BASIC FORMULAS.

Relation b/w angular momentum $\vec{L}$ and angular velocity $\vec{\omega}$:

$$\vec{L} = I \cdot \vec{\omega}$$

Inertia tensor:

$$I = \begin{bmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{bmatrix}$$

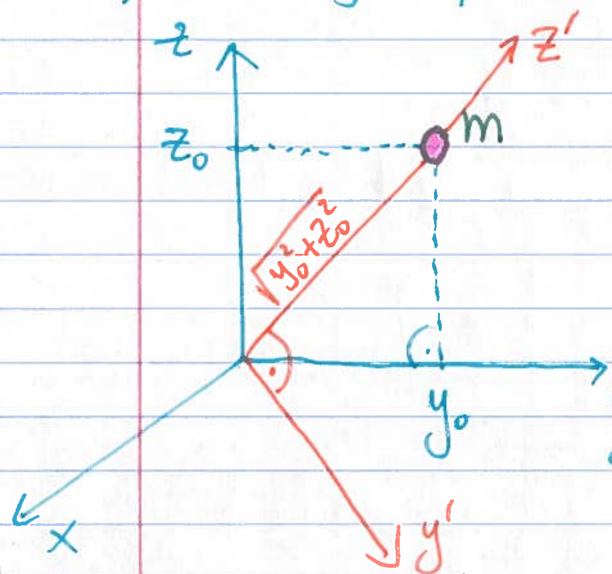
Properties of the inertia tensor.

* symmetric: $I_{xy} = I_{yx}$, $I_{xz} = I_{zx}$, $I_{yz} = I_{zy}$
* if there is an axial symmetry, the two products of inertia for rotation about this axis are $\emptyset$.
* if there is a reflection symmetry, e.g. with respect to $z=0$, then $I_{xz} = I_{yz} = \emptyset$.

MOMENTS OF INERTIA	PRODUCTS OF INERTIA
$I_{xx} = \sum_\alpha m_\alpha (y_\alpha^2 + z_\alpha^2)$	$I_{xy} = -\sum_\alpha m_\alpha x_\alpha y_\alpha$
$I_{yy} = \sum_\alpha m_\alpha (x_\alpha^2 + z_\alpha^2)$	$I_{yz} = -\sum_\alpha m_\alpha y_\alpha z_\alpha$
$I_{zz} = \sum_\alpha m_\alpha (x_\alpha^2 + y_\alpha^2)$	$I_{xz} = -\sum_\alpha m_\alpha x_\alpha z_\alpha$

Example 10.1. Calculate the moments and products of inertia for the following configurations:

a) a single point mass at $(0, y_0, z_0)$.



There is only one term in the sum: $m_\alpha \equiv m$.

$$I_{xy} = -m \cdot \emptyset \cdot y_0 = \emptyset \qquad I_{xx} = m\left(y_0^2 + z_0^2\right)$$

$$I_{xz} = -m \,\emptyset \cdot z_0 = \emptyset \qquad I_{yy} = m\left(0^2 + z_0^2\right) = m z_0^2$$

$$I_{yz} = -m y_0 z_0 \qquad I_{zz} = m\left(0^2 + y_0^2\right) = m y_0^2$$

Inertia tensor:

$$\begin{bmatrix} m\left(y_0^2+z_0^2\right) & 0 & 0 \\ 0 & m z_0^2 & -m y_0 z_0 \\ 0 & -m y_0 z_0 & m y_0^2 \end{bmatrix}$$

The answer is simple enough already but can be further simplified with a better choice of the axes. Rotate about the x-axis until z-axis points in the direction of the point mass.

