Lorentz boost: physical interpretation

Last time we derived the following Lorentz transformation (boost involving the $x_1 = x$ axis):

$$\begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\beta\gamma \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\beta\gamma & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix}$$

where $\gamma = \dfrac{1}{\sqrt{1-\beta^2}}$.

But we did not explain the meaning of the β parameter. For this purpose we need to make the connection to the Galilean transformations which are valid at everyday velocities typically encountered in experiments on Earth.

For future reference, let's write the transformations for x and t:

$$\boxed{\begin{aligned} x' &= \gamma\left(x - \beta c t\right) = \frac{x - \beta c t}{\sqrt{1-\beta^2}} \\ t' &= \gamma\left(t - \frac{\beta}{c}x\right) = \frac{t - \frac{\beta}{c}x}{\sqrt{1-\beta^2}} \end{aligned}}$$

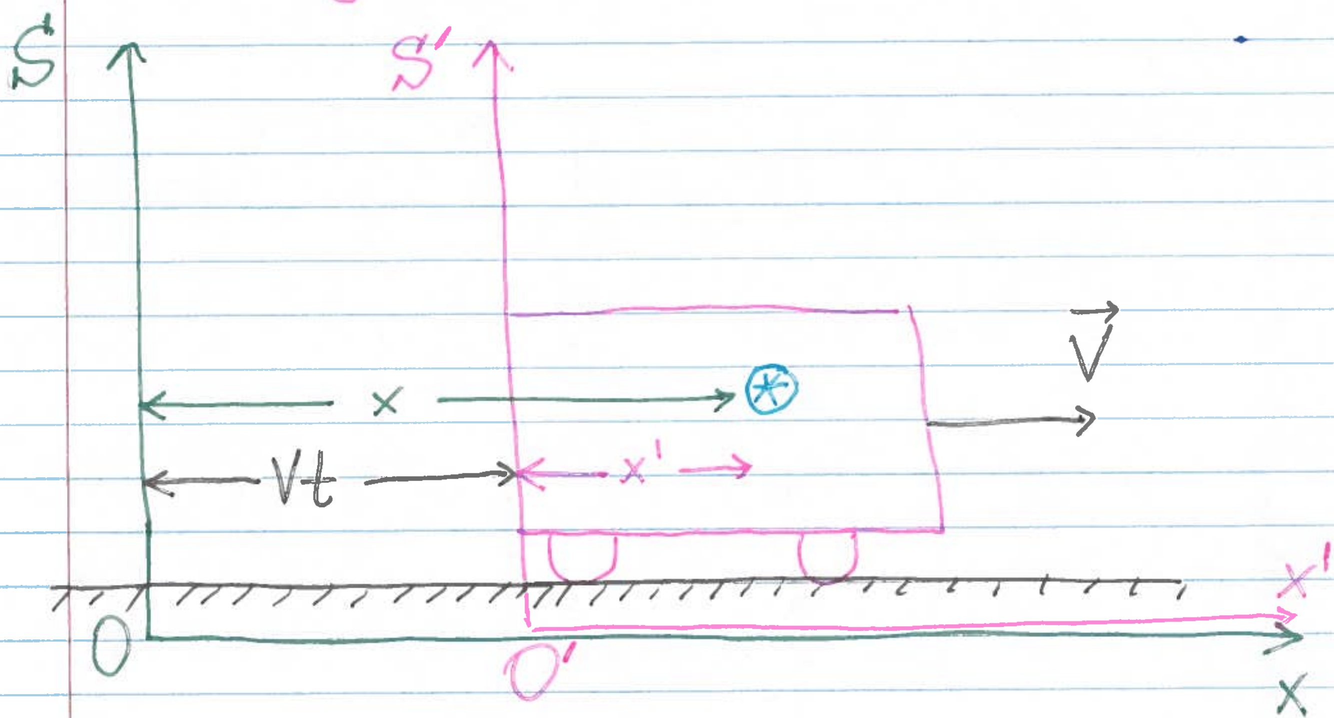
Section 15.2.

