PHY6095/PHZ6166: Final exam: Solutions

due, Tuesday, 04/30, 10 a.m.
submit your work either in person or by e-mail

You must work individually to receive full credit

- **Problem 1: 34 points**
A neutron star can be considered as an ideal and degenerate Fermi gas at $T = 0$. The star is held together by a balance between the outward Pauli pressure and inward gravitational force.Assuming that the star is spherically symmetric, the gravitational potential ϕ satisfies the Newton equation

$$\nabla^2\phi = 4\pi G\rho(r),$$

where G is the gravitational constant and $\rho(r)$ is the neutron mass density. The potential energy of a neutron is $U = m_n\phi$, where m_n is the neutron mass. In equilibrium, $\mu(r) + m_n\phi(r) = \text{const}$, where $\mu(r)$ is the chemical potential of the neutron gas.

  - Find the condition for the neutron gas to be non-relativistic. The condition must be in a form of strong inequality ($\ldots \ll \ldots$). (Hint: Write down the scaling forms of ρ and μ as a function of the distance from the star's center but do not attempt to find the scaling functions explicitly.)
  The chemical potential of a non-relativistic Fermi gas is related to the number density $n = \rho/m_n$ via

$$\mu \sim \frac{\hbar^2 n^{2/3}}{m_n} = \frac{\hbar^2\rho^{2/3}}{m_n^{5/3}}$$

  which gives

$$\rho \sim m_n^{5/2}\mu^{3/2}/\hbar^3. \tag{1}$$

  From the equilibrium condition, $d\mu + m_n d\phi$. Substituting the last two relations into the Newton law, we obtain an equation for μ:

$$-\nabla^2\mu = \alpha\mu^{3/2},$$

  where

$$\alpha = am_n^{7/2}G/\hbar^3$$

  and $a \sim 1$ is a numerical coefficient. The chemical potential can depend only on α, distance for the star center r, and R. Recall that from gravity law

$$U = -G\frac{m_1 m_2}{r},$$

  the units of G are $[G] = [E][L]/[M]^2$. Then, the units of α are

$$[\alpha] = \frac{[M]^{7/2}[E][L]}{[M]^2[E]^3[t]^3} = \frac{[M]^{3/2}[L]}{[E]^2[t]^3} = \frac{1}{[E]^{1/2}[L]^2}$$

  Therefore, the units of μ (energy) can only be formed as

$$[\mu] = \frac{1}{[\alpha]^2[L]^4}$$

  For $[L]$, one can use either r or R, which leaves out one dimensional variable, r/R. Therefore,

$$\mu(r) = \frac{1}{\alpha^2 R^4} f\left(\frac{r}{R}\right),$$

  where $f(x)$ is some scaling function, such that $f(x \sim 1) \sim 1$. At distances $r \sim R$,

$$\mu \sim 1/\alpha^2 R^4. \tag{2}$$

  The gas can be considered as non-relativistic, if $\mu \ll m_n c^2$, or

$$\alpha^2 R^4 m_n c^2 \gg 1 \Rightarrow \frac{m_n^8 G^2 R^4 c^2}{\hbar^6} \gg 1.$$

  Note that the result for μ cannot be determined only by dimensional analysis. Indeed, we have 4 relevant parameters (m_n, $\hbar$, R, and G) but only three independent units ($[M]$,$[L]$, and $[t]$), which means that the exponents of the parameters in the formula for μ are not determined uniquely. The Newton equation specifies uniquely how G enters the result.

– Assuming that the condition above is satisfied, find how the total mass of the star scales with the star's radius, R.
Using relation (1) and equation (2) for μ, we obtain

$$\rho = \frac{\hbar^6}{m_n^8 G^3 R^6} g\left(\frac{r}{R}\right),$$

where g is another scaling function. The total mass is

$$M = 4\pi \int dr r^2 \rho(r) \sim \frac{\hbar^6}{m_n^8 G^3 R^3}.$$

– Suppose now that the neutron gas is ultra-relativistic, i.e., the neutron energy $E_n = pc$, where p is its momentum. Find how the the total mass of the star depends on R in this case.
For a relativistic Fermi gas,

$$\mu \sim \hbar c n^{1/3} \sim \hbar c (\rho/m)^{1/3} \Rightarrow \rho \sim \frac{m}{\hbar^3 c^3}\mu^3.$$

and the Newton law becomes

$$-\nabla^2 \mu = \beta \mu^3,$$

where

$$\beta = b\frac{m^2 G}{\hbar^3 c^3}$$

and $b \sim 1$. Since $[\beta] = 1/[E]^2 L^2$, we can write μ as

$$\mu = \frac{1}{\beta^{1/2} R} \bar{f}\left(\frac{r}{R}\right)$$

and

$$\rho = \frac{m}{\hbar^3 c^3 \beta^{3/2} R^3} \bar{g}\left(\frac{r}{R}\right)$$

Therefore, M does not depend on R

$$M \sim \frac{m}{\hbar^3 c^3 \beta^{3/2}} \sim \frac{1}{m^2}\left(\frac{\hbar c}{G}\right)^{3/2}.$$

This mean that equilibrium is possible only for one value of the total mass.

