PHY6095/PHZ6166: Final exam: Solutions

due, Tuesday, 04/30, 10 a.m.
submit your work either in person or by e-mail

You must work individually to receive full credit
Problem 1: 34 points

A neutron star can be considered as an ideal and degenerate Fermi gas at $T = 0$. The star is held together by a balance between the outward Pauli pressure and inward gravitational force. Assuming that the star is spherically symmetric, the gravitational potential $\phi$ satisfies the Newton equation

$$\nabla^2 \phi = 4\pi G \rho(r),$$

where $G$ is the gravitational constant and $\rho(r)$ is the neutron mass density. The potential energy of a neutron is $U = m_n \phi$, where $m_n$ is the neutron mass. In equilibrium, $\mu(r) + m_n \phi(r) = \text{const}$, where $\mu(r)$ is the chemical potential of the neutron gas.

- Find the condition for the neutron gas to be non-relativistic. The condition must be in a form of strong inequality ($\ldots \ll \ldots$). (Hint: Write down the scaling forms of $\rho$ and $\mu$ as a function of the distance from the star’s center but do not attempt to find the scaling functions explicitly.)

The chemical potential of a non-relativistic Fermi gas is related to the number density $n = \rho/m_n$ via

$$\mu \sim \frac{\hbar^2 k^{2/3}}{m_n} = \frac{\hbar^2 \rho^{2/3}}{m_n^{5/3}},$$

which gives

$$\rho \sim m_n^{5/2} \mu^{3/2} / \hbar^3.$$  

(1)

From the equilibrium condition, $d\mu + m_n d\phi$. Substituting the last two relations into the Newton law, we obtain an equation for $\mu$:

$$-\nabla^2 \mu = \alpha \mu^{3/2},$$

where

$$\alpha = a m_n^{7/2} G / \hbar^3$$

and $a \sim 1$ is a numerical coefficient. The chemical potential can depend only on $\alpha$, distance for the star center $r$, and $R$. Recall that from gravity law

$$U = -G \frac{m_1 m_2}{r},$$

the units of $G$ are $[G] = [E][L]/[M]^2$. Then, the units of $\alpha$ are

$$[\alpha] = \frac{[M]^{7/2}[E][L]}{[M]^2[E]^3[L]^3} = \frac{[M]^{3/2}[L]}{[E]^{1/2}[L]^2} = \frac{1}{[E]^{1/2}[L]^2}$$

Therefore, the units of $\mu$ (energy) can only be formed as

$$[\mu] = \frac{1}{[\alpha]^{2}[L]^4}$$

For $[L]$, one can use either $r$ or $R$, which leaves out one dimensional variable, $r/R$. Therefore,

$$\mu(r) = \frac{1}{\alpha^2 R^4} f \left( \frac{r}{R} \right),$$

where $f(x)$ is some scaling function, such that $f(x \sim 1) \sim 1$. At distances $r \sim R$,

$$\mu \sim 1/\alpha^2 R^4.$$  

(2)

The gas can be considered as non-relativistic, if $\mu \ll m_n c^2$, or

$$\alpha^2 R^4 m_n c^2 \gg 1 \Rightarrow \frac{m_n^8 G^2 R^4 c^2}{\hbar^6} \gg 1.$$  

Note that the result for $\mu$ cannot be determined only by dimensional analysis. Indeed, we have 4 relevant parameters ($m_n$, $\hbar$, $R$, and $G$) but only three independent units ($[M]$, $[L]$, and $[t]$), which means that the exponents of the parameters in the formula for $\mu$ are not determined uniquely. The Newton equation specifies uniquely how $G$ enters the result.
Assuming that the condition above is satisfied, find how the total mass of the star scales with the star’s radius, \( R \).

Using relation (1) and equation (2) for \( \mu \), we obtain

\[
\rho = \frac{\hbar^6}{m_n^6 G^3 R^6} g \left( \frac{r}{R} \right),
\]

where \( g \) is another scaling function. The total mass is

\[
M = 4\pi \int drr^2 \rho(r) \sim \frac{\hbar^6}{m_n^6 G^3 R^3}.
\]

Suppose now that the neutron gas is ultra-relativistic, i.e., the neutron energy \( E_n = pc \), where \( p \) is its momentum. Find how the total mass of the star depends on \( R \) in this case.

For a relativistic Fermi gas,

\[
\mu \sim hc n^{1/3} \sim hc (\rho/m)^{1/3} \Rightarrow \rho \sim \frac{m}{\hbar^3 c^3 \beta^{3/2}}.
\]

and the Newton law becomes

\[-\nabla^2 \mu = \beta \mu^3,
\]

where

\[\beta = b \frac{m^2 G}{\hbar^3 c^3}\]

and \( b \sim 1 \). Since \( [\beta] = 1/[E]^2 L^2 \), we can write \( \mu \) as

\[\mu = \frac{1}{\beta^{1/2} R} f \left( \frac{r}{R} \right)\]

and

\[\rho = \frac{m}{\hbar^3 c^3 \beta^{3/2} R^3} g \left( \frac{r}{R} \right)\]

Therefore, \( M \) does not depend on \( R \)

\[M \sim \frac{m}{\hbar^3 c^3 \beta^{3/2}} \sim \frac{1}{m^2} \left( \frac{hc}{G} \right)^{3/2}.
\]

This means that equilibrium is possible only for one value of the total mass.

