Experimental manifestations of the Josephson effect
Josephson junction in the magnetic field


In-plane magnetic field dependence of junction critical current for a 5±2 nm GdN barrier, showing Fraunhofer-like oscillations at 4.2 and 8.2 K. The solid line shows a fit to the 8.2 K data, where the effective field seen by the junction has an additional 10 Oe originating from the barrier moment, which reverses smoothly at low field. The calculated Fraunhofer patterns at 4.2 K (using a constant field-shift) are shown as dotted lines, to emphasize the suppression of side-lobe critical current $I_{c2}$. 
ac Josephson effect: Voltage oscillations for $I>I_c$

Pairing mechanism in conventional superconductors
Proof of the phonon pairing mechanism?
Isotope effect:
Isotope Effect in the Superconductivity of Mercury
Phys. Rev. 78, 477 – Published 15 May 1950
Emanuel Maxwell;
Superconductivity of Isotopes of Mercury
Phys. Rev. 78, 487 – Published 15 May 1950
C. A. Reynolds, B. Serin, W. H. Wright, and L. B. Nesbitt

<table>
<thead>
<tr>
<th>Sample</th>
<th>Average mass number</th>
<th>$T_s ({}^\circ\text{K})$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>203.4</td>
<td>4.126</td>
</tr>
<tr>
<td>2</td>
<td>202.0</td>
<td>4.143</td>
</tr>
<tr>
<td>3</td>
<td>200.7</td>
<td>4.150</td>
</tr>
<tr>
<td>4</td>
<td>199.7</td>
<td>4.161</td>
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</tbody>
</table>

BCS: $T_c = 1.13 T_D \exp\left(-2 / g_F \lambda\right)$
$T_D = \hbar \omega_D = \hbar s / a_0; s = v_F \sqrt{m / M}$
$T_D \propto 1 / \sqrt{M}$

Deformation potential model:
$\gamma g_F \sim \frac{\Xi^2 (k_F a_0)^3}{M s^2 E_F}$: independent of $M$
Caveats of the isotope effect:

1. In heavy elements, relative changes of masses are small: the case for power-law scaling is not very convincing.
2. In compounds, one can replace only one one element.
3. The coupling constant is independent of $M$ only in a very simple model.
4. The exponent varies dramatically even within “low T\textsubscript{c}” superconductors and sometimes is even negative ("inverse isotope effect").
Isotope effect

$$T_c \propto M^{-\alpha}$$ phonon model: $\alpha = 1/2$

<table>
<thead>
<tr>
<th>Superconductor</th>
<th>$\alpha$</th>
<th>Reference (see [2])</th>
</tr>
</thead>
<tbody>
<tr>
<td>Hg</td>
<td>0.5 ± 0.03</td>
<td>a</td>
</tr>
<tr>
<td>Tl</td>
<td>0.5 ± 0.1</td>
<td>a, and [11]</td>
</tr>
<tr>
<td>Cd</td>
<td>0.5 ± 0.1</td>
<td>b</td>
</tr>
<tr>
<td>Mo</td>
<td>0.33 ± 0.05</td>
<td>c</td>
</tr>
<tr>
<td>Os</td>
<td>0.21 ± 0.05</td>
<td>d,e</td>
</tr>
<tr>
<td>Ru</td>
<td>0.0</td>
<td>e</td>
</tr>
<tr>
<td>Zr</td>
<td>0.0</td>
<td>e</td>
</tr>
<tr>
<td>PdH(D)</td>
<td>-0.25</td>
<td>[33, 34]</td>
</tr>
<tr>
<td>U</td>
<td>-2</td>
<td>[12]</td>
</tr>
<tr>
<td>La$<em>{1.85}$Sr$</em>{0.15}$CuO$_4$</td>
<td>0.07</td>
<td>[55]</td>
</tr>
<tr>
<td>La$<em>{1.89}$Sr$</em>{0.11}$CuO$_4$ (\textsuperscript{16}O - \textsuperscript{18}O subst.)</td>
<td>0.75</td>
<td>[55]</td>
</tr>
<tr>
<td>K$<em>3$C$</em>{60}$ (\textsuperscript{12}C - \textsuperscript{13}C subst.)</td>
<td>0.37 or 1.4</td>
<td>[40]</td>
</tr>
</tbody>
</table>

