

PHY6095/PHZ6166: homework assignment #1

Solutions

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1. **Problem 1** (when answering questions in this problem, always choose the most restrictive condition. For example, out of two possible answers $1 \sim 1.1$ and $1 \approx 1.1$, the last one is considered to be correct.)

- If $a \sim b$, which of the following is true?
 - b) $a + b \sim a$
- If $a \gg b$, which of the following is true?
 - b) $a + b \approx a$
- If $c \ll a \sim b$, which of the following is true?
 - a) $|a - b| \gg c$
- Assuming that
 - a) $1 \leq n \leq N$
 - b) $\varepsilon \ll 1$
 - c) $n^{-1} \ll N\varepsilon \ll n$,

determine all possible situations for n (i.e., n is \ll something but \gg something, etc.)

Solution: First, from c) we deduce that $n \gg n^{-1} \rightarrow n \gg 1$. Since $N \geq n$, it follows that $N \gg 1$ as well. In regard to $N\varepsilon$, we can have three possible situations: 1) $N\varepsilon \sim 1$; 2) $N\varepsilon \ll 1$, and 3) $N\varepsilon \gg 1$. In case 1), condition c) reduces to $n^{-1} \ll 1 \ll n$, and we do not get any constraints on n except for that $n \gg 1$. In case 2), we get a more restrictive condition on n , namely, that $n \gg (N\varepsilon)^{-1} \gg 1$. Likewise, in case 3), we have $n \gg N\varepsilon \gg 1$.

- Estimate the following integrals

$$\begin{aligned}
 \text{a) } I_1 &= \int_0^\infty dx \frac{\ln^2 x}{\sqrt{x}} J_0^2(x), \quad (J_0(x) \text{ is the Bessel function}) \\
 \text{b) } I_2 &= \int_0^1 dx \ln xx^2 \exp(-x) \\
 \text{c) } I_3 &= \int_{-\infty}^\infty dx \frac{x \exp(-x^2)}{1+x^2} \\
 \text{d) } I_4 &= \int_0^\infty dx \frac{x^2}{(1+x^3)^{2/3}} \tag{1}
 \end{aligned}$$

Solution:

- a) the integral is convergent and does not contain any parameters. Therefore, it is a pure (and positive) number which, by the Most Important Theorem, is of order one. This is all one can say without actually calculating the integral. The actual value $I_1 \approx 19.3$, so this is one of those “pathological” cases when the number happens to be large.
- b) The integral is convergent. The integrand is negative within the integration domain, hence $I < 0$ and $I \sim 1$. The actual value is $I_2 = -0.065\dots$, so again it is one of those cases when a pure number is accidentally small.
- c) The integrand is an odd function, hence $I_3 = 0$.
- d) At $x \rightarrow \infty$, the integrand saturates at a constant value, hence $I_4 = \infty$.

If the answer is finite, which region of x does integral come from?

2. **Problem 2** Suppose that particles diffuse in a semi-infinite space bounded by an impenetrable wall (located at $x = 0$). If particles are scattered specularly by the wall, the boundary condition on the distribution function $f(v_x, v_y, v_z; \mathbf{r}, t)$ says the number of particles moving towards the wall (with $v_x > 0$) is equal to the number of particles moving away from the wall (with $v_x < 0$), i.e.,

$$f(v_x, v_y, v_z; x = 0, y, z; t) = f(-v_x, v_y, v_z, x = 0, y, z; t). \tag{2}$$

Consider a one-dimensional geometry, when, as a function of \mathbf{r} , f depends only on the distance along the normal to the wall (x). Assume also that the energy of the particles does not change in the process of scattering, i.e., $|\mathbf{v}| = \text{const}$, so that, as a function of the velocity, f depends only on the angle θ between \mathbf{v} and the x -axis: $f = f(\theta, x, t)$. In this case, the boundary condition (2) reduces to

$$f(\theta, x = 0, t) = f(\pi - \theta, x = 0, t) \tag{3}$$

Starting from (3), derive the boundary condition on the number density $n(x, t) = (1/2) \int_0^\pi d\theta \sin \theta f(\theta, x, t)$ of diffusing particles.