**Problem 2: 33 points**

- Find the asymptotic behavior of the following integral both for \( a \ll 1 \) and \( a \gg 1 \)

\[
I(a) = \int_0^\infty dx \frac{\cos(ax^2)}{\sqrt{x^2 + 1}}
\]

For \( a \ll 1 \), substitute \( y = \sqrt{x} \):

\[
I(a) = 2 \int_0^\infty dy \frac{y \cos(ay)}{y^2 + 1}
\]

Now integrate by parts, noticing that

\[
\left. \frac{d}{dy} \frac{y}{1 + y^2} \right|_{y=0} = 1.
\]
This gives

\[ I(a \gg 1) = -\frac{2}{a^2}. \]

Convergence of the integral at large \( x \) is only due to the cosine factor: If one sets \( a = 0 \), the integral diverges logarithmically. This means that the upper limit can be chosen from the condition when the argument of the cosine becomes of order 1, i.e., \( x \sim 1/a^2 \):

\[ I(a \ll 1) = \int_{1/a^2}^{\infty} dx \frac{1}{x} = 2 \ln \frac{1}{a}. \]

– Estimate the integral

\[ J(a) = \int_{0}^{\infty} dx \sqrt{x} e^{-\left(x + \frac{a}{x}\right)}. \]

for \( a \gg 1 \).

The argument of the exponential is minimal at \( x_m = \sqrt{a} \). Expand the argument around the minimum as

\[ x + \frac{a}{x} = 2\sqrt{a} + \frac{1}{\sqrt{a}} (x - x_m)^2. \]

Since \( x_m \gg 1 \), the pre-exponential factor can be replaced by 1. Shift the variable \( y = x - x_m \) and extend the limits of integration from 0, \( \infty \) to \( -\infty, \infty \):

\[ J(a) = \exp(-2\sqrt{a}) \int_{-\infty}^{\infty} dy \exp \left(-\frac{y^2}{\sqrt{a}}\right) = \sqrt{\pi a^{1/4}} \exp(-2\sqrt{a}). \]

**Problem 3: 33 points**

A regular hexagon has \( C_{6\nu} \) symmetry. The table of characters of this group is given below. According to this table, a polar vector \((x, y, z)\) transforms as \( \Gamma' = A_1 \oplus E_1 \).

– Explain why another 2D representation, \( E_2 \), is not a suitable representation for a polar vector. \( E_2 \) is even under \( C_2 \), while a polar vector must be odd.

– Which transitions can be induced by an external electric field?

<table>
<thead>
<tr>
<th>( C_{6\nu} )</th>
<th>1</th>
<th>2 ( C_2 )</th>
<th>2 ( C_\nu )</th>
<th>3 ( \sigma_v )</th>
<th>3 ( \sigma'_{\nu} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z )</td>
<td>( A_1 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( A_2 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>( B_1 )</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>( B_2 )</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>( (x, y) )</td>
<td>( E_1 )</td>
<td>2</td>
<td>-2</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>( E_2 )</td>
<td>2</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

1. Initial state: \( A_1 \)

\[ \Gamma' \otimes A_1 = \Gamma' = A_1 \oplus E_1 \]

Allowed transition: \( A_1 \rightarrow E_1 \).

2. Initial state: \( A_2 \).

\[ \Gamma' \otimes A_2 = A_1 \otimes A_2 \oplus E_1 \otimes A_2 = A_2 \oplus E_1 \otimes A_2 \]
<table>
<thead>
<tr>
<th>( C_{6v} )</th>
<th>( I )</th>
<th>( C_2 )</th>
<th>( 2C_3 )</th>
<th>( 2C_6 )</th>
<th>( 3\sigma_v )</th>
<th>( 3\sigma'_v )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( z )</td>
<td>( A_1 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>( A_2 )</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>( B_1 )</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>( B_2 )</td>
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<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>( (x,y) )</td>
<td>( E_1 )</td>
<td>2</td>
<td>-2</td>
<td>-1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>( E_2 )</td>
<td>2</td>
<td>2</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( E_1 \otimes \ A_2 )</td>
<td>2</td>
<td>-2</td>
<td>-1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( E_1 \otimes \ B_1 )</td>
<td>2</td>
<td>2</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( E_1 \otimes \ B_2 )</td>
<td>2</td>
<td>2</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( E_1 \otimes \ E_1 )</td>
<td>4</td>
<td>4</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>( E_1 \otimes \ E_2 )</td>
<td>4</td>
<td>-4</td>
<td>1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

Comparing the characters, we see that \( E_1 \otimes A_2 = E_1 \).

Allowed transition \( A_2 \rightarrow E_1 \).

3. Initial state: \( B_1 \).

\[
\Gamma' \otimes B_1 = B_1 \oplus E_1 \otimes B_1.
\]

Comparing the characters, we see that \( E_1 \otimes B_1 = E_2 \).

Allowed transition: \( B_1 \rightarrow E_2 \).

4. Initial state: \( B_2 \).

\[
\Gamma' \otimes B_2 = B_2 \oplus E_1 \otimes B_2 = B_2 \oplus E_2.
\]

Allowed transition: \( B_2 \rightarrow E_2 \).

5. Initial state: \( E_1 \).

\[
\Gamma' \otimes E_1 = E_1 \oplus E_1 \otimes E_1.
\]

Decomposition

\[
E_1 \otimes E_1 = A_1 \oplus A_2 \oplus E_2.
\]

Allowed transitions: \( E_1 \rightarrow A_1, E_1 \rightarrow A_2, E_1 \rightarrow E_2 \).

6. Initial state: \( E_2 \).

\[
\Gamma' \otimes E_2 = E_2 \oplus E_2 \otimes E_1.
\]

Decomposition

\[
E_1 \otimes E_2 = B_1 \oplus B_2 \oplus E_1.
\]

Allowed transitions: \( E_2 \rightarrow B_1, E_2 \rightarrow B_2, E_2 \rightarrow E_1 \).