Radiation field, radiated energy

calculation of magnetic induction

$\vec{B}(t,\vec{r}) = \vec{\nabla}\times\vec{A}^{+}(t,\vec{r})$ complicated because of $t_R = t - \frac{|\vec{r}-\vec{r}\,'|}{c}$ argument

frequency space

$$\vec{B}_\omega(\vec{r}) = \vec{\nabla}_{\vec{r}} \times \frac{\mu_0}{4\pi}\int d^3\vec{r}\,' \frac{\vec{j}_\omega(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} e^{i\frac{\omega}{c}|\vec{r}-\vec{r}\,'|}$$

$\vec{\nabla}\times(\vec{c}\varphi) = -\vec{c}\times\vec{\nabla}\varphi$

$$\stackrel{\downarrow}{=} -\frac{\mu_0}{4\pi}\int d^3\vec{r}\,' \vec{j}_\omega(\vec{r}\,') \times \vec{\nabla}_r \frac{e^{i\frac{\omega}{c}|\vec{r}-\vec{r}\,'|}}{|\vec{r}-\vec{r}\,'|}$$

$$= \frac{\mu_0}{4\pi}\int d^3\vec{r}\,'\Big[\vec{j}_\omega(\vec{r}\,') \times \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^3} e^{i\frac{\omega}{c}|\vec{r}-\vec{r}\,'|} - i\frac{\omega}{c}\vec{j}_\omega(\vec{r}\,') \times \frac{\vec{r}-\vec{r}\,'}{|\vec{r}-\vec{r}\,'|^2} e^{i\frac{\omega}{c}|\vec{r}-\vec{r}\,'|}\Big]$$

FT to time space

$$\vec{B}(t,\vec{r}) = \frac{1}{\sqrt{2\pi}}\int e^{-i\omega t}\vec{B}_\omega(\vec{r})\,d\omega$$

$$= \frac{\mu_0}{4\pi}\int d^3\vec{r}\,'\Big[\frac{\vec{j}(\vec{r}\,',t_R)\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} + \frac{1}{c}\frac{\dot{\vec{j}}(\vec{r}\,',t_R)\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^2}\Big]$$

$\sim\frac{1}{r^2}$: retarded magnetotatics $\qquad \sim\frac{1}{r}$: radiation field

calculation of electric field

$\vec{E}_\omega = -\vec{\nabla}\varphi_\omega + i\omega\vec{A}_\omega$ use: $\vec{\nabla}\vec{j}_\omega - i\omega\rho_\omega = 0$

$\vec{a}\times(\vec{b}\times\vec{c}) = \ldots$

Gauß theorem

then:

$\vec{E}(t,\vec{r}) = \frac{1}{\sqrt{2\pi}}\int e^{-i\omega t}E_\omega(\vec{r})\,d\omega$ $\qquad \int_V \vec{\nabla}\cdot(\ldots)\,dV = \oint_{S(V)}(\ldots)\cdot d\vec{a} \to 0$

retarded electrostatics $\sim \frac{1}{r^2}$

retarded induction $\sim \frac{1}{r^2}$

radiation field $\sim \frac{1}{r}$

$$\vec{E}(t,\vec{r}) = \frac{1}{4\pi\epsilon_0}\int d^3\vec{r}\,'\left[\frac{\rho(\vec{r}\,',t_R)(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3} + \frac{1}{c}\,\frac{\left(\vec{J}(\vec{r}\,',t_R)\cdot(\vec{r}-\vec{r}\,')\right)(\vec{r}-\vec{r}\,') - (\vec{r}-\vec{r}\,')\times\left(\vec{J}(\vec{r}\,',t_R)\times(\vec{r}-\vec{r}\,')\right)}{|\vec{r}-\vec{r}\,'|^4} + \frac{1}{c^2}\,\frac{\left(\dot{\vec{J}}(\vec{r}\,',t_R)\times(\vec{r}-\vec{r}\,')\right)\times(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^3}\right]$$

Note: for integration consider:

$$t_R = t - \frac{|\vec{r}-\vec{r}\,'|}{c} = t_R(\vec{r}\,')$$

Radiated energy

$$U_{rad} = \int_{-\infty}^{\infty} dt \int_{S(V)} \vec{S}(t,\vec{r})\cdot d\vec{a}$$

sphere with radius $R \gg |\vec{r}\,'|$ for localized radiation source

Poynting vector
$$\vec{S} = \vec{E}\times\vec{H} = \frac{1}{\mu_0}\vec{E}\times\vec{B}$$

surface of sphere $A_0 = 4\pi R^2$

for $R \gg |\vec{r}\,'|$: only radiation terms contribute!