- **Problem 2: 33 points**

– Find the asymptotic behavior of the following integral both for $a \ll 1$ and $a \gg 1$

$$I(a) = \int_0^\infty dx \frac{\cos(ax^2)}{\sqrt{x^2+1}}$$

For $a \ll 1$, substitute $y = \sqrt{x}$:

$$I(a) = 2\int_0^\infty dy \frac{y\cos(ay)}{y^2+1}$$

Now integrate by parts, noticing that

$$\frac{d}{dy}\frac{y}{1+y^2}\Big|_{y=0} = 1.$$

This gives

$$I(a \gg 1) = -\frac{2}{a^2}.$$

Convergence of the integral at large x is only due to the cosine factor: If one sets $a = 0$, the integral diverges logarithmically. This means that the upper limit can be chosen from the condition when the argument of the cosine becomes of order 1, i.e., $x \sim 1/a^2$:

$$I(a \ll 1) = \int_1^{1/a^2} \frac{dx}{x} = 2 \ln \frac{1}{a}.$$

– Estimate the integral

$$J(a) = \int_0^\infty dx \sqrt{\frac{x}{1+x}} e^{-\left(x + \frac{a}{x}\right)}.$$

for $a \gg 1$.

The argument of the exponential is minimal at $x_m = \sqrt{a}$. Expand the argument around the the minimum as

$$x + \frac{a}{x} = 2\sqrt{a} + \frac{1}{\sqrt{a}}(x - x_m)^2.$$

Since $x_m \gg 1$, the pre-exponential factor can be replaced by 1. Shift the variable $y = x - x_m$ and extend the limits of integration from $0, \infty$ to $-\infty, \infty$:

$$J(a) = \exp(-2\sqrt{a}) \int_{-\infty}^{\infty} dy \exp\left(-\frac{y^2}{\sqrt{a}}\right) = \sqrt{\pi} a^{1/4} \exp(-2\sqrt{a}).$$

- **Problem 3: 33 points**

A regular hexagon has C_{6v} symmetry. The table of characters of this group is given below. According to this table, a polar vector (x, y, z) transforms as $\Gamma' = A_1 \oplus E_1$.

– Explain why another 2D representation, E_2, is not a suitable representation for a polar vector.

E_2 is even under C_2, while a polar vector must be odd.

– Which transitions can be induced by an external electric field?

C_{6v}		I	C_2	$2C_3$	$2C_6$	$3\sigma_v$	$3\sigma'_v$
z	A_1	1	1	1	1	1	1
	A_2	1	1	1	1	-1	-1
	B_1	1	-1	1	-1	-1	1
	B_2	1	-1	1	-1	1	-1
(x, y)	E_1	2	-2	-1	1	0	0
	E_2	2	2	-1	-1	0	0

1. Initial state: A_1

$$\Gamma' \otimes A_1 = \Gamma' = A_1 \oplus E_1$$

Allowed transition: $A_1 \to E_1$.

2. Initial state: A_2.

$$\Gamma' \otimes A_2 = A_1 \otimes A_2 \oplus E_1 \otimes A_2 = A_2 \oplus E_1 \otimes A_2$$

C_{6v}		I	C_2	$2C_3$	$2C_6$	$3\sigma_v$	$3\sigma_v'$
z	A_1	1	1	1	1	1	1
	A_2	1	1	1	1	-1	-1
	B_1	1	-1	1	-1	-1	1
	B_2	1	-1	1	-1	1	-1
(x, y)	E_1	2	-2	-1	1	0	0
	E_2	2	2	-1	-1	0	0
	$E_1 \otimes A_2$	2	-2	-1	1	0	0
	$E_1 \otimes B_1$	2	2	-1	-1	0	0
	$E_1 \otimes B_2$	2	2	-1	-1	0	0
	$E_1 \otimes E_1$	4	4	1	1	0	0
	$E_1 \otimes E_2$	4	-4	1	-1	0	0

Comparing the characters, we see that $E_1 \otimes A_2 = E_1$.
Allowed transition $A_2 \to E_1$.

3. Initial state: B_1.

$$\Gamma' \otimes B_1 = B_1 \oplus E_1 \otimes B_1.$$

Comparing the characters, we see that $E_1 \otimes B_1 = E_2$.
Allowed transition: $B_1 \to E_2$.

4. Initial state: B_2.

$$\Gamma' \otimes B_2 = B_2 \oplus E_1 \otimes B_2 = B_2 \otimes E_2.$$

Allowed transition: $B_2 \to E_2$.

5. Initial state: E_1.

$$\Gamma' \otimes E_1 = E_1 \oplus E_1 \otimes E_1.$$

Decomposition

$$E_1 \otimes E_1 = A_1 \oplus A_2 \oplus E_2.$$

Allowed transitions: $E_1 \to A_1$, $E_1 \to A_2$, $E_1 \to E_2$.

6. Initial state: E_2.

$$\Gamma' \otimes E_2 = E_2 \oplus E_2 \otimes E_1.$$

Decomposition

$$E_1 \otimes E_2 = B_1 \oplus B_2 \oplus E_1.$$

Allowed transitions: $E_2 \to B_1$, $E_2 \to B_2$, $E_2 \to E_1$.