With the new choice of axes the point mass is located at

$$\left(0, 0, \sqrt{y_0^2+z_0^2}\right).$$

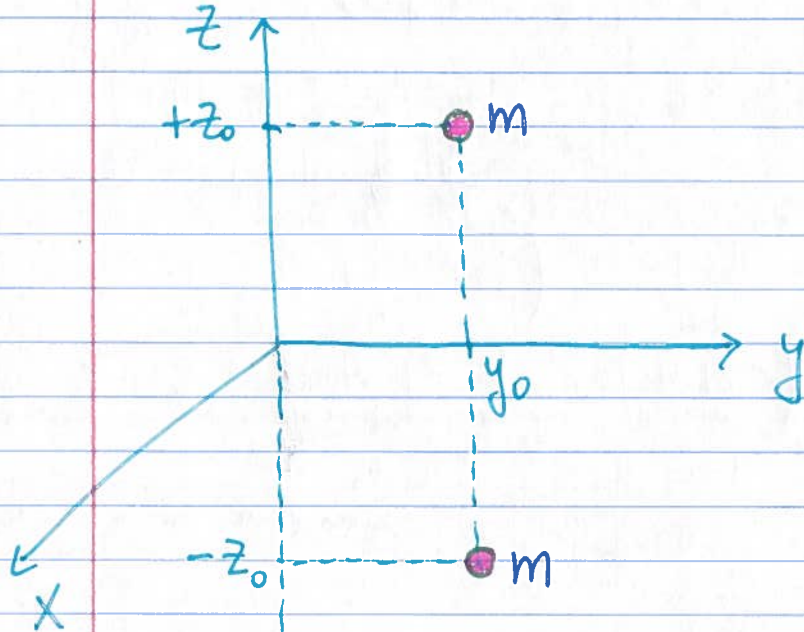
With this choice of the axes, the inertia tensor becomes.

$$\begin{bmatrix} m(y_0^2+z_0^2) & 0 & 0 \\ 0 & m(y_0^2+z_0^2) & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

It costs nothing to spin a point mass around itself.

☺

b) two point masses at $(0, y_0, z_0)$ and $(0, y_0, -z_0)$.



reflection symmetry with respect to $z=0$.

$$I_{xy} = -m\,0\,y_0 - m\,0\,y_0 = 0.$$

$$I_{xz} = -m\,0\,z_0 - m\,0\,(-z_0) = 0.$$

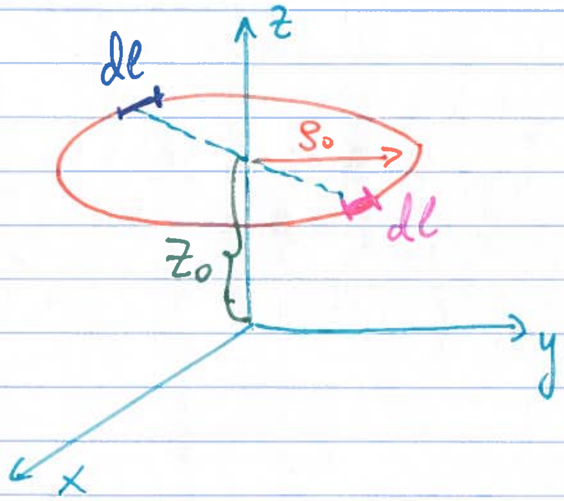
$$I_{yz} = \underbrace{-m\,y_0 z_0 - m\,y_0(-z_0)}_{\text{cancellation} \rightarrow \text{reflection symmetry.}} = 0.$$

$$I_{xx} = m\left(y_0^2 + z_0^2\right) + m\left(y_0^2 + (-z_0)^2\right) = 2m\left(y_0^2 + z_0^2\right).$$

$$I_{yy} = m\left(0^2 + z_0^2\right) + m\left(0^2 + (-z_0)^2\right) = 2m z_0^2$$

$$I_{zz} = m\left(0^2 + y_0^2\right) + m\left(0^2 + y_0^2\right) = 2m y_0^2.$$

c) a hoop of radius ρ_0 centered on the z-axis at $z = z_0$.



