text{ and } g\text{)}$$

Convolution theorem:

given $f(x)$, $g(x)$ and their Fourier transformed $\tilde{f}(k)$ and $\tilde{g}(k)$ and $F(x)$ the convolution of f and g. Calculation of

$$\tilde{F}(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(x)\, e^{-ikx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy\, f(y)\, g(x-y)\, \underbrace{e^{-ikx}}_{e^{-iky}\; e^{-ik(x-y)}}$$

substitution $s = x - y$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dy\, f(y)\, e^{-iky} \int_{-\infty}^{\infty} ds\, g(s)\, e^{-iks}$$

$$= \sqrt{2\pi}\; \tilde{f}(k)\; \tilde{g}(k)$$

Convolution of f and

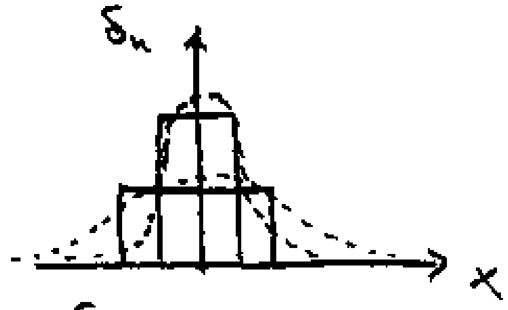
δ-function

Reminder : $\int_{-\infty}^{\infty} f(x)\, \delta(x-a)\, dx = f(a)$

$$\delta(x) = 0 \quad \forall\ x \neq 0 \,, \quad \int_{-\infty}^{\infty} \delta(x)\, dx = 1$$

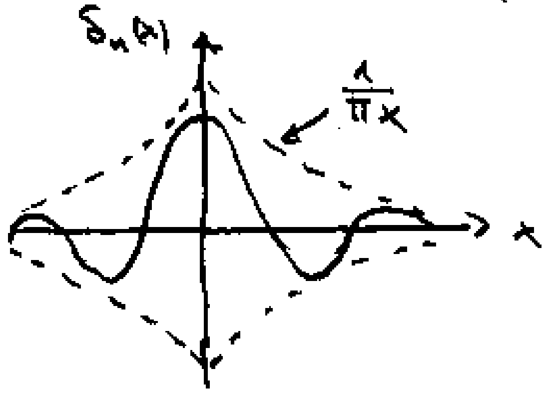
$$\delta(x) = \lim_{n\to\infty} \delta_n(x)$$

$$\delta_n(x) = \begin{cases} \frac{n}{2} & |x| < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$$



$$\delta_n(x) = \sqrt{\frac{n}{\pi}}\; e^{-nx^2}$$

$$\delta_n(x) = \frac{1}{\pi} \frac{\sin(nx)}{x}$$



(more oscillations for larger n)

$$\int_{-\infty}^{\infty} \frac{1}{\pi} \frac{\sin(nx)}{x}\, dx = 1$$

filter property : use Taylor expansion for finite n

Rewrite δ-function

$$\delta(x) = \lim_{n\to\infty} \frac{1}{\pi} \frac{\sin(nx)}{x} = \lim_{n\to\infty} \frac{1}{\pi} \frac{1}{2ix}\left(e^{inx} - e^{-inx}\right)$$

$$= \lim_{n\to\infty} \frac{1}{2\pi} \int_{-n}^{n} e^{ikx}\, dk = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ikx}\, dk$$

Theorem: Inverse Fourier transform

Given $f(x)$ and its Fourier transform $\tilde{f}(k)$

Then:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k)\, e^{ikx}\, dk$$

Proof: Use definition of FT and representation of $\delta(x)$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{f}(k)\, e^{ikx}\, dk = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk \int_{-\infty}^{\infty} dy\, f(y)\, e^{ik(x-y)}$$

$$= \int_{-\infty}^{\infty} dy\, f(y)\, \underbrace{\frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ik(x-y)}\, dk}_{\delta(x-y)} = f(x) \quad \checkmark$$

Question: What is the FT of $\delta(x)$?