The Galilean Transformation

Consider two inertial reference frames:

S: stationary observer on the ground

S': railroad car travelling with constant velocity $\vec{V}$ in the +x direction:



Let the two origins O and O' coincide at $t=0$.
Then the horizontal coordinate of an event ⊛ is given by x in O and x' in O', related by

$$\boxed{\begin{aligned} x' &= x - Vt \\ t' &= t \end{aligned}}$$

Galilean transformations.

The β parameter

At low enough speeds V,

Lorentz $\qquad$ Galilean

$$x' = \frac{x - \beta ct}{\sqrt{1-\beta^2}} \quad \text{must reduce to} \quad x' = x - Vt$$

Upon inspection, we identify:

$$V = \beta c \quad \Rightarrow \quad \boxed{\beta = \frac{V}{c}}$$

Since the speed V is low, i.e. $V \ll c$, β is a small parameter, and then γ becomes approximately

$$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \left(1-\beta^2\right)^{-1/2} \approx 1 + \frac{1}{2}\beta^2 = 1 + \frac{1}{2}\frac{V^2}{c^2} \approx 1$$

and up to linear terms in β, the Lorentz transformation is the same as the Galilean. ✓

Conclusion: β is the relative speed of the two reference frames S and S', measured in units of the fundamental velocity c, thus β is dimensionless and so is γ.

15.9. The Invariant Scalar Product.

We can obtain Lorentz-invariant quantities by taking two 4-vectors and "dotting" them into each other with the help of the Minkowski metric g:

$$a^T \cdot g \cdot b \equiv a_1 b_1 + a_2 b_2 + a_3 b_3 - a_4 b_4$$

Under a general Lorentz transformation Λ

$$a' = \Lambda \cdot a \quad , \quad b' = \Lambda \cdot b ,$$

this quantity transforms into

$$(a')^T \cdot g \cdot b' = (\Lambda a)^T \cdot g \cdot \Lambda \cdot b$$

$$= a^T \cdot \underbrace{\Lambda^T \cdot g \cdot \Lambda} \cdot b = a^T \cdot g \cdot b .$$

this is equal to g because Λ satisfied the equation

$$\Lambda^T \cdot g \cdot \Lambda = g .$$

Section 15.9.

The Invariant Scalar Product in Spacetime

As an example, let's take the 4-vector to be the position of an event in spacetime, i.e.

$$a = b = \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$$

The invariant dot product is

$$a^T \cdot g \cdot a = x_1^2 + x_2^2 + x_3^2 - x_4^2$$

or in the usual physics notation,

$$S \equiv x^2 + y^2 + z^2 - c^2t^2 = r^2 - c^2t^2$$

radial coordinate in spherical coordinates.

If two observers related by a Lorentz transformation measure some event (x, y, z, ct), they may disagree on the individual values of x, y, z, t, but then if they sit down and calculate the quantity $S = x^2 + y^2 + z^2 - c^2t^2$ they will both obtain the same value for S.

4-vector of energy-momentum.

In classical mechanics, we have the 3-vector of momentum:

$$\vec{p} = m\vec{v}$$

In relativity we have to promote it to a 4-vector by adding an extra component with units of momentum:

$$\text{4-vector} = (p_x, p_y, p_z, ?)$$

what could this be?

Wait! We have c at our disposal, and we can use it to adjust the units just like we switched from t to $x_4 = ct$.

What differs from p by units of velocity?

$$p = mv \qquad E = \frac{1}{2}mv^2$$

extra factor of v.