The contributions from dl and dl cancel: they have the same value of z: $z = z_0$, but opposite values of x & y. Therefore

$$\boxed{I_{xz} = I_{yz} = 0.}$$

(axial symmetry).

$$I_{zz} = \sum_\alpha m_\alpha \underbrace{(x_\alpha^2 + y_\alpha^2)}_{\rho_0^2} = \rho_0^2 \sum_\alpha m_\alpha = \boxed{M\rho_0^2}$$

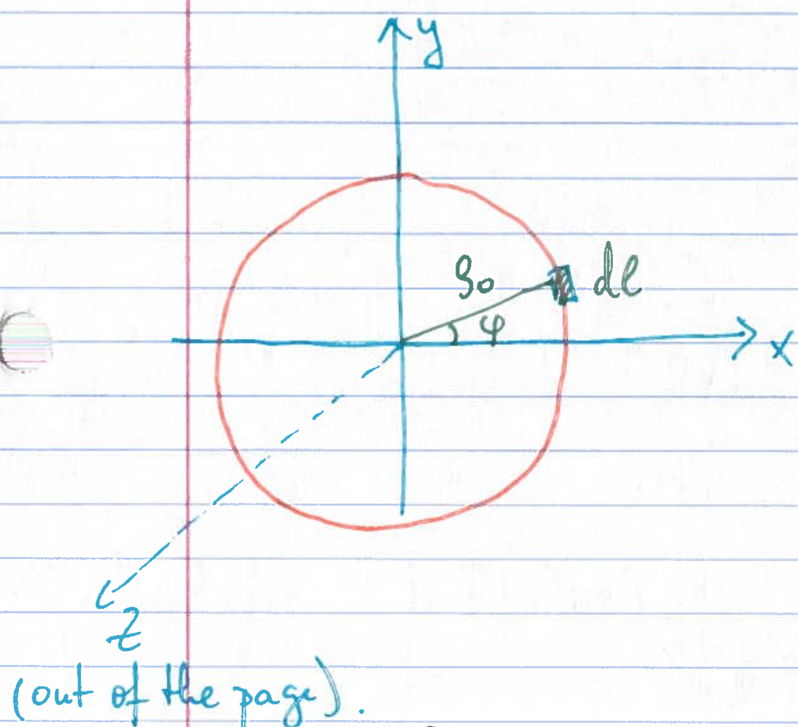
Alternative argument:

* $x=0$ is a plane of reflection symmetry
  ⇒ any product of inertia involving x is $\emptyset$
* $y=0$ is a plane of reflection symmetry
  ⇒ any product of inertia involving index y is $\emptyset$.

Although the book didn't do this, for completeness let's derive the remaining I_{xx} and I_{yy}. It is obvious that due to the symmetry of the problem they are equal:

$$I_{xx} = I_{yy}$$

so we need to derive only one of them.



Linear mass density

$$\frac{M}{2\pi\varrho_0} = \varkappa$$

Useful relations:

$dl = \varrho_0\, d\varphi$

$x = \varrho_0 \cos\varphi$

$y = \varrho_0 \sin\varphi$

$$I_{xx} = \int (y^2 + \underset{\overset{\parallel}{\emptyset}}{z^2})\, \varkappa\, dl = \varkappa \int_0^{2\pi} \varrho_0^2 \sin^2\varphi \cdot \varrho_0\, d\varphi$$

$$= \varkappa \varrho_0^3 \underbrace{\int_0^{2\pi} \sin^2\varphi\, d\varphi}_{\overset{\parallel}{\pi}} = \pi \varkappa \varrho_0^3 = \pi \frac{M}{2\pi\varrho_0} \varrho_0^3$$

$$= \frac{1}{2} M \varrho_0^2 .$$

Final answer for the inertia tensor of a thin hoop of radius R:

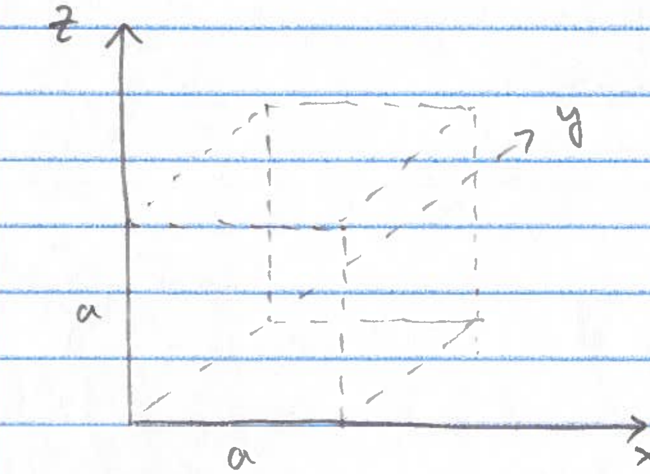
$$\begin{bmatrix} \frac{1}{2}MR^2 & 0 & 0 \\ 0 & \frac{1}{2}MR^2 & 0 \\ 0 & 0 & MR^2 \end{bmatrix}$$