define $\hat{k} = \frac{\vec{r}}{r}$, $\vec{k} = \hat{k}\frac{\omega}{c}$

then (radiated energy per solid angle in freq. interval $[\omega,\omega+d\omega]$)

$$\frac{dU_{rad}}{d\Omega}\,d\omega = \frac{1}{4\pi}\sqrt{\frac{\mu_0}{\epsilon_0}}\left|\frac{1}{\sqrt{2\pi}}\vec{J}_\omega(\vec{k})\times\vec{k}\right|^2 d\omega$$

FT of $\vec{J}_\omega(\vec{r})$

Special case: monochromatic source

$$\vec{j}(\vec{r},t) = \vec{j}(\vec{r})\cos(\omega_0 t)$$

FT: $$\vec{j}_\omega(\vec{k}) = \frac{1}{\sqrt{2\pi}}\vec{j}(\vec{k})\ \frac{1}{2}\ 2\pi\left[\delta(\omega-\omega_0)+\delta(\omega+\omega_0)\right]$$

$$\vec{j}(\vec{k}) = \frac{1}{\sqrt{2\pi}^3}\int d^3\vec{r}\ \vec{j}(\vec{r})\,e^{-i\vec{k}\cdot\vec{r}}$$

integration over all frequencies

$$\frac{dU_{rad}}{d\Omega} = \int_0^\infty \frac{dU_{rad}}{d\Omega}\,d\omega = \delta(\omega=0)\,\frac{1}{16\pi}\sqrt{\frac{\mu_0}{\epsilon_0}}\left|\vec{j}(\vec{k})\times\vec{k}\right|^2$$

formally divergent; source radiates continously at all times

→ consider radiated energy averaged over period $\tau = \frac{2\pi}{\omega_0}$

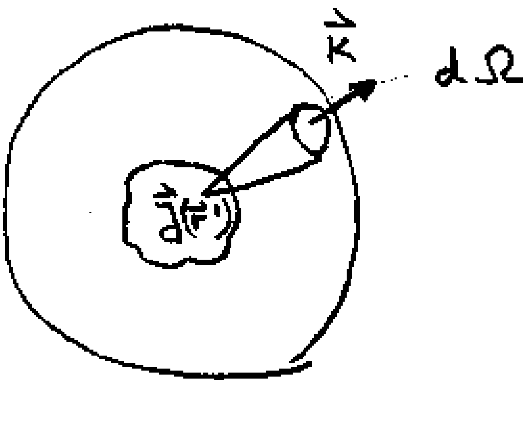
$$\frac{d\overline{P}_{rad}}{d\Omega} = \frac{1}{\tau}\int_0^\tau d\tau \oint_{S(V)} \vec{S}\cdot d\vec{a}$$

radiated power per solid angle

reminder $$2\pi\,\delta(\omega) = \lim_{T\to\infty}\int_{-\frac{T}{2}}^{\frac{T}{2}} dt\, e^{i\omega t}$$

$$2\pi\,\delta(0) = \lim_{T\to\infty} T$$

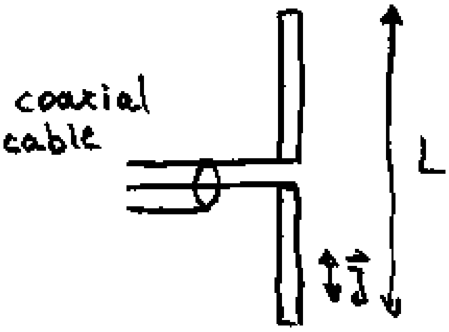
then $$\frac{dP_{rad}}{d\Omega} = \lim_{T\to\infty}\left(\frac{dU_{rad}}{d\Omega\, T}\right) = \frac{1}{32\pi^2}\underbrace{\sqrt{\frac{\mu_0}{\epsilon_0}}}_{R_0}\ \underbrace{\left|\vec{j}(\vec{k})\times\vec{k}\right|}_{\text{angular dependence}}$$



$$R_0 = \sqrt{\frac{\mu_0}{\epsilon_0}} \approx 377\,\Omega$$

(impedance of free space)

