$D_4 = C_{4v}$
(tetragonal)

D_4 (C_{4v}) already contains all symmetries of D_2 (C_{2v}). $\Rightarrow$ At least, we must have

$$\hat{\sigma}_{\text{tetra}} = \begin{pmatrix} \sigma_{xx} & 0 \\ 0 & \sigma_{yy} \end{pmatrix} \tag{12}$$

However, we have additional symmetries: 180 rotations about diagonals (or reflections in the diagonal vertical planes).
Reflection in a diagonal vertical plane: $x \to y$, $y \to x$

$$\hat{R} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\hat{R}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ x \end{pmatrix} \tag{13}$$

$$\begin{aligned} \begin{pmatrix} \sigma_{xx} & 0 \\ 0 & \sigma_{yy} \end{pmatrix} &= \hat{R}\hat{\sigma}\hat{R} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} \sigma_{xx} & 0 \\ 0 & \sigma_{yy} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma_{xx} \\ \sigma_{yy} & 0 \end{pmatrix} = \begin{pmatrix} \sigma_{yy} & 0 \\ 0 & \sigma_{xx} \end{pmatrix} \\ &\Rightarrow \sigma_{xx} = \sigma_{yy} \end{aligned} \tag{14}$$

$$\hat{\sigma}_{\text{tetra}} = \begin{pmatrix} \sigma_{xx} & 0 \\ 0 & \sigma_{xx} \end{pmatrix} \tag{15}$$