With our choice of axes, the tensor is already diagonal, therefore these are the principal axes as well, and the three principal moments are the diagonal entries:

$$\frac{1}{2}MR^2 , \ \frac{1}{2}MR^2 , \ MR^2 .$$

Example 10.2. Inertia Tensor for a solid cube.
A uniform solid cube of mass M and side a.
Find $\vec{L}$ for the cases when $\vec{\omega} = \omega \hat{x}$ and
$\vec{\omega} = \omega \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \right)$.

(a) choose axes along the edges:



density $\rho = \frac{M}{a^3} \equiv d$

not to be confused with the cylindrical coordinate.

$$I_{xx} = \underbrace{\int dx\, dy\, dz\, d}_{\text{mass } m_\alpha} \cdot \left(y^2 + z^2 \right).$$

$$= d \int_0^a dx \int_0^a dy \cdot y^2 \int dz + d \int_0^a dx \int_0^a dy \int_0^a dz\, z^2$$

$$= d \cdot a \cdot \frac{a^3}{3} \cdot a + d \cdot a \cdot a \cdot \frac{a^3}{3} = da^3 \left(\frac{a^2}{3} + \frac{a^2}{3} \right)$$

$$= M \cdot \frac{2}{3} a^2 = \boxed{\frac{2}{3} M a^2}$$

$$I_{xy} = -\int dx\, dy\, dz\, d \cdot xy.$$

$$= -\int_0^a dx \cdot x \int_0^a dy \cdot y \int_0^a dz\, d = -\frac{a^2}{2}\frac{a^2}{2} a \cdot d = -\frac{1}{4} Ma^2$$

By symmetry the others are the same.

$$I = Ma^2 \begin{pmatrix} 2/3 & -1/4 & -1/4 \\ -1/4 & 2/3 & -1/4 \\ -1/4 & -1/4 & 2/3 \end{pmatrix} = \frac{Ma^2}{12} \begin{pmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{pmatrix}$$

If rotating by x-axis: $\vec{L} = I \circ \vec{\omega}$

$$\vec{L} = \frac{Ma^2}{12} \begin{pmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{pmatrix} \begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix} = \frac{Ma^2\omega}{12}(8, -3, -3)$$

$$= Ma^2\omega \left(\frac{2}{3}, -\frac{1}{4}, -\frac{1}{4}\right).$$

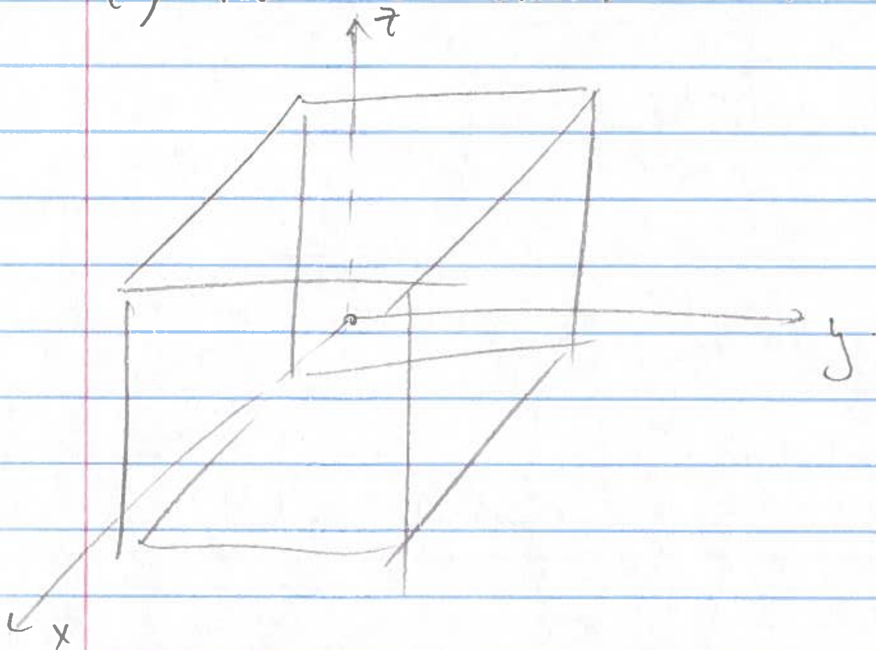
About a diagonal:

$$\vec{L} = \frac{Ma^2}{12} \begin{pmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{pmatrix} \cdot \frac{\omega}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{Ma^2\omega}{12\sqrt{3}} \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

$$\vec{L} = \frac{Ma^2}{6} \underbrace{\omega \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)}_{\vec{\omega}} = \frac{Ma^2}{6} \vec{\omega}$$

Same direction!

(b) The cube rotates about its center.



Reflection symmetry with respect to $z=0$ $\Rightarrow$ $I_{xz} = I_{yz} = 0$

— " — " — " — $x=0$ $\Rightarrow$ $I_{yx} = I_{zx} = 0$

— " — " — " — $y=0$ $\Rightarrow$ $I_{xy} = I_{zy} = 0$

$$I_{xx} = \int dx\, dy\, dz \cdot d \cdot (y^2+z^2)$$

$$= \int_{-a/2}^{+a/2} dx \int_{-a/2}^{+a/2} dy \cdot y^2 \cdot \int_{-a/2}^{a/2} dz \cdot d + \int_{-\frac{a}{2}}^{+a/2} dx \int_{-\frac{a}{2}}^{+a/2} dy \cdot d \int_{-\frac{a}{2}}^{+a/2} dz\, z^2$$

$$= a \cdot \frac{2}{3}\left(\frac{a}{2}\right)^3 \cdot a \cdot d + \underset{\text{result}}{\text{same}} = d\,\frac{a^5}{12} \cdot 2 = \frac{Ma^2}{6}.$$

$$I = \frac{Ma^2}{6} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \frac{Ma^2}{6}\, \mathbb{1}$$

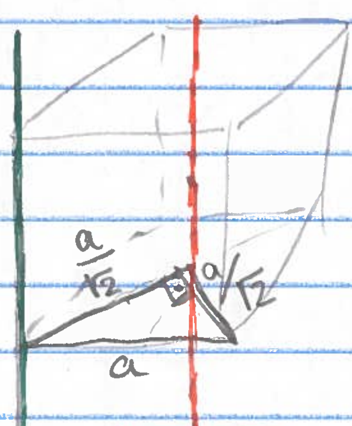
unit matrix

The angular momentum is in the same direction as $\vec{\omega}$:

$$\vec{L} = I\,\vec{\omega} = \frac{Ma^2}{6}\cdot\mathbb{1}\cdot\vec{\omega} = \frac{Ma^2}{6}\,\vec{\omega}.$$

This is the same result as before when the cube was rotating around the main diagonal through the corner of the cube.

Parallel axis theorem?



