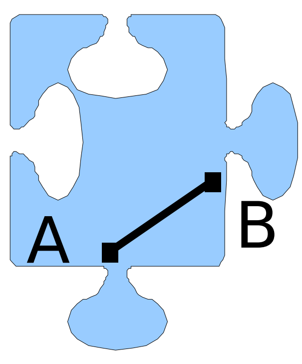
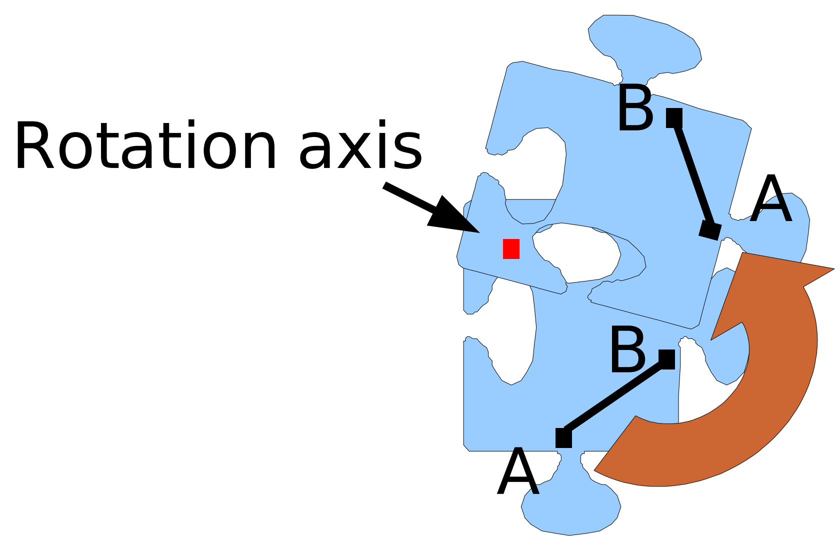
Chapter 10: Rotation

Rotation:

- Solid Body:
  - Distance between any two points does not change
  - All points rotate around one axis by exactly the same angle

