Special Relativity

Relativistic invariance implies invariance with respect to space-time transformations belonging to the Lorentz group. The key ideas for this lecture are therefore:

* Lorentz group. This is the group of transformations in 4 dimensions which consists of:
  - ordinary rotations in the 3-d subspace involving only the spatial coordinates and leaving time unaffected. Previously (see lectures on Feb. 19 & 21) we saw that rotations in 3d form their own group, SO(3), and are described by orthogonal matrices O:

$$\vec{r}\,' = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} O_{11} & O_{12} & O_{13} \\ O_{21} & O_{22} & O_{23} \\ O_{31} & O_{32} & O_{33} \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = O \cdot \vec{r}$$

    The group invariant is the dot product

$$\vec{r}^{\,T} \cdot \vec{r} = (x, y, z) \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{the Euclidean metric.}} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = x^2 + y^2 + z^2$$

  - Lorentz boosts, which involve the time coordinate plus one of the spatial coordinates. We shall derive them today.

* Space-time. We shall introduce this today.

$\boxed{\text{Space-time.}}$

See Section 15.8.

Let's add the time coordinate to the position coordinates $\vec{r} = (x, y, z)$ [in Cartesian coordinates]. If we naively try

$$(x, y, z, t)$$

this doesn't quite work because of units mismatch. We can fix this by introducing a new fundamental constant c with units of velocity:

$$[c] = \frac{m}{s}$$

and then add ct as the extra coordinate:

$$(x, y, z, ct).$$

At this point we do not know the meaning of c, but it makes sense that it is a universal constant, because all observers must use it in the same way to convert their measured time intervals to distances $c\Delta t$.

For convenience, we will use indices 1 through 4:

$$\text{Spacetime} \longrightarrow (\underset{\substack{\downarrow \\ x}}{x_1}, \underset{\substack{\downarrow \\ y}}{x_2}, \underset{\substack{\downarrow \\ z}}{x_3}, \underset{\substack{\downarrow \\ ct}}{x_4}).$$

This allows us to treat x_4 on equal footing with x_1, x_2, x_3. But we must not forget that x_4 is in principle "special" and different from the other three coordinates, so we should not be surprised when something odd happens to it (minus sign in the metric).

Section 15.8.

"Rotations" in spacetime.

Once we have formed spacetime, we can consider arbitrary linear transformations of the 4-vectors in it

$$\underbrace{q \equiv \underbrace{(q_1, q_2, q_3}_{\text{spatial components}}, \underbrace{q_4}_{\text{time component}})}_{\text{of the 4-vector } q.}$$

The conventional notation for the "rotation" matrix is Λ, so we have

$$q' = \begin{pmatrix} q_1' \\ q_2' \\ q_3' \\ q_4' \end{pmatrix} = \begin{pmatrix} \Lambda_{11} & \Lambda_{12} & \Lambda_{13} & \Lambda_{14} \\ \Lambda_{21} & \Lambda_{22} & \Lambda_{23} & \Lambda_{24} \\ \Lambda_{31} & \Lambda_{32} & \Lambda_{33} & \Lambda_{34} \\ \Lambda_{41} & \Lambda_{42} & \Lambda_{43} & \Lambda_{44} \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} \equiv \Lambda \cdot q$$

Here q is any 4-vector. As a special case, it can be the position 4-vector ~~of~~ spacetime, whose 3d analogue is $\vec{r}$. But it could also be a 4-vector which is the 4-dimensional generalization of a 3-vector like momentum $\vec{p}$, electric field $\vec{E}$, etc.

Next we need to specify the properties of Λ.

Sections 15.8 & 15.9

Lorentz transformations.

In analogy to 3d rotations, the Lorentz transformations are defined so that they preserve an invariant quantity - the "length" of a 4-vector in spacetime. But how do you define "length" in spacetime? This is where we must remember that x_4 is special and change the sign signature just for x_4:

$$\begin{pmatrix} q_1 & q_2 & q_3 & q_4 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{pmatrix} = q_1^2 + q_2^2 + q_3^2 - q_4^2$$

This matrix is the "Minkowski" metric

this is what makes time special.

If the last sign was "+", we would get the standard group SO(4) of ordinary rotations in 4 dimensions.

