I. NOTATIONS

\( k_B = 1 \).

II. DC JOSEPHSON EFFECT

If the current applied to a superconductor does not exceed the critical value, it does not generate voltage, i.e., the resistance is equal to zero. Suppose now that we have two superconductors connected by an insulating layer. If the layer is not too thick, Cooper pairs will tunnel through the barrier and, as a result, two superconductors will be in a single macroscopic state, i.e., they can be considered as one superconductor. If the current through the junction does not exceed the critical value, the voltage drop is absent, as in the bulk case. This is the dc Josephson effect. The critical current, however, is much smaller than the bulk critical value because superconductors are coupled only weakly (this is why phenomena occurring in Josephson junctions are referred to as “weak superconductivity”).

We now prove (following mostly Ref. [1]) that two weakly coupled superconductors allow for supercurrent flowing through the junction. The magnitude of the current is related to the phase difference between two superconductors. Although Josephson himself worked out the current-phase relationship within the microscopic (BCS) theory, the same relationship follows from the Ginzburg-Landau equation, which is what I will do here. Recall that, when minimizing the free energy functional, we got the surface terms and the bulk terms. The vanishing of the bulk terms gives the equation of motion for the order parameter (the non-linear Schroedinger equation). The vanishing of the surface terms gives the boundary condition

\[
\hat{n} \cdot \left( \vec{\nabla} - \frac{2ie}{\hbar c} \mathbf{A} \right) |_{S} = 0,
\]

where \( S \) denotes the sample boundary and \( \hat{n} \) is the unit normal vector. At the same time, minimizing the free energy with respect to the vector potential gives the equation for the current density

\[
j = \frac{e\hbar}{2mi} \left( \Psi^* \vec{\nabla} \Psi - \Psi \vec{\nabla} \Psi^* \right) - \frac{2e^2}{mc} |\Psi|^2 \mathbf{A}.
\]

Condition (1) refers to the boundary with vacuum or insulator. When the second superconductor is present on the other side of the barrier, the order parameter “leaks out”. In the GL regime, the order parameters in both superconductors are small, and one can modify the boundary condition by replacing 0 on the RHS by a quantity proportional to the order parameter on the other side. If \( \Psi_{1,2} \) are the
order parameters in superconductor 1(2), the normal to the junction is along the $x$ axis, and the set of boundary conditions reads

$$
\left( \partial_x - \frac{2ieA_x}{\hbar c} \right) \Psi_1 = \frac{\Psi_2}{\ell},
$$

(3)

$$
\left( \partial_x - \frac{2ieA_x}{\hbar c} \right) \Psi_2 = \frac{\Psi_1}{\ell},
$$

(4)

where $\ell$ is a (real) phenomenological parameter of units of inverse length. In reality, it is proportional to the tunneling resistance of the barrier. To have any sizable transmission, the insulating layer must be very thin, a few nm max. Such a length is much smaller than both the correlation length of the order parameter and the penetration length, and thus the junction can be treated as an infinitely thin.

Since we are looking at the $dc$ case now, the current is the same at any distance from the junction and we can evaluate it, e.g., in superconductor 1:

$$
j = \frac{e\hbar}{2mi} (\Psi_1^* \partial_x \Psi_1 - \Psi_1 \partial_x \Psi_1^*) - \frac{2e^2}{mc} |\Psi_1|^2 A_x.
$$

(5)

Excluding $\partial_x \Psi_1$ with the help of Eq. (3), we find that the vector-potential term in Eq. (5) cancels out, and the equation for the current reads

$$
j = \frac{e\hbar}{2mi\ell} (\Psi_1^* \Psi_2 - \Psi_1 \Psi_2^*).
$$

(6)

If the two superconductors have the same magnitude by different phases of the order parameter, the last equation becomes

$$
j = \frac{e\hbar|\Psi|^2}{m\ell} \sin(\phi_2 - \phi_1) \equiv j_c \sin(\phi_2 - \phi_1)
$$

(7)

where $j_c = \frac{e\hbar|\Psi|^2}{m\ell}$ is the critical current of the junction. All Eq. (7) says is that a supercurrent of magnitude $j < j_c$ can flow through a junction.

The microscopic theory$^2$ gives for $j_c$

$$
j_c = \frac{\pi \Delta^2}{2eR},
$$

(8)

where $\Delta$ is the superconducting gap and $R$ is the junction resistance measured above $T_c$. For a tunneling barrier, $1/R$ is proportional to the square of the transmission coefficient; for a rectangular barrier of height $U$ and width $d$, $1/R \propto \exp(-2\sqrt{2m(U - E_F)}\hbar)$. One of the initial counter-arguments against Josephson’s prediction was that the critical current must be proportional to $1/R^2$ (because one has to transfer two electrons forming a Cooper pair rather than a single electron.
through the junction) and thus immeasurably small. This is (fortunately) not the case: the Cooper pair tunnels coherently, as a whole, without disintegrating into two single electrons.