r_{AB}=constant





Equations of motion

Translation

coordinate: $x(t)$

velocity: $v = dx/dt$

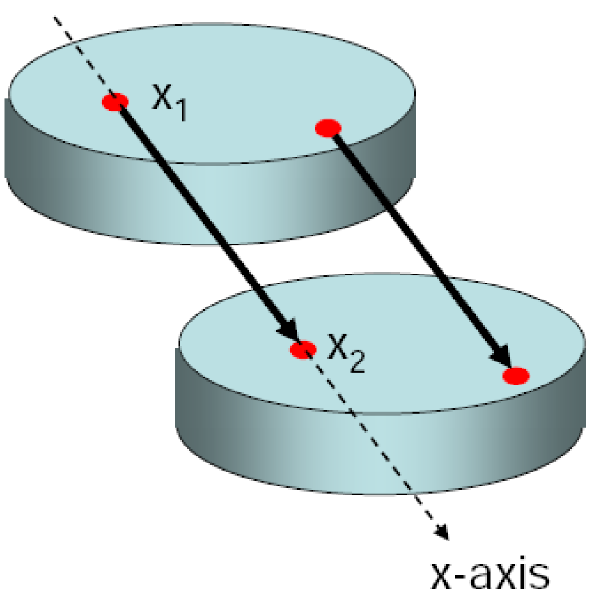
acceleration: $a = d^2x/dt^2$

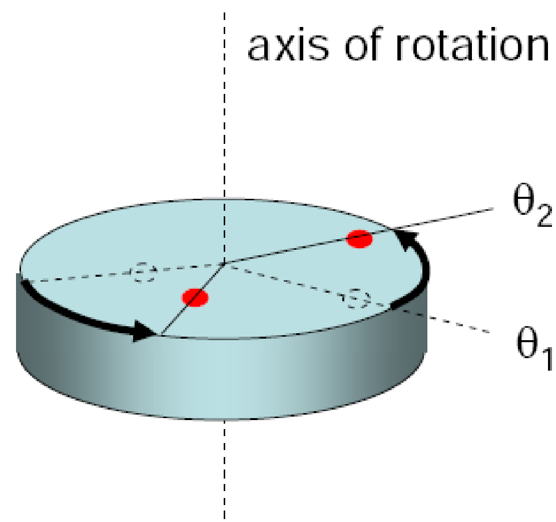
Rotation

angle: $\theta(t)$

angular velocity: $\omega = d\theta/dt$

angular acceleration: $\alpha = d^2\theta/dt^2$

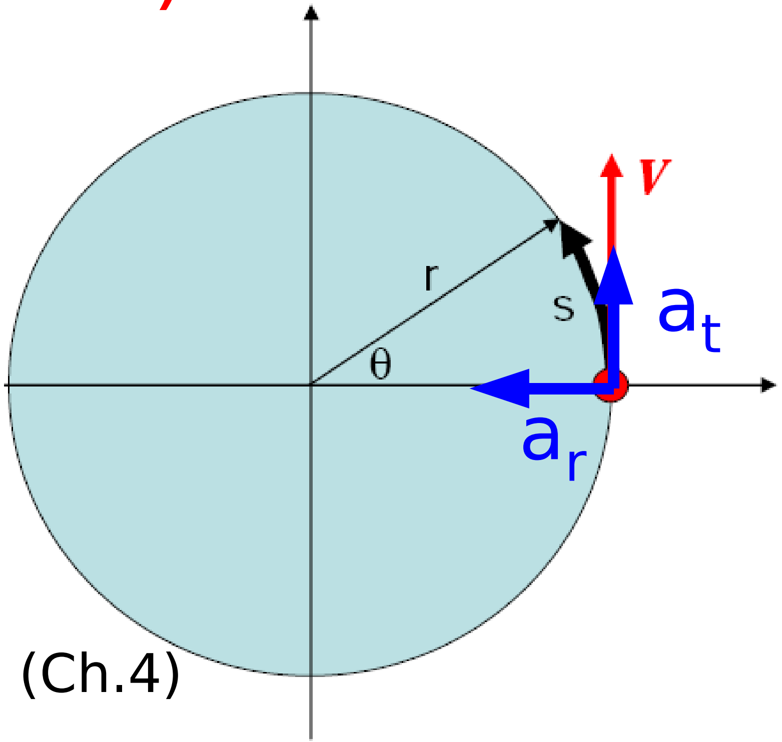




Velocity and Angular Velocity

Rotation (no translation):

- $\theta = s/r$
- $\omega = d\theta/dt = v_t/r$
  - v_t=tangential velocity
- $\alpha = d\omega/dt = a_t/r$
  - a_t=tangential acceleration
- $a_r = v_t^2/r = \omega^2 r$
  - a_r=centripetal acceleration (Ch.4)

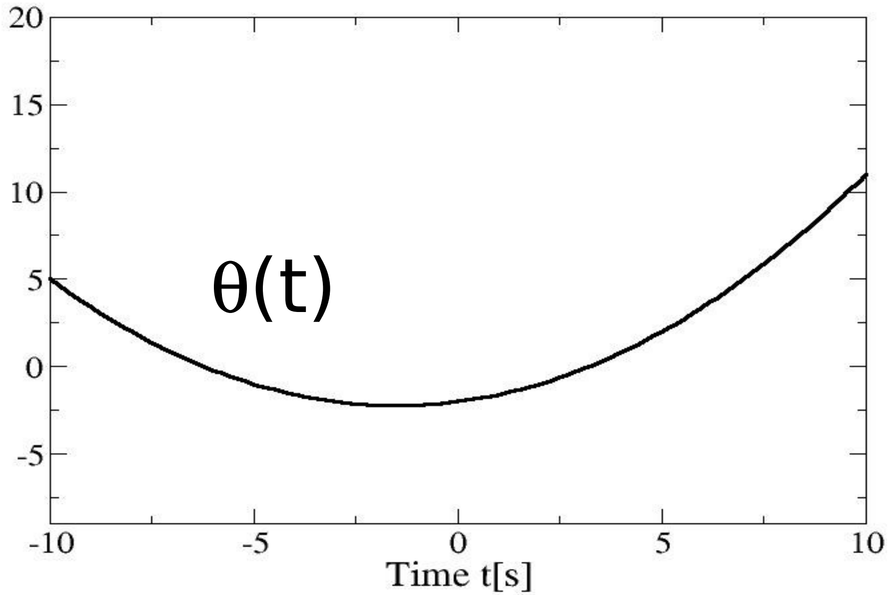


This only works if you use radian to measure angles! No degrees, no revolutions!