Solution Since the angular distribution of particles is almost isotropic,

$$f(\theta, x, t) = n(x, t) + \frac{2j(x, t)}{v} \cos \theta$$

The boundary condition reduces to

$$f(\theta, 0, t) = n(0, t) + \frac{2j(0, t)}{v} \cos \theta = f(\pi - \theta, 0, t) = n(0, t) - \frac{2j(0, t)}{v} \cos \theta,$$

hence $n(0, t)$ is undetermined while the boundary condition on j is $j(0, t) = 0$.

3. Problem 3

Consider diffusion on the semi-axis $0 \leq x < \infty$. At $t = 0$, a particle starts its motion at point $x' > 0$. The diffusion coefficient is D . Consider two situations:

a) As soon as the particle reaches the boundary ($x = 0$) it disappears (for example, it may happen if the semi-axis is connected to a reservoir, which is so large that once the particle gets there it never comes back). In this case, the appropriate *boundary condition* is $n(x = 0, t) = 0$.

b) The boundary is totally impenetrable for the particle. Therefore, at $x = 0$ the outgoing flux is equal to the incoming one, and as a result the net flux is equal to zero. The appropriate boundary condition is $\partial_x n(x = 0, t) = 0$.

Find the profiles of $n(x, t)$ in the two case described above.

Solution A solution for the semi-infinite medium can be obtained as a linear superposition of the solutions for infinite medium. The coefficients in the linear superposition are chosen to satisfy the boundary conditions. This gives

a)

$$n(x, t) = \frac{1}{\sqrt{\pi Dt}} [\exp\{-(x - x')^2/4Dt\} - \exp\{-(x + x')^2/4Dt\}] \quad (4)$$

and

b)

$$n(x, t) = \frac{1}{\sqrt{\pi Dt}} [\exp\{-(x - x')^2/4Dt\} + \exp\{-(x + x')^2/4Dt\}] \quad (5)$$

4. Problem 4

Following up on the discussion of the scattering cross-section for a power-law potential $V(r) = Ar^{-a}$, combine quantum- and classical-mechanical arguments to estimate the *total* cross-section.

Solution: In Classical Mechanics, the total cross-section is infinite for any continuous potential (see *Landau & Lifshits, Classical Mechanics*). Indeed, the differential cross-section is defined as $d\Sigma = 2\pi dbb$, where b is the impact parameter. The total cross-section

$$\Sigma = 2\pi \int_0^\infty dbb = \infty.$$

Thus the estimate for Σ obtained by dimensional analysis within Classical Mechanics, i.e., $\Sigma = (A/E)^{2/a}$ (where E is the energy of an incoming particle) is incorrect (strictly speaking that it must be multiplied by a dimensionless number which happens to be equal to infinity). This is one of the paradoxes of Classical Mechanics which, had it been noticed earlier, might have put some “clouds in the sky” before people really started to worry about atoms. The resolution of the paradox is that the integral diverges for large values of the impact parameter which correspond to small scattering angles. For very small scattering angles, quantum diffraction becomes important and Classical Mechanics becomes inapplicable. Let's estimate the maximum value of b beyond which the classical description becomes invalid. For large b , the classical trajectory is almost

a straight $x = vt$ line passing at distance b from the scattering center. The classical force acting on the particle in the direction transverse to its trajectory is

$$F_{\perp} = \left. \frac{\partial V}{\partial y} \right|_{y=b} \sim -\frac{bA}{(b^2 + v^2t^2)^{3/2}}$$

The change of the transverse momentum

$$|\delta p_{\perp}| = \left| \int_{-\infty}^{\infty} dt F_{\perp} \right| \sim \frac{A}{vb^a}$$