CaC$_6$ ($T_c \approx 11$ K): $\alpha_B \approx 0.5$

(Ba,K)BiO$_3$ ($T_c \approx 30$ K): $\alpha_O \approx 0.4$

MgB$_2$ ($T_c \approx 40$ K): $\alpha_B \approx 0.3$

A. Bill, V.Z. Kresin, and S.A. Wolf
in: \textit{Many Fermion Systems}, edited by V.Z. Kresin,
Iron-based superconductors: inverse isotope effect

\[ T_c \propto M^{-\alpha_{Fe}} \] phonon model: \( \alpha = 1/2 \)
Another confirmation of e-ph pairing: Strong-coupling (Eliashberg) theory

\[ \bar{\lambda} = g_F \lambda / 2 = \begin{cases} 
1.6, \text{ Hg} \\
1.4, \text{ Pb} \\
2.1, \text{ Pb}_{0.65} \text{Bi}_{0.35}
\end{cases} \]

Treats electron-phonon interaction explicitly (rather than as a model attraction in the BCS model)

McMillan-Dynes formula

\[ T_c = \frac{\hbar \omega_D}{1.12} \exp \left( - \frac{1.04(1+\bar{\lambda})}{\bar{\lambda} - \mu_c (1+0.62\bar{\lambda})} \right) \]; \mu_c = "Coulomb pseudopotential"

\[ \bar{\lambda} = 2 \int \frac{d\omega}{\omega} \left( \alpha^2(\omega) \text{ matrix element} \right) g_{\text{ph}}(\omega) \text{ phonon density of states} \]

First-principle calculations of \( \alpha^2(\omega), g_{\text{ph}}(\omega) \Rightarrow T_c \)

Satisfactory explanations of the experiment (\( T_c \), tunneling spectra) but not a single (correct) prediction for a new superconductor

Strong critique of the e-ph coupling model (and BCS in general):
Unconventional superconductivity

s-wave superconductors:
Continuum: $\Delta(k)=$const
Lattice: $\Delta(k)>0$ for all $k$
Non-s-wave: nodes $\Delta(k_n)=0$

Non-s-wave symmetry of the order parameter: p-wave ($l=1$), d-wave ($l=2$), f-wave ($l=3$)...

Spin singlet $\rightarrow$ even orbital momentum (s,d...)
Spin triplet $\rightarrow$ odd orbital momentum (p,f...)

Triplet superconductivity: He$^3$ (superfluid), Sr$_2$RuO$_4$, UPt$_3$, UGe$_2$ (?)

Spin-singlet, d-wave: cuprates

One possibility: s±

NB: unconventional≠HTC (Sr$_2$RuO$_4$, UPt$_3$, UGe$_2$: $T_c \sim 1K$)
Unconventional superfluidity: $\text{He}^3$

J. Wheatley, RMP 47, 415 (1975)
Spin triplet: \((l=1)\) p-wave orbital symmetry

B-phase: \(L=1, S=1\), superposition of the \(S_z=-1,0,1\) states

A-phase: \(L=1, S=1, S_z=+1/-1\)
How do we know that this is a triplet?

Singlet state: spin susceptibility = 0 at T=0
B-phase: $\chi = (1/3)\chi_N$
A-phase: $\chi = \chi_N$

UPt$_3$

Gannon et al.
PRL 86, 104510 (2012)

Sr$_2$RuO$_4$ [B=1T, $\kappa=$(002)]

Mackenzie & Maeno, RMP 75, 657

Wheatley, 1975
Cuprates: spin singlet (NMR) → s or d.  
How do we know that this is a d-wave?

Normal quasiparticles reside near nodal lines even at T=0: 
Thermodynamics of a nodal superconductor 
is very much different from s-wave.

s-wave: 
\[ \Lambda \propto 1/\sqrt{n_s(T)} \] 
\[ n - n_s \propto \exp(-2\Delta/T) \] 
\[ \Lambda = \Lambda(0) + c \exp(-2\Delta/T) \]

d-wave: 
\[ n - n_s \propto T \]  
\[ \Lambda = \Lambda(0) + cT \]

Hardy et al. PRL 70, 3999 (1993)
Which d-wave?

Possible symmetries of the order parameter on a square lattice

\( \Delta(k) = \text{const} \)

\( s^- : \quad \Delta(k) = \Delta \left[ \cos k_x a + \cos k_y a \right] \)

\( d_{x^2-y^2} : \quad \Delta(k) = \Delta \left[ \cos k_x a - \cos k_y a \right] \)

\( d_{xy} : \quad \Delta(k) = \Delta \sin k_x a \sin k_y a \)
Smoking-gun: corner junction experiment
Urbana experiment: Wollman et al. PRL 71, 2134 (1993)

\[ \phi_a - \phi_b = 2\pi \frac{\Phi}{\Phi_0} + \delta_{ab} \]

s-wave: \( \delta_{ab} = 0 \)

d-wave: \( \delta_{ab} = \pi \)