Example

A disk is rotating about its central axis.
The angular position of a reference line on the disk is given by:

$$\theta(t) = -2.0 + 0.3t + 0.1t^2$$



$\theta(t=-10s) = 5 \text{ rad} = 286^o$

$\theta(t=-5s) = -1 \text{ rad} = -57^o$

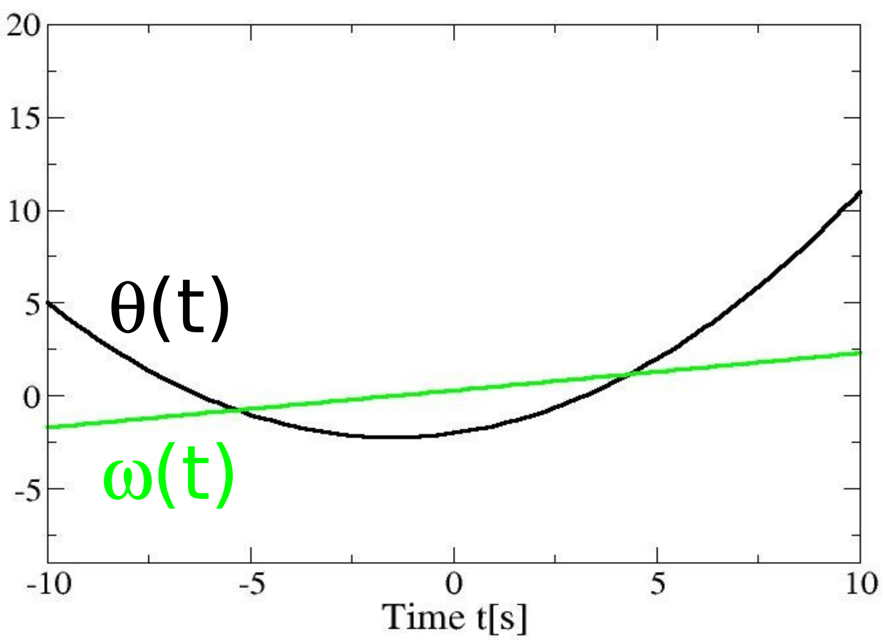
$\theta(t=0) = -2 \text{ rad} = -115^o$

$\theta(t=5s) = 2 \text{ rad} = 115^o$

$\theta(t=10s) = 11 \text{ rad} = 630^o$

Example

A disk is rotating about its central axis.
The angular position of a reference line on the disk is given by: $\theta(t) = (-2.0 + 0.3t + 0.1t^2)$ rad



What is the angular velocity as a function of time?

$\omega(t) = (0.3 + 0.2t)$ rad/s

What is the minimum angular position?

$\omega(t_{min}) = 0 \;\; \rightarrow \;\; t_{min} = -1.5$s

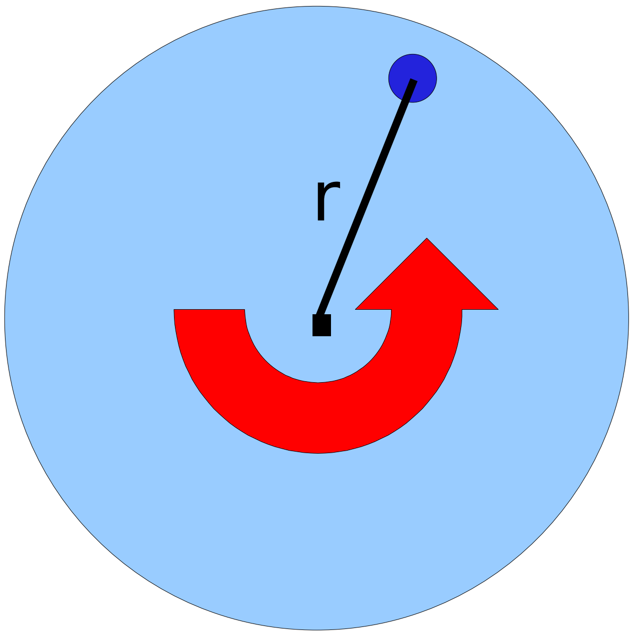
$\rightarrow \;\; \theta(t_{min}) = -2.225$ rad

What is the minimum angular acceleration? $\alpha(t) = 0.2$ rad/s^2

Example

A disk is rotating about its central axis.
The angular position of a reference line on the disk is given by:

$$\theta(t) = (-2.0 + 0.3t + 0.1t^2) \text{ rad}$$



Assume a point at a distance r=2cm from the center. What is the length of the segment on the circle limited by the location of the point at t=-3s and t= 3s?