Perhaps $p_4 = \frac{E}{c}$? Let's try it:

4-vector of energy-momentum is given by:

$$\boxed{P \equiv \left(p_x, p_y, p_z, \frac{E}{c}\right) = \left(\vec{p}, \frac{E}{c}\right)}$$

Eq. (15.85)

Invariant mass

Once we have a 4-vector we can build an invariant quantity by dotting it into itself:

$$p^T \cdot g \cdot p = \begin{pmatrix} p_x & p_y & p_z & \frac{E}{c} \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} p_x \\ p_y \\ p_z \\ E/c \end{pmatrix}$$

$$= p_x^2 + p_y^2 + p_z^2 - \frac{E^2}{c^2} = \text{constant}.$$

Recall that $p^2 \sim \text{mass}^2 \times \text{velocity}^2$

Therefore we can use the c in place of velocity and define the constant in terms of mass:

$$\vec{p}^{\,2} - \frac{E^2}{c^2} = \,?\, m^2 c^2$$

what should be the sign?

Which is larger? $|\vec{p}|$ or $\frac{E}{c}$?

Obviously there are objects with $\vec{p} = 0$ and nonzero energy, therefore $\frac{E}{c} > |\vec{p}|$ and the sign must be minus:

$$\vec{p}^{\,2} - \frac{E^2}{c^2} = -m^2 c^2$$

Equivalently: $E^2 = m^2 c^4 + p^2 c^2$ (15.85)

In particle physics conventions: $E^2 - p^2 = m^2$
(with $c = 1$ in natural units)

Problem 15.60. A particle of mass m_a decays at rest into two identical particles each of mass m_b. Use conservation of momentum and energy to find their momenta and energies. Use natural units in which the value of c is equal to 1.

BEFORE: m_a 4-momentum $(\emptyset, E_a)$

AFTER: $\longleftarrow m_b \quad m_b \longrightarrow$

4-momenta: $(\vec{p}_{b_1}, E_{b_1}) \; (\vec{p}_{b_2}, \vec{E}_{b_2})$

First let's trade all energies for masses:

$$E_a^2 - \underbrace{\vec{p}_a^{\,2}}_{=0} = m_a^2 \quad \Rightarrow \quad \boxed{E_a = m_a}$$

0 because a is at rest

$$E_{b_1}^2 - \vec{p}_{b_1}^{\,2} = m_b^2 \quad \Rightarrow \quad E_{b_1} = \sqrt{m_b^2 + \vec{p}_{b_1}^{\,2}}$$

$$E_{b_2}^2 - p_{b_2}^2 = m_b^2 \quad \Rightarrow \quad E_{b_2} = \sqrt{m_b^2 + \vec{p}_{b_2}^{\,2}}$$

Now let's use momentum conservation:

$$\vec{p}_{b_1} + \vec{p}_{b_2} = \vec{p}_a = 0 \Rightarrow \vec{p}_{b_2} = -\vec{p}_{b_1}$$

and E_{b_2} becomes

$$E_{b_2} = \sqrt{m_b^2 + \vec{p}_{b_1}^{\,2}}$$

We expressed everything in terms of $|\vec{p}_{b_1}|$.

It remains to use energy conservation to solve for it:

$$E_a = E_{b_1} + E_{b_2}$$

$$m_a = \sqrt{m_b^2 + p_{b_1}^2} + \sqrt{m_b^2 + p_{b_1}^2} = 2\sqrt{m_b^2 + p_{b_1}^2}$$

Square both sides:

$$m_a^2 = 4\left(m_b^2 + p_{b_1}^2\right)$$

$$p_{b_1}^2 = \frac{m_a^2}{4} - m_b^2 \Rightarrow \boxed{p_{b_1} = \sqrt{\frac{m_a^2}{4} - m_b^2}}$$

What happens when $m_a < 2m_b$?

The energies are:

$$E_{b_1} = E_{b_2} = \sqrt{m_b^2 + p_{b_1}^2} = \sqrt{\cancel{m_b^2} + \frac{m_a^2}{4} - \cancel{m_b^2}} = \frac{m_a}{2}$$

The daughter particles are identical and equally share the energy of the mother particle