Theorem of Parsival: $\int_{-\infty}^{\infty} |\tilde{f}(k)|^2\, dk = \int_{-\infty}^{\infty} |f(x)|^2\, dx$

Proof: use def. of FT and inverse FT

Fourier transform of derivative(s):

Given $f(x)$ and its FT $\tilde{f}(k)$ and

$$\tilde{f}_n(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx}\, \frac{d^n f(x)}{dx^n}\, dx \quad \text{(FT of nth derivative)}$$

If $\lim_{|x| \to \infty} f^{(n)}(x) = 0 \quad \forall\, n = 0, 1, \ldots$ (derivatives vanish at $\pm\infty$)

then $\tilde{f}_n(k) = (ik)^n\, \tilde{f}(k)$

Proof: Iteration / induction:

$$\tilde{f}_1(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} \frac{df}{dx} dx$$

$$= \frac{1}{\sqrt{2\pi}} e^{-ikx} f(x) \Big|_{-\infty}^{\infty} - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (-ik) e^{-ikx} f(x$$

$$= ik \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-ikx} f(x) dx = (ik) \tilde{f}(k)$$

For the derivatives it then holds:

$$\frac{d^n f(x)}{dx^n} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (ik)^n \tilde{f}(k) e^{ikx} dk$$

Applications of the Fourier transformation

a) derivatives → multiplicative factors
(partial) differential equations → algebraic equatio
→ use in solving differential equations

b) physics: analysis and manipulation of measurement data

c) IT: compression of acoustic or visual data
idea: knowledge of $f(x)$ or $\tilde{f}(k)$ contains the same information
instead of storing $f(x)$, one stores $\tilde{f}(k)$, but only for relevant k

Example: 2d Helmholz equation

$$\left(\Delta_t + \gamma^2\right) E'_z(x,y) = 0 \qquad \text{differential eqn.}$$

$$E'_z(x,y) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \; e^{ik_x x} \, e^{ik_y y} \, \tilde{E}'_z(k_x,k_y) \, dk_x \, dk_y$$

$$\Rightarrow \frac{1}{\sqrt{2\pi}^2} \int_{-\infty}^{\infty} dk_x \int_{-\infty}^{\infty} dk_y \left(-k_x^2 \; k_y^2 + \gamma^2\right) \overbrace{\tilde{E}'_z(k_x,k_y)}^{e^{ik_x x}\, e^{ik_y y}} dk_x \, dk_y = 0$$

or in short

$$\left(-k_x^2 - k_y^2 + \gamma^2\right) \tilde{E}'_z = 0 \qquad \text{(algebraic equation)}$$

note: $[k_x x] = 1$ dimension of x length
dimension of k_x $\frac{1}{\text{length}}$

other examples $[\omega t] = 1$ t : time
ω : $\frac{1}{\text{time}}$

Fourier transform and method of Green functions

Reminder: Method for solving inhomogeneous D.E.

Example: $\Delta \varphi = -\frac{\rho}{\epsilon_0}$ Poisson equation

1) Solve inhom. D.E for δ-function as inhomogeneity:

$$\Delta \, G(\vec{r} - \vec{r}\,') = \delta(\vec{r} - \vec{r}\,')$$

Linear differential operator

Green function

δ-function

2) Build up solution of D.E. by a convolution of the Green function and the inhomogeneity

$$\varphi(\vec{r}) = \int_{V'} G(\vec{r}-\vec{r}\,')\, S(\vec{r}\,')\, d^3\vec{r}\,' \qquad (7P2)$$

Calculate $G(\vec{r}-\vec{r}\,')$ using Fourier transform:

1) $$\Delta G(\vec{r}) = \delta^{(3)}(\vec{r})$$

$$G(\vec{r}) = \frac{1}{\sqrt{2\pi}^3} \int dk_x \int dk_y \int dk_z\, e^{i\vec{k}\cdot\vec{r}}\, \tilde{G}(\vec{k})$$

$$\delta^{(3)}(\vec{r}) = \frac{1}{(2\pi)^3} \int dk_x \int dk_y \int dk_z\, e^{i\vec{k}\cdot\vec{r}}$$

$$\Rightarrow \quad -\vec{k}^2\, \tilde{G}(\vec{k}) = \frac{1}{\sqrt{2\pi}^3}$$

$$\Rightarrow \quad \tilde{G}(\vec{k}) = -\frac{1}{\sqrt{2\pi}^3}\, \frac{1}{\vec{k}^2}$$ (Green function in Fourier space)

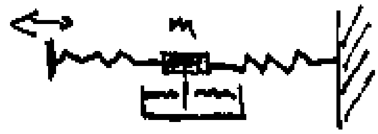
$$G(\vec{r}) = \frac{1}{\sqrt{2\pi}^3} \int e^{-i\vec{k}\cdot\vec{r}} \left(-\frac{1}{\sqrt{2\pi}^3}\, \frac{1}{\vec{k}^2}\right) d^3\vec{k}$$

actually use the inverse FT together with a regularization: $\frac{e^{-\lambda r}}{r}$

$$= -\frac{1}{4\pi|\vec{r}|}$$

2) Driven damped harmonic oscillator

$$m\ddot{x} + \alpha\dot{x} + kx = F(t)$$



$$\underbrace{\left(\partial_t^2 + 2\beta\,\partial_t + \omega_0^2\right)}_{\hat{L}\,:\ \text{linear differential operator}} x(t) = \frac{F(t)}{m} \qquad 2\beta = \frac{\alpha}{m}, \quad \omega_0^2 = \frac{k}{m}$$

$$\left(\partial_t^2 + 2\beta\partial_t + \omega_0^2\right) G(t-t') = \delta(t-t')$$

1) solve for $G(t-t')$
2) construct solution by convolution

$$x(t) = \int dt'\, G(t-t')\, \frac{F(t')}{m}$$

$$G(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \tilde{G}(\omega)\, e^{i\omega t}\, d\omega \quad , \quad \delta(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} d\omega$$

$$\Rightarrow \left(-\omega^2 + 2i\beta\omega + \omega_0^2\right)\tilde{G}(\omega) = \frac{1}{\sqrt{2\pi}}$$

$$\Rightarrow \tilde{G}(\omega) = \frac{1}{\sqrt{2\pi}}\, \frac{1}{-\omega^2 + 2i\beta\omega + \omega_0^2}$$

$\to$ solve two more integrals to get $x(t)$.

consider periodic driving $\frac{F(t)}{m} = f_0 \cos(\omega_d t)$
with driving frequency ω_d

$$x(t) = \frac{f_0}{2\pi} \int dt' \int_{-\infty}^{\infty} \frac{e^{i\omega(t-t')}}{-\omega^2 + 2i\beta\omega + \omega_0^2} \cos(\omega_d t')\, d\omega$$

$$= \frac{f_0}{2\pi}\int_{-\infty}^{\infty} d\omega\, \frac{1}{-\omega^2 + 2i\beta\omega + \omega_0^2} \int dt'\, \underbrace{e^{i\omega(t-t')}\, \tfrac{1}{2}\left(e^{i\omega_d t'} + e^{-i\omega_d t'}\right)}_{e^{i\omega t}\left(e^{i(\omega_d+\omega)t'} + e^{i(\omega-\omega_d)t'}\right)}$$

$$= f_0 \int_{-\infty}^{\infty} \frac{d\omega\, e^{i\omega t}}{-\omega^2 + 2i\beta\omega + \omega_0^2}\, \frac{1}{2\pi}\, \frac{1}{2}\int dt' \left(e^{i(\omega_d+\omega)t'} + e^{i(\omega-\omega_d)t'}\right)$$

$$= f_0 \int_{-\infty}^{\infty} \frac{d\omega\, e^{i\omega t}}{-\omega^2 + 2i\beta\omega + \omega_0^2}\, \frac{1}{2}\left[\delta(\omega_d+\omega) + \delta(\omega_d-\omega)\right]$$

$$= f_0\, \mathrm{Re}\left[\frac{e^{i\omega_d t}}{-\omega_d^2 + 2i\beta\omega_d + \omega_0^2}\right]$$

$$x(t) = \frac{f_0}{(\omega_0^2-\omega_d^2)^2 + 4\beta^2\omega_d^2}\left[(\omega_0^2-\omega_d^2)\cos(\omega_d t) + 2\beta\omega_d \sin(\omega_d t)\right]$$