$$s(t) = \theta(t)r = (-4.0 + 0.6t + 0.2t^2) \text{ cm}$$

This point travels on a circle. Travel in the counterclockwise direction is positive, in the clockwise is negative!

$$s(t=3s) - s(t=-3s) =$$
$$-0.4 \text{ cm} - (-4\text{cm}) = 3.6\text{cm}$$

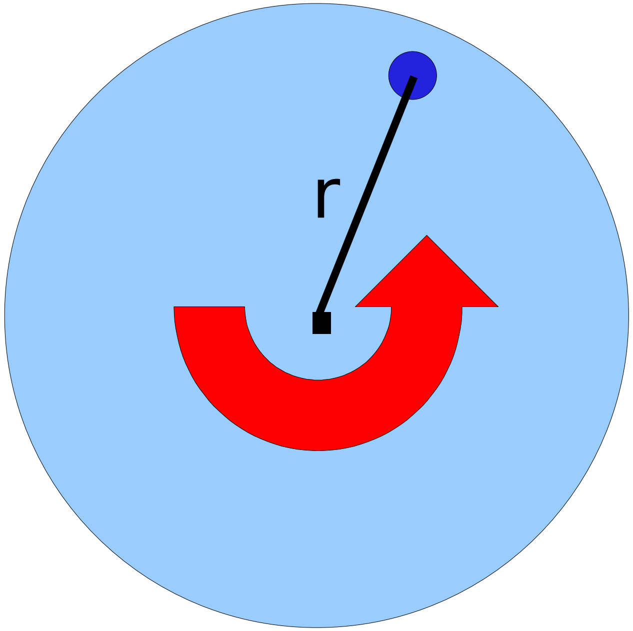
Example

A disk is rotating about its central axis.
The angular position of a reference line on the disk is given by:

$$\theta(t) = (-2.0 + 0.3t + 0.1t^2)\ \text{rad}$$

$$\omega(t) = (0.3 + 0.2t)\ \text{rad/s}$$

$$\alpha(t) = 0.2\ \text{rad/s}^2$$



What is the tangential velocity as a function of time of a point at a distance r=2cm from the center?

$$v(t) = \omega(t)r = (0.6 + 0.4t)\ \text{cm/s}$$

Tangential acceleration?

$$a_t(t) = \alpha(t)r = 0.4\ \text{cm/s}^2$$

Centripetal acceleration?

$$a_r(t) = v^2(t)/r = \omega^2(t)r = (0.18 + 0.24t + 0.08t^2)\ \text{cm/s}^2$$

Example

A disk is rotating about its central axis.
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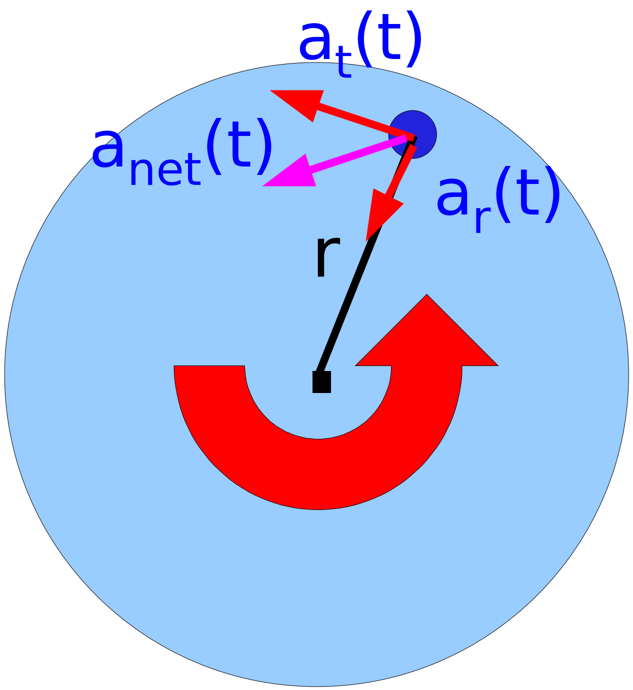
Tangential acceleration?