Example of radiation source: antenna



here: ideal antenna
(ideal conductor $\sigma \to \infty$)

$$\vec{j}(\vec{r}',t) = \vec{e}_z I_0 \, \delta(x')\,\delta(y') \, \frac{\sin\left(k\left(\frac{L}{2} - |z|\right)\right)}{\sin\left(k\frac{L}{2}\right)} \cos(\omega t)$$

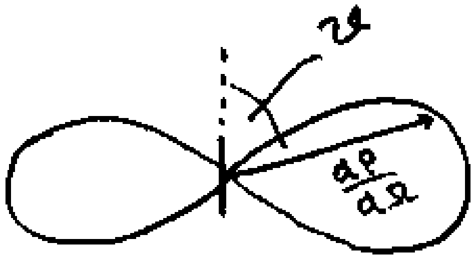
$\omega = ck$

I_0: maximal current

calculate $\vec{j}(\vec{k}) = \frac{1}{\sqrt{2\pi}^3}\int d^3 r' \, e^{-i\vec{k}\cdot\vec{r}'} \, I_0 \, \delta(x')\,\delta(y') \, \frac{\sin\left(k\left(\frac{L}{2} - |z'|\right)\right)}{\sin\left(k\frac{L}{2}\right)}$

$$= \frac{2 I_0 \vec{e}_z}{\sqrt{2\pi}^3 \sin\left(k\frac{L}{2}\right)} \int_0^{L/2} dz' \, \sin\left[k\left(\tfrac{L}{2} - z'\right)\right] \cos(k z' \cos\vartheta)$$

$$\frac{dP_{rad}}{d\Omega} = \frac{1}{32\pi^2} R_0 \, 4 I_0^2 \underbrace{\left(\frac{\cos\left(\frac{kL}{2}\cos\vartheta\right) - \cos\left(\frac{kL}{2}\right)}{\sin\left(\frac{kL}{2}\right)\sin\vartheta}\right)^2}_{\approx \frac{(kL)^2}{8} \frac{\cos^2\vartheta - 1}{\underbrace{\sin\left(\frac{kL}{2}\right)}_{\approx \frac{kL}{2}} \sin\vartheta} \approx \frac{kL}{4}\sin\vartheta}$$



typical dipolar radiation
(only parallel to dipole moment)

$$\frac{dP_{rad}}{d\Omega} \approx \frac{I_0^2}{8\pi^2} R_0 \left(\frac{kL}{4}\sin\vartheta\right)^2$$

integration over solid angle: $\int_0^{\pi} \sin^3\vartheta \, d\vartheta = \frac{4}{3}$

power needed from source

$$P_{rad} = \int \frac{dP_{rad}}{d\Omega} d\Omega = \frac{I_0^2 k^2 L^2 R_0}{48\pi} = \frac{1}{2} R_{rad} I_0^2$$

$R_{rad} = \frac{k^2 L^2}{24\pi} R_0$ (radiation impendance)

Some remarks for approximation schemes (monochromatic sources)

$$\left.\begin{aligned} \rho(\vec r, t) &= \rho(\vec r)\, e^{-i\omega t} \\ \vec j(\vec r, t) &= \vec j(\vec r)\, e^{-i\omega t} \end{aligned}\right\} \to$$ all quantities have the same time dependence

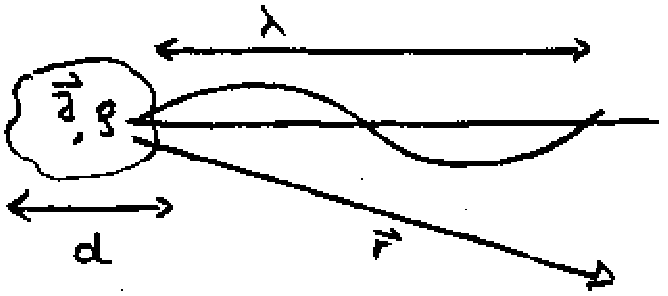
especially $\vec A(t, \vec r) = A(\vec r)\, e^{-i\omega t}$