(discussion: see TP1)

3) inhomogeneous wave equation of electrodynamics

$$\Box A^\mu = j^\mu$$

question: given $j^\mu = \left(\frac{\rho(\vec r, t)}{c}, \vec j_{free}(\vec r, t)\right)$ what are the electromagnetic fields?

first step: obtain Green function of the wave equation

$$\Box\, G(\vec r - \vec r', t-t') = -\,\delta(\vec r - \vec r')\,\delta(t-t')$$

Note: we have used that $\Box$ is translational invariant (in $\vec r$ and t) and set $\vec r - \vec r' = \vec R$, $t - t' = \tau$

Now: Fourier transformation in τ only

$$G(\vec R, \tau) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} d\omega\, e^{-i\omega\tau}\, G_\omega(\vec R)$$

Fourier transformed with respect to τ!

now, our wave equation reads

$$\left(\Delta_{\vec R} + \frac{\omega^2}{c^2}\right) G_\omega(\vec R) = -\,\frac{\delta(\vec R)}{\sqrt{2\pi}} \qquad (*)$$

Side remark: $G_\omega(\vec{R})$ cannot depend on the direction of $\vec{R}$!

Proof by contradiction: $G_\omega(\vec{R})$ depends on the direction, i.e. there exists a rotation matrix D_ϕ such that $G_\omega(\vec{R}) \neq G_\omega(D_\phi \vec{R})$

but:

$$\left(\Delta_R + \frac{\omega^2}{c^2}\right) \tilde{G}(\vec{K}) = \frac{1}{(2\pi)^2}$$

$$\Rightarrow \left(-\vec{K}^2 + \frac{\omega^2}{c^2}\right) \tilde{G}(\vec{K}) = \frac{1}{(2\pi)^2} \quad \text{and} \quad \Big[-\underbrace{(D_\phi \vec{K})^2}_{\vec{K}^2} + \frac{\omega^2}{c^2}\Big] \tilde{G}(D_\phi \vec{K})$$

$$\text{i.e. } \tilde{G}(\vec{K}) = \tilde{G}(D_\phi \vec{K}) \quad \text{thus } G(\vec{R}) = G(D_\phi \vec{R})\ !$$

with $G_\omega(\vec{R}) = G_\omega(R, \cancel{\vartheta}, \cancel{\varphi})$ we get for $R \neq 0$

$$0 = \left(\Delta_{\vec{R}} + \frac{\omega^2}{c^2}\right) G_\omega(R) = \left[\underbrace{\frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2 \frac{\partial G_\omega(R)}{\partial R}\right)}_{\text{radial part of } \Delta_{\vec{R}}} + \frac{\omega^2}{c^2} G_\omega(R)\right]$$

$$\Rightarrow \frac{d^2}{dR^2}\left(R\, G_\omega\right) + \frac{\omega^2}{c^2}\left(R\, G_\omega\right) = 0$$

two linear independent solutions:

$$R\, G_\omega = A\, e^{\pm \frac{\omega}{c} R}$$

we define two different Green functions

$$G_\omega^\pm(R) = \frac{A}{R}\, e^{\pm i \frac{\omega}{c} R}$$

for $\omega = 0$, we get back the Poisson equation, thus we fix the parameter $A = \frac{1}{\sqrt{2\pi}} \frac{1}{4\pi}$
(because of the extra $\frac{1}{\sqrt{2\pi}}$ in (*))