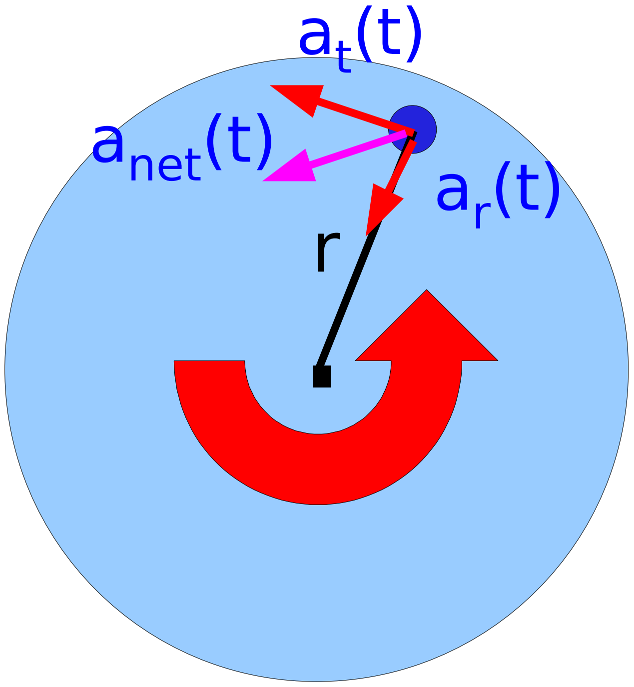
$$a_t(t) = \alpha(t) r = 0.4\ \text{cm/s}^2$$

Centripetal acceleration?

$$a_r(t) = v^2(t)/r = (0.18 + 0.24t + 0.08t^2)\ \text{cm/s}^2$$

The net acceleration $a_{net}(t)$ is the vector sum of the two.

Example



Assume we put an ice cube at r=2cm on this disk and the cube flies off the disk at t=3s.
What is the coefficient of static friction between the cube and the disk?

$$ma_{net}(t=3s) = \mu_s mg$$

$$a_t(t) = \alpha(t)r = 0.4\ cm/s^2$$

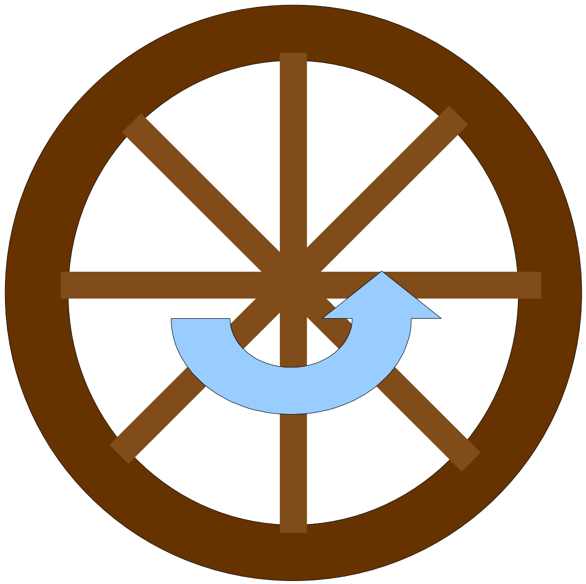
$$a_r(t=3s) = 1.62\ cm/s^2$$

$$a_{net} = (a_t^2 + a_r^2)^{1/2} = 1.67\ cm/s^2 = 0.0167 m/s^2$$

$$\mu_s = a_{net}/g = 0.0017$$

Example

Problem 7: A wheel with 8 equally spaced spokes and a radius of 30cm rotates with 2.5rev./s. You want to shoot a 20-cm long arrow straight through the wheel without hitting the spokes. Assume that the arrow and the spokes are very narrow. What is the minimum speed the arrow will have to have?

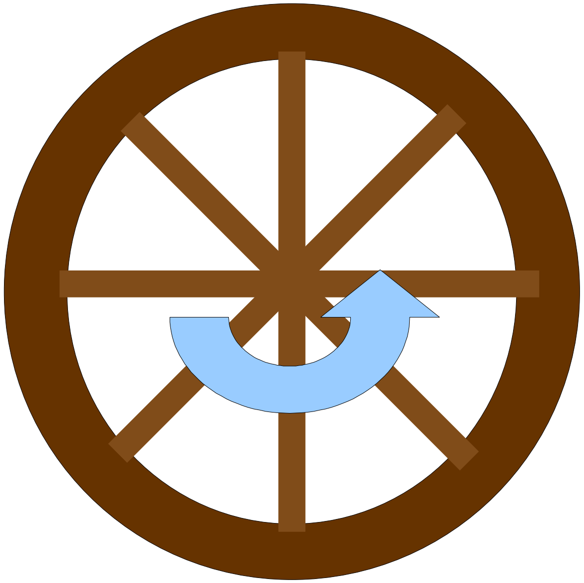


Plan:
Step 1: We need the time it takes each spoke to rotate from its original angle to the angle the next spoke is at the start time.
That is the maximum time the arrow has to squeeze through.