$$\vec A(\vec r) = \frac{\mu_0}{4\pi} \int d^3 r' \, \frac{\vec j(\vec r\,')}{|\vec r - \vec r\,'|} \, e^{i \frac{\omega}{c} |\vec r - \vec r\,'|}$$

then, we can calculate $\vec B = \vec\nabla \times \vec A$

and further $\vec E = i \frac{c^2}{\omega} \vec\nabla \times \vec B$ (use $\omega = c k$)

geometric limits ("small parameter")



small source

$$\frac{d}{\lambda} = \frac{kd}{2\pi} = \frac{\omega d}{2\pi c} \ll 1$$

far away

$d \ll |\vec r|$

approximations in $\vec A(\vec r)$

small source + far away

$$|\vec r - \vec r\,'| = r \sqrt{1 - \frac{2 \vec r \cdot \vec r\,'}{r^2} + \frac{r'^2}{r^2}} \approx r - \hat k \cdot \vec r\,'$$

then $$\frac{e^{ik|\vec r - \vec r\,'|}}{|\vec r - \vec r\,'|} \approx \frac{e^{ikr}}{r} \left(1 + \hat k \cdot \vec r\,'\right)\left(\frac{1}{r} - ik\right) \quad (*)$$

$$\frac{e^{ik|\vec r - \vec r\,'|}}{|\vec r - \vec r\,'|} \approx \frac{1}{r} e^{ikr} e^{-i \vec k \cdot \vec r\,'}$$ (only far away, but arbitrary size)

(*) → expansion in
a) electric dipole radiation
b) magnetic dipole + electric quadrupole radiation
c) (...) Multipole expansion ....

Radiation of a single accelerated charge

given : trajectory of point charge q $\vec{R}(t)$

charge density $\varrho(\vec{r},t) = q\,\delta(\vec{r}-\vec{R}(t))$

$\vec{j}(\vec{r},t) = q\,\vec{v}(t)\,\delta(\vec{r}-\vec{R}(t))$

goal : calculate $\vec{E}(t,\vec{r})$, $\vec{B}(t,\vec{r})$

use : retarded potentials

$$\varphi^+(t,\vec{r}) = \frac{1}{4\pi\epsilon_0}\int_{-\infty}^{\infty} dt' \int d^3r' \,\frac{\varrho(\vec{r}\,',t')}{|\vec{r}-\vec{r}\,'|}\,\delta\left(t'-t+\frac{|\vec{r}-\vec{r}\,'|}{c}\right)$$

do not use $\delta(t'\ldots)$

$$= \frac{1}{4\pi\epsilon_0}\int_{-\infty}^{\infty} dt' \int d^3r' \,\frac{q\,\delta(\vec{r}\,'-\vec{R}(t'))}{|\vec{r}-\vec{r}\,'|}\,\delta\left(t'-t+\frac{|\vec{r}-\vec{r}\,'|}{c}\right)$$

$$= \frac{q}{4\pi\epsilon_0}\int_{-\infty}^{\infty} dt' \,\frac{\delta\left(t'-t+\frac{|\vec{r}-\vec{R}(t')|}{c}\right)}{|r-\vec{R}(t')|}$$

evaluation of the t' integral

Reminder $\delta(f(t')) = \sum_i \frac{\delta(t'-t_i)}{|f'(t_i)|}$ t_i : simple zeros of function $f(t')$

we calculate the derivative

$$f(t') = t'-t+\frac{|\vec{r}-\vec{R}(t')|}{c}$$

$$f'(t') = 1-\frac{1}{c}\,\frac{(\vec{r}-\vec{R}(t'))\cdot\dot{\vec{r}}(t')}{|\vec{r}-\vec{R}(t')|} \qquad \dot{\vec{r}}(t') = \vec{v}(t')$$