Because the sign is "−" we get the Lorentz group, which is different from SO(4), and is often denoted as

$$SO(3,1) \equiv \text{Lorentz group.}$$

3 plusses; one minus in the metric

Explicit form of the Lorentz transformations

We just defined a Lorentz transformation Λ as a linear transformation

$$q' = \Lambda \cdot q$$

which preserves the Minkowski metric

$$g \equiv \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

which we use to form the "dot product" in spacetime of two arbitrary 4-vectors a and b

$$a^T \cdot g \cdot b \equiv (a_1\ a_2\ a_3\ a_4) \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{pmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3 - a_4 b_4 .$$

If we transform the two vectors:

$$a' = \Lambda a \quad \Rightarrow \quad a'^T = a^T \Lambda^T \qquad (*)$$

$$b' = \Lambda b \qquad (**)$$

then the transformed dot product $a'^T \cdot g \cdot b'$ should be the same as the original $a^T \cdot g \cdot b$:

$$a'^T \cdot g \cdot b' = a^T \cdot g \cdot b \qquad (***)$$

Now substitute (*) and (**) into (***):

$$a^T \cdot \boxed{\Lambda^T \cdot g \cdot \Lambda} \cdot b = a^T \cdot \boxed{g} \cdot b$$

must be equal

This is supposed to be true for any vectors a and b, therefore we must have:

alternative definition. (*)

$$\boxed{\Lambda^T \cdot g \cdot \Lambda = g} \qquad g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

It is in this sense that we say that the Lorentz transformations "preserve" the Minkowski metric g.

Equation (*) is an alternative, "linear algebra" definition of a Lorentz transformation: any matrix Λ which satisfies (*) is a valid Lorentz transformation, i.e. a transformation belonging to the Lorentz group.

So what could Λ be?

* one possibility is a pure 3d rotation in space leaving the time (fourth) component unaffected:

$$\Lambda_R = \left(\begin{array}{c|c} \mathbb{R} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \hline \begin{matrix} 0 & 0 & 0 \end{matrix} & 1 \end{array}\right)$$

where $\mathbb{R}$ is a 3×3 orthogonal matrix from $SO(3)$

* the remaining possibility is ..... (next page)

Lorentz Boosts

A Lorentz boost is a transformation which affects the time coordinate as well as one space coordinate. For example, a boost along x_1 looks like this:

$$\Lambda_x = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \qquad \text{Eq. (15.43)}$$

Clearly this leaves x_2 and x_3 the same:

$$x_1' = \gamma x_1 - \gamma\beta x_4 = \gamma x_1 - \gamma\beta ct$$

$$x_2' = x_2 \qquad \text{Eq. (15.39)}$$

$$x_3' = x_3$$

$$ct' = x_4' = -\gamma\beta x_1 + \gamma x_4 = -\gamma\beta x_1 + \gamma ct$$

but scrambles x_1 and t among themselves. At this point, eq. (15.43) above was simply postulated. It remains to:

1. figure out what is γ
2. figure out what is β
3. Prove that (15.43) is indeed a Lorentz transformation, i.e. that it satisfies the definition (*)

The Lorentz factor γ

Start with the basic definition:

$$\Lambda^T \cdot g \cdot \Lambda = g$$

and take the determinant of both sides:

$$\det\left(\Lambda^T \cdot g \cdot \Lambda\right) = \det g$$

use that $\det(AB) = \det A \cdot \det B$

$$\det \Lambda^T \cancel{\det g} \det \Lambda = \cancel{\det g}$$

$$\left(\det \Lambda\right)^2 = 1 \qquad \text{because } \det \Lambda^T = \det \Lambda$$

$$\boxed{\det \Lambda = 1}$$

Let's check if this is true for our ansatz

$$\det \begin{Vmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{Vmatrix} = \gamma^2 - (-\gamma\beta)^2 = \gamma^2(1-\beta^2) = 1$$

This provides a relationship between γ and β

$$\boxed{\gamma = \frac{1}{\sqrt{1-\beta^2}}}$$

Now we know γ in terms of β. But what is β?

(next page)

Proof of $\Lambda^T \cdot g \cdot \Lambda = g$

Let's check whether the conjectured transformation Λ_x in Eq. (15.43) satisfies the defining equation

$$\Lambda^T \cdot g \cdot \Lambda = g.$$

Start with the LHS and multiply through:

$$\Lambda^T \cdot g \cdot \Lambda = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & 0 & 0 & \gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & -\gamma \end{pmatrix} \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix}$$

$$= \begin{pmatrix} \gamma^2 - \gamma^2\beta^2 & 0 & 0 & -\gamma^2\beta + \gamma^2\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma^2\beta + \gamma^2\beta & 0 & 0 & \gamma^2\beta^2 - \gamma^2 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma^2(1-\beta^2) & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -\gamma^2(1-\beta^2) \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

where in the last step we used the previously derived relation

$$\gamma^2(1-\beta^2) = 1 \iff \gamma = \frac{1}{\sqrt{1-\beta^2}}.$$

SUMMARY.

* In special relativity, spatial coordinates and time are combined in spacetime.

* Lorentz transformations are linear transformations of the spacetime coordinates which preserve the Minkowski metric

$$g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

in the sense that

$$\Lambda^T \cdot g \cdot \Lambda = g$$

* There is a fundamental constant c with dimensions of velocity which allows us to incorporate time together with the spatial coordinates.

* The Lorentz group $SO(3,1)$ consists of 3d rotations plus boosts.

* A boost involving the x_1 coordinate is given by

$$\Lambda = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \quad \text{where} \quad \gamma = \frac{1}{\sqrt{1-\beta^2}}$$