Step 2: The speed has to be high enough that the 20-cm arrow makes it through the wheel within that time.

Example

Problem 7: A wheel with 8 equally spaced spokes and a radius of 30cm rotates with 2.5rev./s. You want to shoot a 20-cm long arrow straight through the wheel without hitting the spokes. Assume that the arrow and the spokes are very narrow. What is the minimum speed the arrow will have to have?



Step 1:
The spokes split the wheel into angular segments of 1/8 revolution.
The time it takes the wheel to rotate one revolution:

$$T = (1/2.5\text{rev./s}) = 0.4\text{s}$$

The time it takes for 1/8 revolution:

$$\tau = T/8 = 50\text{ms}$$

Step 2: $v = 20\text{cm}/50\text{ms} = 4\text{m/s}$

Independent from the radial distance from the axle!

Are angular quantities vectors?

'Nes' or 'Yo'

We can define a vector for the quantities
• angular velocity
• angular acceleration

The direction is parallel to the axis of rotation
+ right hand rule

The magnitude is equal to the magnitude of the quantity.

All laws of vector manipulation we discussed in Chapter 3 apply (and we will use them later).

But the angular displacement is NOT.
Two rotations do not commute!

Are angular quantities vectors?

But the angular displacement is NOT.
Two rotations do not commute!

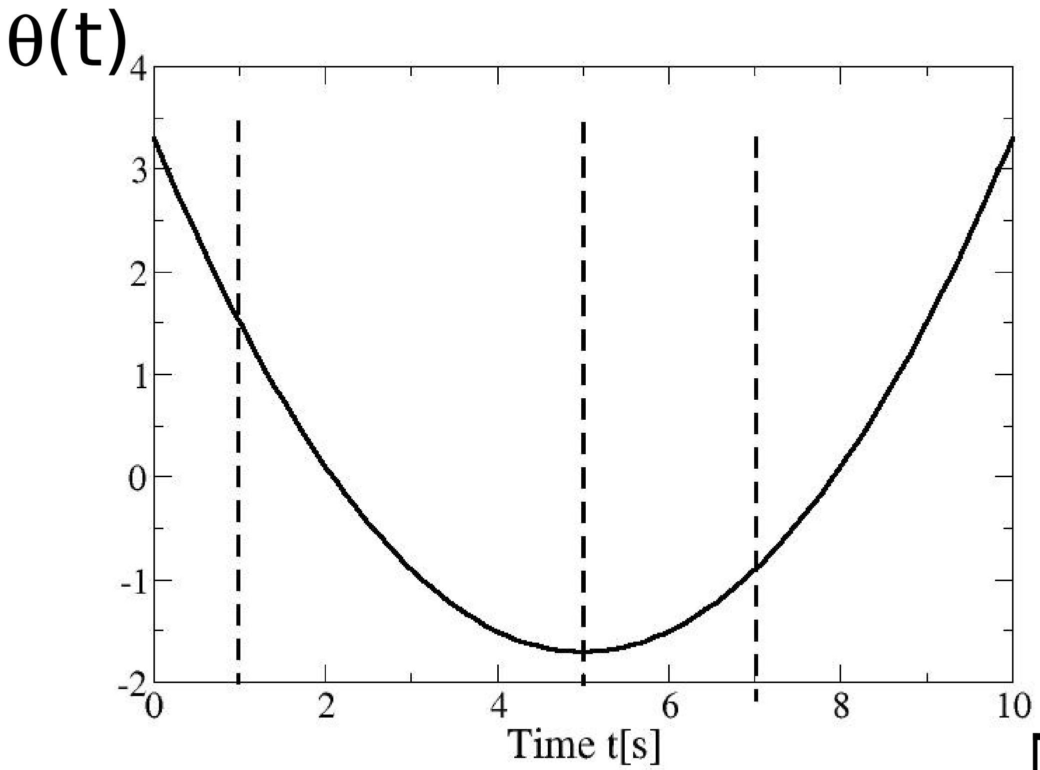
How can the angular velocity be a vector but the displacement not?
Both are related by differentiation.

$$\omega = d\theta/dt$$

The answer:

- Very small angular displacements commute!
- Angular velocities are proportional to very, very small angular displacements!