A. A Josephson junction in the magnetic field

Notice that in deriving Eq. (7) we did allow for a magnetic field in the junction. Now, we can consider its effect explicitly. Consider geometry shown in Fig. 2, with \( \mathbf{B} \) along the \( z \) axis and varying with \( x \) (across the junction) and with the vector potential along the \( x \) axis: \( \mathbf{A} = (B_z(x)y, 0, 0) \). The field penetrates into each of the superconductors within the penetration length from the junction boundaries. We know that a gauge-invariant combination of the phase and the vector-potential is \( \vec{\nabla}\phi - 2e\mathbf{A}/hc \). Let’s choose some point \( y \) and integrate the gauge-invariant combination along the vertical line passing through \( y \), starting in one of the superconductors and ending up in another.

\[
\int d\mathbf{l} \cdot \left( \vec{\nabla}\phi - 2e\mathbf{A}/hc \right) = \phi_2 - \phi_1 - \frac{2e}{hc} \int_{-\infty}^{+\infty} dx B_z(x)y
\]  

(9)
Therefore, the phase difference in Eq. (7) needs to be replaced by the RHS of the equation above and the current becomes

\[ j = j_c \sin \left( \delta \phi - \frac{2e}{\hbar c} \int_{-\infty}^{+\infty} dB_z(x)y \right) \]  (10)

where \( \delta \phi = \phi_2 - \phi_1 \) can now be understood as the phase difference in the absence of the field. If the junction length in the \( y \) direction is \( L \), the average current density is

\[ \bar{j} = \frac{j_c}{L} \int_0^L dy \sin \left( \delta \phi - \frac{2e}{\hbar c} \int_{-\infty}^{+\infty} dB_z(x)y \right) \]

\[ = \frac{j_c}{2e L} \int_{-\infty}^{+\infty} dB_z(x) \left[ \cos \left( \delta \phi - \frac{2e}{\hbar c} \int_{-\infty}^{+\infty} dB_z(x)L \right) - \cos \delta \phi \right] \]  (11)

Introducing the total flux through the junction \( \Phi = L \int_{-\infty}^{+\infty} dB_z(x) \) and the flux quantum \( \Phi_0 = \frac{hc}{2|e|} = \frac{\pi \hbar c}{e} \) and using the trigonometric identity \( \cos x - \cos y = 2 \sin \left( \frac{x+y}{2} \right) \sin \left( \frac{y-x}{2} \right) \), we re-write the last equation as

\[ \bar{j} = j_c \frac{\Phi_0}{\pi \Phi} \sin \left( \pi \frac{\Phi}{\Phi_0} \right) \sin \left( \frac{\delta \phi + \pi \Phi}{\Phi_0} \right). \]  (12)

This relation is known as the Josephson-Fraunhofer pattern. The current oscillates with the field and vanishes each time the flux the junction is equal to an integer multiple of \( \Phi_0 \).

Notice that the consideration presented above is valid only for a narrow junction, when can one neglect the magnetic field produced by the current itself.

**B. Superconducting Quantum Interference Device (SQUID)**

A \( dc \) superconducting interferometer or SQUID, invented by Robert Jaklevic, John J. Lambe, James Mercereau, and Arnold Silver of Ford Research Labs, consists of two Josephson junctions in parallel and placed in a magnetic field, see Fig. ??.

Inside the superconductor, the magnetic induction and thus the current are equal to zero, therefore

\[ \nabla \phi - \frac{2e}{\hbar c} A = 0 \]  (13)

Integrating this equation along the contour shown the in the Figure, we obtain

\[ \delta \phi_1 - \delta \phi_2 + 2\pi \frac{\Phi}{\Phi_0} = 2\pi n, \]  (14)
where $\delta \phi_1$ and $\delta \phi_2$ are the phase jumps at each of the junctions. Then we can parameterize each of the phase jumps by a continuous variable $\phi$ such that
\begin{align*}
\delta \phi_1 &= \phi + \pi \frac{\Phi}{\Phi_0} - 2\pi n \\
\delta \phi_2 &= \phi - \pi \frac{\Phi}{\Phi_0}
\end{align*}
(15)
(16)

The total current through the ring is the sum of the currents through each of the junctions. Assuming that the critical currents of junctions are the same,
\[
I = I_c \left[ \sin \delta \phi_1 + \sin \delta \phi_2 \right] = I_c \left[ \sin \left( \phi + \pi \frac{\Phi}{\Phi_0} \right) + \sin \left( \phi - \pi \frac{\Phi}{\Phi_0} \right) \right] = 2I_c \sin \phi \cos \pi \frac{\Phi}{\Phi_0}.
\]
(17)

The current through the ring oscillates with the field and vanishes each time when $\Phi = (n+1/2)\Phi_0$. Because $\Phi_0$ is very small ($= 2 \times 10^{-15}$ Tm$^2$), SQUIDs are extremely sensitive even to minuscule variations of the magnetic field.