triangle equation

$\vec{a}\cdot\vec{b} \le ab$

$-ab \le \vec{a}\cdot\vec{b}$

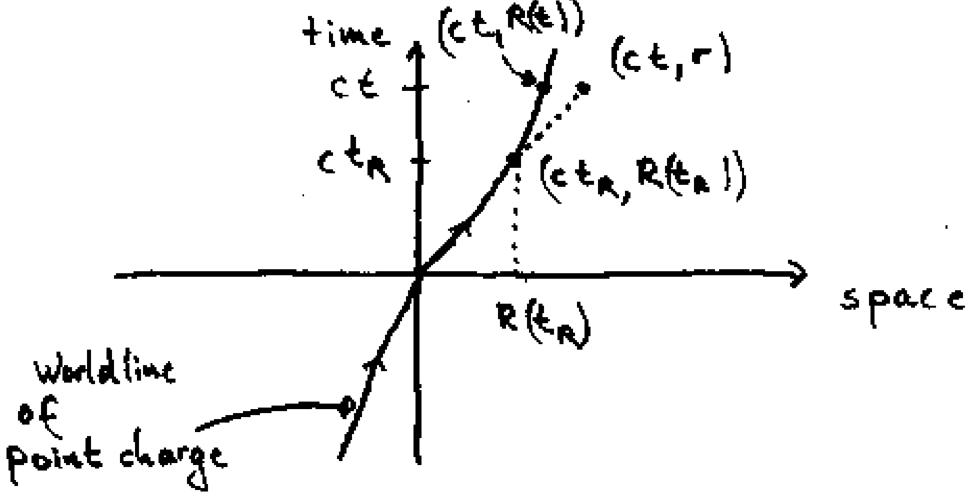
$$1-\frac{v(t')}{c} \le f'(t') \le 1+\frac{v(t')}{c}$$

especially : $0 \le f'(t')$, i.e. $f(t')$ is monotonously increasing and has at most one (simple) zero , actually it should have exactly one zero (phys. arguments

$f(t_R) = 0$ yields $t_R = t - \frac{|\vec{r} - \vec{R}(t_R)|}{c}$

(retarded time)

Interpretation in Minkowski diagram



inv. distane $s^2 = (ct - ct_R)^2 - (r - R(t_R))^2 = 0$

(lightlike)

We rewrite:

$f'(t_R) = 1 - \vec{\beta} \cdot \frac{\vec{D}}{D}$ with $\vec{\beta}(t_R) = \frac{\vec{v}(t_R)}{c}$

$\vec{D} = \vec{r} - \vec{R}(t_R)$

retarded potential

$$\varphi^+(t, \vec{r}) = \frac{q}{4\pi\epsilon_0} \int_{-\infty}^{\infty} \frac{\delta(t' - t_R)\, dt'}{D(t_R, \vec{r}) \left[1 - \vec{\beta}(t_R) \cdot \frac{\vec{D}(t_R, \vec{r})}{D(t_R, \vec{r})}\right]}$$

$$= \frac{q}{4\pi\epsilon_0} \frac{1}{D(t_R, \vec{r}) - \vec{\beta}(t_R) \cdot \vec{D}(t_R, \vec{r})}$$

introducing $S(t_R, \vec{r}) = D(t_R, \vec{r}) - \vec{\beta}(t_R) \cdot \vec{D}(t_R, \vec{r})$

we obtain

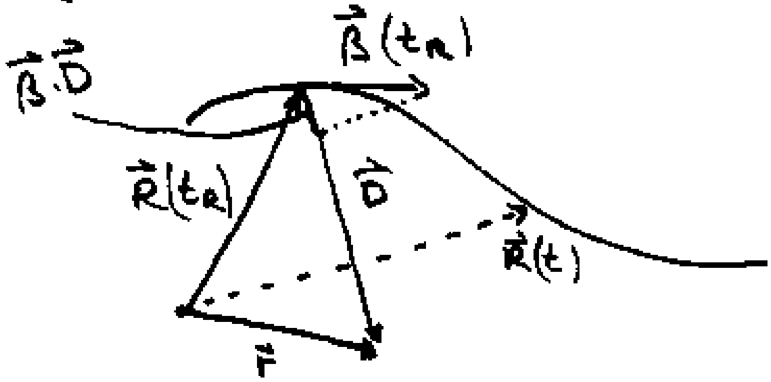
$$\varphi^+(t, \vec{r}) = \frac{q}{4\pi\epsilon_0\, S(t_R, \vec{r})}$$