Edge junction: both junctions on the same side

NB: R is 90 shifted compared to I

FIG. 5. (a) Calculated critical current vs flux for a YBCO-Pb single corner junction for s-wave and d-wave symmetry. (b) Measured resistance vs flux for a corner junction and a single junction on one edge (shifted vertically for clarity).
ARPES


Observation of a $d$-wave nodal liquid in highly underdoped $\text{Bi}_2\text{Sr}_2\text{CaCu}_2\text{O}_{8+\delta}$
Attraction from repulsion: Kohn-Luttinger mechanism

To second order in a short-range repulsive interaction, the effective interaction in the Cooper channel is **ATTRACTION** at least for odd \( L \) and for \( L \gg 1 \).


Recent review: S. Maiti (UF) and A. V. Chubukov, [arXiv:1305.4609](https://arxiv.org/abs/1305.4609)

Original KL result: pairing amplitude in \( L \)-channel: \[ \Gamma^L_c = U \delta_{L,0} + \left(-\right)^L U^2 \frac{C_L}{L^4}, \quad L \gg 1 \]
Origin of the effect: the same as in Friedel oscillations
Myth and truth about the KL mechanism

Myth: the mechanism produces only ridiculously low $T_c$

Origin of the myth: KL obtained the result valid only for $L>>1$ and applied it to $L=2$.

$$U(r) = \frac{4\pi a}{m} \delta^3(r)$$

$a =$ scattering length.

For He$^3$: $k_Fa \approx 2$

Truth: Values of $T_c$ are reasonable for $L\sim1$.

Lay and Layzer, PRL 20, 187 (1968)

$L = 1: T_c / E_F \approx 10^{-3}$

Superfluid He$^3$ (Lee, Oscheroff, Richardson 1972):

$$T_c / E_F \approx 10^{-3}!$$
Nodes are a consequence of repulsive interaction

Suppose that the pairing interaction is anisotropic. BCS equation

$$\Delta(k) = \sum_{k'} (-V_{k,k'}) \frac{\Delta(k')}{2E_{k'}} (1 - 2n_{k'})$$

If $V_{k,k'} > 0$, $\Delta(k)$ must change sign for the eq-n to have a solution

Another consequence of repulsion: enhanced magnetism

Fermi liquid interaction function

$$F_{\alpha\gamma,\beta\delta} (\theta) = F_s (\theta) \delta_{\alpha\gamma} \delta_{\beta\delta} + F_a (\theta) \vec{\sigma}_{\alpha\gamma} \cdot \vec{\sigma}_{\beta\delta}$$

$$F_{\alpha\gamma,\beta\delta} (\theta) =$$

$$\sum_{\alpha, \beta}$$

Another consequence of repulsion: enhanced magnetism
Singlet vs triplet

Singlet: the orbital part of the wave function is even

Triplet: the orbital part of the wavefunction is odd

Repulsive interaction favors larger distances between particles: triplet pairing is preferred

Strong ferromagnetic fluctuations favor triplet pairing (He$^3$, Sr$_2$RuO$_4$...
URhGe (a) and UGe$_2$ (b)

Superconductivity: A different class?
Gilbert G. Lonzarich
What about *antiferromagnetism*?

![Diagram showing the phase diagram of cuprates and Fe-based superconductors (pnictides)]
Doped transition-metal oxide (example: cuprate)

When Ba concentration is 0
Electrons are immobilized owing to strong interaction

Hole doping
(La$^{3+}$ $\rightarrow$ Ba$^{2+}$)

Observation of fluctuation of stripes using X-rays
Cuprates: pairing via antiferromagnetic fluctuations

Doped antiferromagnetic insulator

Staggered spin susceptibility

\[ \chi(q) = \sum_r <\vec{S}_r \cdot (-\vec{S}_{r+a})> e^{iq \cdot r} \sim \frac{1}{(q - q_\pi)^2 + \xi^{-2}} \]

Effective interaction

\[ H_{int} = \sum_{k,q,\alpha,\beta} c_{\alpha,k+q}^\dagger \vec{\sigma}_{\alpha\beta} c_{\beta,k} \cdot S_{-q} + \sum_q \frac{S_q \cdot S_{-q}}{\chi(q)} \Rightarrow \]

\[ \Rightarrow \sum_{k,q,\alpha,\beta} c_{\alpha,k+q}^\dagger \vec{\sigma}_{\alpha\beta} c_{\beta,k} \cdot c_{\gamma,p-q}^\dagger \vec{\sigma}_{\gamma\delta} c_{p,\delta} \chi(q) \]

The interaction is repulsive and strongest along the diagonals: the \( d_{x^2-y^2} \) state wins!