III. AC JOSEPHSON EFFECT

If the current through a Josephson junction exceeds the critical value, a voltage drop occurs. In this case, we have the “ac Josephson effect”: a $dc$ voltage $V$ across the junction generated electromagnetic radiation with frequency $\omega_J = 2|e|/\hbar$. To see how it works, we need to establish a relation between the time-derivative of the phase of the order parameter and the voltage applied to a superconductor. In the absence of the voltage, the phase is constant: $\partial_t \phi = 0$. However, this is not a gauge-invariant relation. To see what kind of constraints are imposed by gauge invariance, we recall that a simultaneous transformation of the scalar and vector potentials
\begin{align*}
V \rightarrow V - \frac{1}{c} \partial_t \chi; \\
A \rightarrow A + \vec{\nabla} \chi
\end{align*}
(18)
(19)
does not change the electric and magnetic fields. Indeed,
\[
E = -\vec{\nabla} V - \frac{1}{c} \partial_t A
\]
(20)
is not affected by such a transformation. We have already seen how Eq. (19) modifies the relation between the supercurrent and the phase gradient. To explore consequences of Eq. (18), to turn to a time-dependent Schroedinger equation
\[
i\hbar \partial_t \psi = (H_0 + eV) \psi
\]
(21)
It is easy to see that we if apply transformation Eq (18) and simultaneously change the phase of the wavefunction as $\theta \to \theta + (e/\hbar c)\chi$, the Schroedinger equation preserves its form. Since the macroscopic wave function is composed of the product of two single-particle wave functions, the phase of the order parameter changes as

$$\phi \to \phi + (2e/\hbar c)\chi.$$  \hspace{1cm} (22)

Therefore, the phase-voltage relation is gauge-invariant, if we write it as

$$\partial_t \phi + \frac{2eV}{\hbar} = 0.$$  \hspace{1cm} (23)

For $V = \text{const}$,

$$\phi(t) = \phi(0) - \frac{2eVt}{\hbar}$$  \hspace{1cm} (24)

and the current through a junction oscillates in time?

$$j = j_c \sin(\phi(t)) = \sin \left( \phi(0) - \frac{2eVt}{\hbar} \right).$$  \hspace{1cm} (25)
The frequency of oscillations (Josephson frequency) \( \omega_J = 2eV/\hbar \) contains a ratio of fundamental constants \( e/\hbar \). A factor of 2 is a direct manifestation of the pairing nature of the superconducting state.

Because the current is larger than the critical value, a normal current also flows through the junction, and the total current \( I \) is the sum of the supercurrent \( I_s = I_c \sin \phi \) and normal current \( I_n = V/R = -(\hbar/2eR)\partial_t \phi \), where we used Eq. (23). We thus obtain a differential equation that describes dynamics of the Josephson junction for \( I > I_c \):

\[
I = I_c \sin \phi - \frac{\hbar}{2eR} \partial_t \phi. \tag{26}
\]

This equation can be easily solved in the limit \( I \gg I_c \). In this case, we neglect the supercurrent and obtain the zeroth-order approximation for \( \phi \): \( \phi_0 = -(2eRI/\hbar)t \). Substituting this back into Eq. (26), we obtain an equation for the first-order correction:

\[
\frac{\hbar}{2eR} \partial_t \phi_1 = I_c \sin \phi_0 = -I_c \sin \left( \frac{2eRI}{\hbar}t \right). \tag{27}
\]

This immediately gives a time-dependent correction to voltage

\[
V_1(t) = -(\hbar/2e)\partial_t \phi = I_c R \sin \left( \frac{2eRI}{\hbar}t \right). \tag{28}
\]

The total voltage is

\[
V(t) = V + V_1(t) = R \left[ I + I_c \sin \left( \frac{2eRI}{\hbar}t \right) \right]. \tag{29}
\]

The voltage oscillate with the frequency \( \omega = 2|e|RI/\hbar \).

The full solution can be found, e.g., in Ref. [2]). The result for the voltage reads:

\[
V(t) = R \frac{I^2 - I_c^2}{I + I_c \cos(\omega t - \phi_0)}, \tag{30}
\]

where \( \omega = (2eR/\hbar)\sqrt{I^2 - I_c^2} \) and \( \phi_0 = \cos^{-1}(I_c/I) \). In general, the oscillations in \( V \) are non-sinusoidal.

If the junction is connected to a \( dc \) voltmeter, it will measure the average value of the voltage. Averaging Eq. (30) over one period gives

\[
\langle V \rangle = \frac{\hbar}{2e} \omega = R \sqrt{I^2 - I_c^2}. \tag{31}
\]

This is an average (over time) current-voltage characteristic of a Josephson junction.

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