(Liénard - Wichert potential)

vector potential : $\rho \to \mu_0 \vec{j} = q\vec{v}(t)\, \delta(\vec{r} - \vec{R}(t))$

$$\vec{A}^+(t, \vec{r}) = \frac{q\, \vec{v}(t_R)}{4\pi\epsilon_0\, S(t_R, \vec{r})} = \vec{\beta}(t_R) \frac{\varphi^+(t, \vec{r})}{c}$$

geometrical interpretation



$\vec{D} = \vec{r} - \vec{R}(t_R)$

$S = D - \vec{\beta}\cdot\vec{D}$

reduced distance between position at retarded time and observer

potential's contain 2 effects

a) retarded time (em. wave needs to be created at earlier time to reach observer)

b) moving charge density towards observer will be "compressed" when evaluated at t_R $S = D - \vec{\beta}\cdot\vec{D} < D$, and diluted if charge density moves away

Note: $\vec{A} = \vec{\beta}\,\frac{\varphi}{c}$ (Lorentz transformation from Σ' where particle is at rest $\vec{A}' = 0$)

→ contains physics of Lorentz force

fields:

$$\vec{E} = -\vec{\nabla}\varphi(t,\vec{r}) - \frac{\partial}{\partial t}\vec{A}(t,\vec{r})$$

(use chain rule for $t_A = t_A(t, \vec{r}, \vec{R})$)

$$E(\vec{r},t) = \vec{E}_v(\vec{r},t) + E_a(\vec{r},t)$$

only contains $\vec{v}$

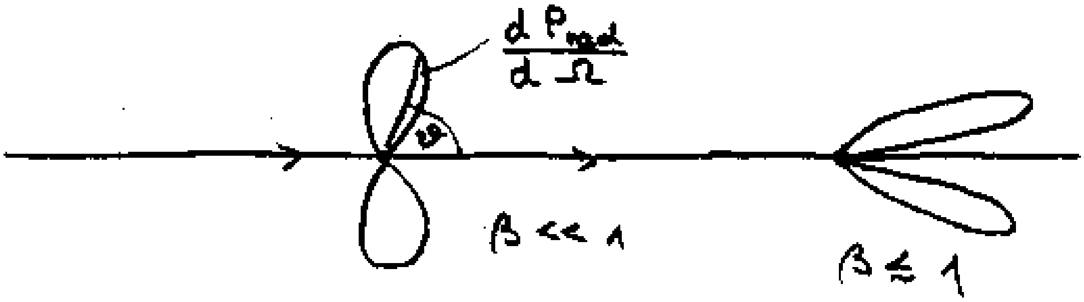
(similar for $\vec{B}(\vec{r},t)$)

also contains $\dot{\vec{v}} = \vec{a}$: radiation (why?)

Examples :

1) (accelerated) straight line motion

$\vec{v} = v(t)\,\vec{e}_z \qquad \dot{\vec{v}} = \dot{v}(t)\,\vec{e}_z$



$$\frac{dP_{rad}}{d\Omega} = \frac{q^2\,\dot{v}^2}{16\pi^2\epsilon_0 c^3}\;\frac{\sin^2\vartheta}{(1-\beta\cos\vartheta)^5}$$ (radiated power)

$\beta \ll 1$:

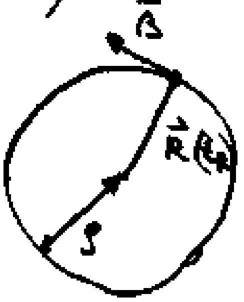
$$\frac{dP_{rad}}{d\Omega} \approx \frac{q^2\,\dot{v}^2}{16\pi^2\epsilon_0 c^3}\,\sin^2\vartheta$$ (Lamor formula)

(describes radiation (X-ray) of decelerated electrons in metals)

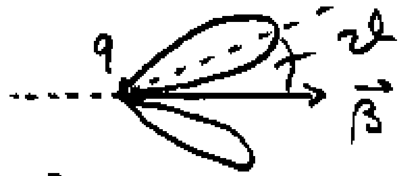
2) circular motion (with constand angular frequency

$v = \rho\,\omega_0 \qquad \vec{v} = v\,\vec{e}_\varphi$

$\dot{v} = \rho\,\omega_0^2 \qquad \dot{\vec{v}} = -\dot{v}\,\vec{e}_\rho$



use coordinate system with $\vec{\beta} \parallel \vec{e}_z$



$$\frac{dP_{rad}}{d\Omega} = \frac{q^2\,\dot{v}^2}{16\pi\epsilon_0 c^3}\;\frac{(1-\beta\cos\vartheta)^2 - (1-\beta^2)\sin^2\vartheta\cos^2\varphi}{(1-\beta\cos\vartheta)^5}$$