

PHY 6648: Quantum Field Theory I
Fall Term 2005
Homework Set 7

Due Friday, October 14 2005

Reading: Chapter 3 from the textbook and class notes.

In this homework we will work out the discrete symmetries of the Dirac theory and derive the assignments for the Dirac bilinears which are listed in the table on page 71 (section 3.6).

Problem 1. Parity. In this problem we will consider parity transformations, i.e. mirror reflections in spacetime:

$$x^\mu \rightarrow x'^\mu = (t, -\vec{x}). \quad (1)$$

(a) Let $\psi(x)$ satisfy the Dirac equation

$$(i\gamma^\mu \partial_\mu - m)\psi(x) = 0. \quad (2)$$

Show that

$$\psi'(x') \equiv \gamma^0 \psi(x) \quad (3)$$

satisfies the Dirac equation in the parity-reflected world:

$$(i\gamma^\mu \partial'_\mu - m)\psi'(x') = 0, \quad (4)$$

where

$$\partial'_\mu = \frac{\partial}{\partial x'^\mu} = \left(\frac{\partial}{\partial t}, -\frac{\partial}{\partial \vec{x}} \right).$$

(b) Now we can derive the P -parity assignments of the bilinears in the table on page 71, simply by comparing each one to its cousin in the parity-reflected world. Show that

$$\bar{\psi}'(x')\psi'(x') = \bar{\psi}(x)\psi(x), \quad (5)$$

$$i\bar{\psi}'(x')\gamma^5\psi'(x') = -i\bar{\psi}(x)\gamma^5\psi(x), \quad (6)$$

$$\bar{\psi}'(x')\gamma^\mu\psi'(x') = g^{\mu\mu}\bar{\psi}(x)\gamma^\mu\psi(x), \quad (7)$$

$$\bar{\psi}'(x')\gamma^\mu\gamma^5\psi'(x') = -g^{\mu\mu}\bar{\psi}(x)\gamma^\mu\gamma^5\psi(x), \quad (8)$$

$$\bar{\psi}'(x')\sigma^{\mu\nu}\psi'(x') = g^{\mu\mu}g^{\nu\nu}\bar{\psi}(x)\sigma^{\mu\nu}\psi(x), \quad (9)$$

where in the last three equations I have used $g^{\mu\mu}$ (no sum over μ) instead of the confusing $(-1)^\mu$ defined in the text.

Problem 2. Charge conjugation. In QED the interaction between the Dirac fermions and the electromagnetic field is introduced through the covariant derivative (4.5)

$$\partial_\mu \rightarrow D_\mu \equiv \partial_\mu + ieA_\mu(x) \quad (10)$$

and the Dirac equation becomes

$$[i\gamma^\mu(\partial_\mu + ieA_\mu(x)) - m]\psi(x) = 0. \quad (11)$$

(a) Introduce the so called charge-conjugated spinor

$$\psi_c \equiv C\gamma^0\psi^*(x) \quad (12)$$

where the operator C is defined to satisfy

$$\gamma^0\gamma^{\mu*}\gamma^0 = -C^{-1}\gamma^\mu C. \quad (13)$$

This latter equation does look complicated, but this is because it is written in a basis-independent form. If we choose a basis for the gamma matrices where γ^0 is real (such as the Weyl basis or the Dirac basis) it will simplify to

$$\gamma^{\mu T} = -C^{-1}\gamma^\mu C \quad (14)$$

by using the complex conjugated version of the identity

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0. \quad (15)$$

From here on let us agree to use a basis with a real γ^0 and therefore use the simplified definition of C as in (14).

Show that ψ_c satisfies the same Dirac equation, but with the opposite sign of the charge e :

$$[i\gamma^\mu(\partial_\mu - ieA_\mu(x)) - m]\psi_c(x) = 0, \quad (16)$$

thus justifying its name.

(b) Show that the operator C can be represented by

$$C = \gamma^2\gamma^0, \quad (17)$$

i.e. check that this satisfies (14). *Hint: In the Dirac and Weyl bases γ^2 is the only imaginary γ matrix, the other three are real.* In the remainder of this problem we will be using this explicit form of C .

(c) Show that if we twice charge-conjugate $\psi(x)$, we get back $\psi(x)$.

(d) Find the relation between C , C^\dagger , C^T and C^{-1} .

(e) Now we are ready to derive the C -parity assignments of all the bilinears in the table on page 71. Just as we did in problem 1b, compare the bilinears made up of ψ 's to their corresponding cousins made up of ψ_c 's and derive the C -parity assignments given in the table.

Problem 3. Time reversal. Now consider the time reversal transformation

$$x^\mu \rightarrow x'^\mu = (-t, \vec{x}). \quad (18)$$

By now you get the idea - we will look for a transformation relating the Dirac spinor $\psi(x)$ and its cousin $\psi'(x')$ in the time-reversed world, then we compare the bilinears made up of the two

types of Dirac spinors. However, in the case of time reversal, this procedure becomes somewhat tricky, since time reversal also involves complex conjugating all c -numbers as well as γ -matrices (see eq. (3.133)). For our purposes this means that the Dirac equation in the time reversed world will be

$$(-i\gamma^{\mu*}\partial'_\mu - m)\psi'(x') = 0, \quad (19)$$

where

$$\partial'_\mu = \frac{\partial}{\partial x'^\mu} = \left(-\frac{\partial}{\partial t}, \frac{\partial}{\partial \vec{x}}\right).$$

(a) Show that in the Dirac or Weyl bases the required relation is

$$\psi'(x') = \gamma^1\gamma^3\psi(x), \quad (20)$$

i.e. show that $\psi'(x')$ satisfies eq. (19). *Hint: In the Dirac and Weyl bases γ^2 is the only imaginary γ matrix, the other three are real.*

(b) Derive the T -parity assignments in the table on page 71 by comparing the bilinears made up of ψ 's and their analogues in the T -reflected world. *Hint: In the latter case, do not forget to complex-conjugate any c -numbers and γ -matrices entering the bilinear expressions.*