$\boxed{\frac{1}{6}Ma^2}$

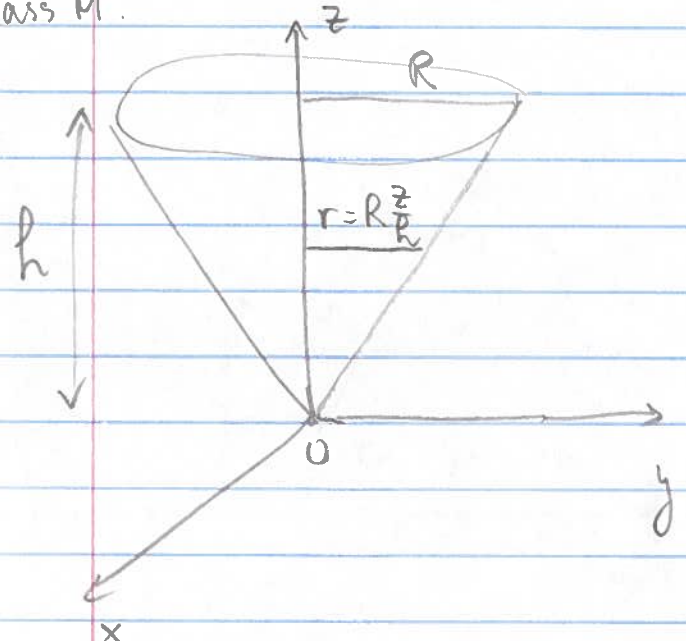
$\boxed{\frac{2}{3}Ma^2}$ how are these answers related?

$$\frac{2}{3}Ma^2 = \frac{1}{6}Ma^2 + M\left(\frac{a}{\sqrt{2}}\right)^2 = \left(\frac{1}{6}+\frac{1}{2}\right)Ma^2 = \frac{2}{3}Ma^2$$

Problem 10.26.

Example 10.3 Inertia tensor of a Solid Cone.

Mass M.



$$I_{zz} = \int dV \cdot d \cdot (x^2+y^2)$$

$$= d \int_0^h dz \int_0^{2\pi} d\varphi \int_0^r d\rho\, \rho \cdot \rho^2$$

$$= d \cdot 2\pi \cdot \int_0^h dz \cdot \frac{r^4}{4}$$

$$= \frac{\pi d}{2} \int_0^h dz\, \frac{R^4}{h^4} \cdot z^4$$

$$= \frac{\pi d}{2} \frac{R^4}{h^4} \cdot \frac{h^5}{5} = \frac{\pi}{10} d R^4 h$$

$$= \frac{\pi}{10} \cdot \frac{M}{\frac{1}{3}\pi R^2 h} \cdot R^4 h = \frac{3MR^2}{10}$$

$I_{xx} = I_{yy}$ because of the symmetry.

$$I_{xx} = \int_V dV \cdot d \cdot (y^2+z^2) = \underbrace{\int dV \cdot d \cdot y^2}_{\text{this is just } \frac{I_{zz}}{2} = \frac{3}{20}MR^2} + \int dV \cdot d \cdot z^2$$

Second term gives

$$d\int_0^h dz\, z^2 \cdot \int_0^{2\pi} d\varphi \int_0^{r} d\rho \cdot \rho = d\int_0^h dz \cdot z^2 \cdot \cancel{2}\pi \cdot \frac{r^2}{\cancel{2}}$$

$$= \pi d \int_0^h dz \cdot z^2 \cdot \frac{R^2}{h^2} z^2$$

$$= \pi d \cdot \frac{R^2}{h^2} \frac{h^5}{5} = \frac{\pi}{5} d R^2 h^3$$

$$= \frac{\cancel{\pi}}{5} \cdot \frac{M}{\frac{1}{3}\cancel{\pi R^2 h}} \cdot R^2 h^3 = \frac{3}{5} M h^2$$

The off-diagonal products are all zero:

$x=0$ is a plane of symmetry $\Rightarrow I_{yx} = I_{zx} = 0$.

$y=0$ —"— "— "—— $\Rightarrow I_{xy} = I_{zy} = 0$.

$$\mathbb{I} \equiv \begin{pmatrix} \frac{3}{20}MR^2 + \frac{3}{5}Mh^2 & 0 & 0 \\ 0 & \frac{3}{20}MR^2 + \frac{3}{5}Mh^2 & 0 \\ 0 & 0 & \frac{3}{10}MR^2 \end{pmatrix}$$

$$= \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}$$

$$\vec{L} = \mathbb{I}\vec{\omega} = (\lambda_1 \omega_x, \lambda_2 \omega_y, \lambda_3 \omega_z).$$