(Notice that the last estimate can be obtained solely by dimensional analysis.) By uncertainty principle,

$$\hbar \leq \delta p_{\perp} \delta y \ll \delta p_{\perp} b \sim \frac{A}{vb^{a-1}}$$

For $a > 1$, this implies that the classical description breaks down for $b \gtrsim (A/\hbar v)^{\frac{1}{a-1}}$. Using this value of b as the upper limit in the integral for Σ , we obtain

$$\Sigma \sim \left(\frac{A}{\hbar v} \right)^{\frac{2}{a-1}}.$$

5. Problem 5

In *non-linear diffusion*, the diffusion coefficient, D , depends on the number density of diffusing particles. (A more realistic and quite common situation is *diffusion of heat*. In this case, temperature $T(r, t)$ plays the role of the number density whereas the heat diffusion coefficient depends on the local value of T . But let's stick to diffusion of number density for the sake of simplicity.) Suppose that

$$D = D_0 n^a.$$

a) write down the diffusion equation in this case starting from the continuity equation $\partial_t n + \vec{\nabla} \cdot \mathbf{j} = 0$ and Fick's law $\mathbf{j} = -D \vec{\nabla} n$.

b) using the dimensional analysis, determine the dependence of the "characteristic front" of the density profile on t . (Incidentally, for $a \neq 0$, this "front" happens to be a real one: there is a sharp boundary between the regions of finite and zero densities.

c) Analyze the behavior of front for all possible a (both positive and negative). Discuss the physical origin of all divergences or irregularities you may encounter.

Solution:

a)

$$\partial_t n = \vec{\nabla} \cdot (D \vec{\nabla} n) = D \nabla^2 n + \vec{\nabla} D \cdot \vec{\nabla} n \quad (6)$$

$$= D_0 n^a \nabla^2 n + D_0 a n^{a-1} (\vec{\nabla} n)^2. \quad (7)$$

b)

$$R \sim (D_0 t)^{\frac{1}{2+da}}, \quad (8)$$

where d is the spatial dimensionality.

c) Solution collapses when $a \leq -2/d$. Instability occurs because for $a < 0$, diffusion is faster in regions of smaller concentration: particles in regions remote from the source diffuse faster and come back to the origin.

6. Problem 6

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(None of the questions in this problem requires any deep knowledge of the field the problem comes from. Not all parameters which determine the result are mentioned in the formulation of the problem, some of the mentioned parameters may be irrelevant. Use your best judgement and intuition to determine the minimum set of relevant parameters. **Even if you happen to know the exact solution to the problem, still use qualitative methods to estimate the result.**)

- Gravitational collapse (black hole formation) occurs if the radius of a gravitating body of mass M is less than some critical value, r_g . Estimate r_g .

$$r_g \sim GM/c^2 \quad (9)$$

- Under certain conditions, stars can be described as *ultrarelativistic Fermi gases*, i.e., ensembles of free fermions moving at speeds very close to the speed of light, c . The dispersion relation at such speeds is $\varepsilon \approx cp$, where p is the electron momentum. Estimate the dependence of the specific heat (per unit volume) of such a gas on its density n .

$$c_v \sim k_B^2 n^{2/3} T / \hbar c, \quad (10)$$

- An ultra-relativistic free Fermi-gas is placed in a box of side L . The temperature is equal to zero. Estimate the pressure that the gas exerts on the walls.

$$p \sim \hbar n^{4/3} c \quad (11)$$

- A hydrogen atom is placed into a strong magnetic field B . The orbital effect of the field is to squeeze the electron orbit. Estimate the field when this effect becomes well-pronounced.

$$\ell_B = \sqrt{\hbar c / eB} \sim a_B = \hbar^2 / m e^2 \rightarrow B \sim \frac{m^2 e^3}{\hbar^3 c}. \quad (12)$$

- Free massive bosons condense into a state in which all particles have zero energy (Bose-Einstein condensation). For a given density of bosons, estimate the temperature of Bose-Einstein condensation.

$$k_B T_{BE} \sim \hbar^2 n^{2/3} / m \quad (13)$$