Diagonalizing Real Symmetric Matrices.

Let us review how to find the eigenvalues and the eigenvectors of a given constant matrix. The matrices which we shall encounter in mechanics 4222 are real and symmetric, but the same procedure applies in general.

(The Appendix in the textbook is concerned with proofs of the existence of the real eigenvalues and the corresponding orthonormal set of eigenvectors. Here we are more interested in the practical applications.).

Main topics for today:

1. Finding the eigenvalues.
2. Finding the eigenvectors
3. Diagonalizing the matrix.

We shall illustrate the procedure with the following example.

$$M = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

1 Finding the eigenvalues.

$M = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ is a 2×2 matrix, therefore it has 2 eigenvalues and 2 eigenvectors. The eigenvalues are often denoted with λ:

λ_1 first eigenvalue

λ_2 second eigenvalue.

To find them, 1 subtract λ from each diagonal entry 2 take the determinant of the resulting matrix 3 set it equal to zero 4 solve the resulting characteristic equation:

Solve this : $\begin{vmatrix} 1-\lambda & 2 \\ 2 & 4-\lambda \end{vmatrix} = 0$

$$(1-\lambda)(4-\lambda) - 2\times2 = 0$$

$$4 - 4\lambda - \lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 - 5\lambda = 0$$

$$\lambda(\lambda-5) = 0 \implies \boxed{\text{the roots are: } \lambda_1 = 0,\ \lambda_2 = 5}$$

The two eigenvalues are 0 and 5.

2 Finding the eigenvectors

Now take each of the just found eigenvalues λ_i and solve for the components of the respective eigenvector

$$\vec{r}_i = \begin{pmatrix} x_i \\ y_i \end{pmatrix} \quad ; \quad i = 1, 2 .$$

the eigenvalue equation

$$M \vec{r}_i = \lambda_i \vec{r}_i$$

or equivalently,

$$\left(M - \lambda_i \underset{\uparrow}{\mathbb{I}} \right) \vec{r}_i = 0 .$$

unit matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

First eigenvector

$\lambda_1 = 0$, so the eigenvalue equation becomes

$$\begin{pmatrix} 1 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = 0$$

This is a system of 2 equations for 2 unknowns, x_1 and y_1.

$$\left\| \begin{array}{l} x_1 + 2y_1 = 0 \\ 2x_1 + 4y_1 = 0 \end{array} \right.$$

It is easy to see that the equations are <u>not</u> independent. This is expected since the determinant of the matrix is zero.

Any vector of the form

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \begin{pmatrix} -2d \\ d \end{pmatrix}$$

is a solution. Let's unit-normalize:

$$\boxed{\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} -2 \\ 1 \end{pmatrix}} \boxed{= \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}}$$

Second eigenvector.

$\lambda_2 = 5$ and the eigenvalue equation becomes

$$\begin{pmatrix} 1-5 & 2 \\ 2 & 4-5 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} -4 & 2 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = 0$$

The system of equations is

$$\left\| \begin{array}{l} -4x_2 + 2y_2 = 0 \\ 2x_2 - y_2 = 0 \end{array} \right.$$

Again, this is the same equation, with solution $y_2 = 2x_2$. The eigenvector is of the form

$$\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \begin{pmatrix} \alpha \\ 2\alpha \end{pmatrix}$$

for any α. Again, unit-normalize the vector:

$$\boxed{\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}}$$

Final Result

We found that for the matrix

$$M = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$$

the eigenvalues and the eigenvectors are:

$$\lambda_1 = 0 \implies \vec{r}_1 = \begin{pmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$$

$$\lambda_2 = 0 \implies \vec{r}_2 = \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$$

But how can we be sure that we got the right answer? What if we made a mistake?

CONSISTENCY CHECKS! (next page)

Consistency Checks.

① $\vec{r}_1$ and $\vec{r}_2$ are orthogonal:

$$\vec{r}_1 \cdot \vec{r}_2 = \left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right) \cdot \begin{pmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{pmatrix}$$

$$= -\frac{2}{\sqrt{5}} \cdot \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{5}} \cdot \frac{2}{\sqrt{5}} = -\frac{2}{5} + \frac{2}{5} = \emptyset \quad \checkmark$$

② Sum of the eigenvalues is the trace of M:

$$\text{Trace } M = 1 + 4 = 5 = \lambda_1 + \lambda_2 \quad \checkmark$$

③ Product of the eigenvalues is the determinant of M:

$$\det M = 1 \times 4 - 2 \times 2 = 4 - 4 = 0 = 0 \times 5 \quad \checkmark$$

3 $\boxed{\text{Diagonalizing the matrix}}$

The matrix formed out of the eigenvectors is an orthogonal matrix which diagonalizes our original matrix M:

$$R \equiv \left[\left(\vec{r}_1 \right) \left(\vec{r}_2 \right) \right] = \begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$$

First verify that R is orthogonal:

$$R^T \cdot R = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \frac{1}{\sqrt{5}} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 4+1 & -2+2 \\ -2+2 & 1+4 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

YES.

Now check that it diagonalizes M as follows

$$R^T \; M \; R$$

$$= \left(\frac{1}{\sqrt{5}}\right)^2 \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix}}_{\text{multiply these first}}$$

$$= \frac{1}{5} \begin{bmatrix} -2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 0 & 5 \\ 0 & 10 \end{bmatrix}$$

$$= \frac{1}{5} \begin{bmatrix} 0 & 0 \\ 0 & 25 \end{bmatrix} = \begin{bmatrix} \overset{\lambda_1}{0} & 0 \\ 0 & \underset{\lambda_2}{5} \end{bmatrix}$$