We calculate the inverse FT:

$$G^{\pm}(R,\tau) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, e^{-i\omega\tau} \frac{1}{\sqrt{2\pi}} \frac{1}{4\pi R} e^{\pm i\frac{\omega}{c}R}$$

$$= \frac{1}{4\pi R} \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega(\tau \mp \frac{R}{c})}$$

$$= \frac{1}{4\pi R} \delta(\tau \mp \frac{R}{c})$$

Solution of the inhomogeneous wave equation by convolution of Green function and inhomogeneity:

$$\varphi^{\pm}(t,\vec{r}) = \phi_0(t,\vec{r}) + \int d^3\vec{r}\,' \int dt'\, G^{\pm}(\vec{r}-\vec{r}\,', t-t') \frac{\rho(\vec{r}\,',t')}{\epsilon_0}$$

general solution of homogeneous D.E.

special solution of inhom. D.E.

$\frac{\rho}{\epsilon_0}$ is inhomogeneity in wave eqn.

$$= \varphi_0(t,\vec{r}) + \int d^3\vec{r}\,' \int dt' \frac{\delta\left(t-t' \mp \frac{|\vec{r}-\vec{r}\,'|}{c}\right)}{4\pi\epsilon_0 |\vec{r}-\vec{r}\,'|} \rho(\vec{r}\,',t')$$

$$= \varphi_0(t,\vec{r}) + \int d^3\vec{r}\,' \frac{\rho\left(\vec{r}\,', t \mp \frac{|\vec{r}-\vec{r}\,'|}{c}\right)}{4\pi\epsilon_0 |\vec{r}-\vec{r}\,'|}$$

physical interpretation

$\varphi^{+}(t,\vec{r})$ potential at $\vec{r}$ and time t which is generated from a charge density at positions $\vec{r}\,'$ at an earlier time $t' = t - \frac{|\vec{r}-\vec{r}\,'|}{c}$
e.m. wave needs $\frac{|\vec{r}-\vec{r}\,'|}{c}$ to reach point $\vec{r}$ from $\vec{r}\,'$ (causality)

$G^{+}(R,\tau)$: causal or retarded Green function
$G^{-}(R,\tau)$: advanced Green function

same for the vector potential

$$\vec{A}^{+}(t,\vec{r}) = \vec{A}_0(t,\vec{r}) + \int d^3\vec{r}\,' \, \frac{\mu_0 \vec{j}(\vec{r}\,', t \mp \frac{|\vec{r}-\vec{r}\,'|}{c})}{4\pi\, |\vec{r}-\vec{r}\,'|}$$

for potentials $\vec{A}^{+}, \varphi^{+}$, we have to consider the (earlier) retarded time

$$t_R = t - \frac{|\vec{r}-\vec{r}\,'|}{c} \qquad \text{note: } t_R = t_R(\vec{r}\,', \vec{r}, t)$$

Note: If the charge and current density is known in "frequency space" (FT with respect to time), we obtain

$$\varphi_\omega(\vec{r}) = \varphi_{0,\omega}(\vec{r}) + \frac{1}{4\pi\epsilon_0} \int d^3\vec{r} \, \frac{\rho_\omega(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \, e^{i\frac{\omega}{c}|\vec{r}-\vec{r}\,'|}$$

$$A_\omega(\vec{r}) = \vec{A}_{0,\omega}(\vec{r}) + \frac{\mu_0}{4\pi} \int d^3\vec{r} \, \frac{\vec{j}_\omega(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \, e^{i\frac{\omega}{c}|\vec{r}-\vec{r}\,'|}$$

Calculation of e.m. fields from potentials

$$\vec{B}(t,\vec{r}) = \vec{\nabla} \times \vec{A}^{+}(t,\vec{r})$$

$$\vec{E}(t,\vec{r}) = -\vec{\nabla}\varphi^{+}(t,\vec{r}) - \frac{\partial}{\partial t}\vec{A}^{+}(t,\vec{r})$$

or in frequency space

$$\vec{B}_\omega(\vec{r}) = \vec{\nabla} \times A_\omega(\vec{r})$$

$$\vec{E}_\omega(\vec{r}) = -\vec{\nabla}\varphi_\omega(\vec{r}) + i\omega \vec{A}_\omega(\vec{r})$$