HITT 1



The plot shows the angular position of a reference point on a rotating disk as a function of time.

What are the signs of the angular velocities
at times t=1s, t=5s, t=7s?

Answer/time	1s	5s	7s
A	pos	neg	zero
B	neg	zero	pos
C	zero	pos	neg
D	neg	zero	neg
E	pos	zero	neg

Rotation

A few other useful definitions:

- Period T: Time for one complete revolution
- Period T = circumference / tangential velocity

$$T = 2\pi\, r/v_t$$

Note: v_t : average velocity during that revolution

Use $\omega = v_t/r$ → $T = 2\pi/\omega$

- Instead of T, we will occasionally use the Frequency: $f = 1/T$ Unit: Hz=1/s

- Note: $\omega = 2\pi f$ Unit of ω = rad/s

ω is sometimes called angular frequency

Rotation

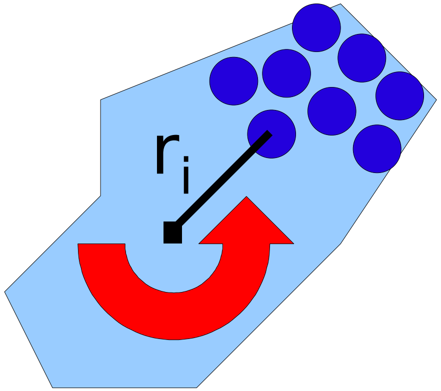
Comparison
linear motion vs angular motion
for constant acceleration

	Linear	Angular
Displacements:	$x-x_0$	$\theta-\theta_0$
Velocities:	v	ω
Accelerations:	a	α
Velocities:	$v = v_0+at$	$\omega = \omega_0+\alpha t$
Displacements:	$x-x_0 = v_0t+0.5at^2$	$\theta-\theta_0 = \omega_0 t+0.5\alpha t^2$
Velocities:	$v^2 = v_0^2+ 2a(x-x_0)$	$\omega^2 = \omega_0^2+ 2\alpha\,(\theta-\theta_0)$

Rotation

Kinetic energy of a rotating body:

- Lets split up the body into a collection of particles



Each particle has a certain mass m_i and a certain tangential velocity v_i

Each particle has a the a kinetic energy of
$K_i=0.5m_i\, v_i^2$

The kinetic energy of the entire body is the sum over all these energies $v_i=wr_i$

$$\mathbf{K = \Sigma K_i = \Sigma 0.5 m_i\, v_i^2 = 0.5\omega^2(\Sigma m_i\, r_i^2)}$$

Rotation

Kinetic energy of a rotating body:

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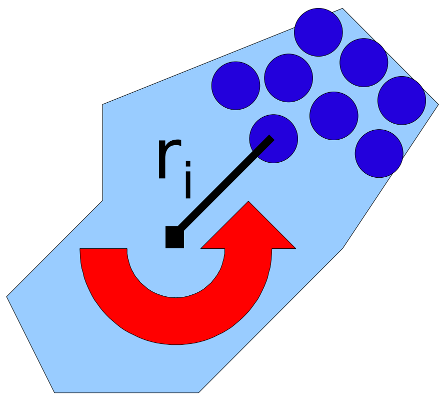
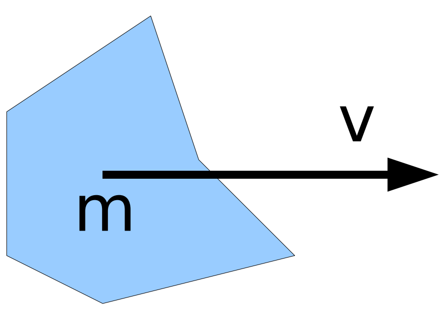


$K = 0.5\ (\Sigma m_i\ r_i^2)\ \omega^2$

Rotational Inertia:

$I = (\Sigma m_i\ r_i^2)$

$K = 0.5\ I\ \omega^2$



Compare with translation:

$K = 0.5\ m\ v^2$

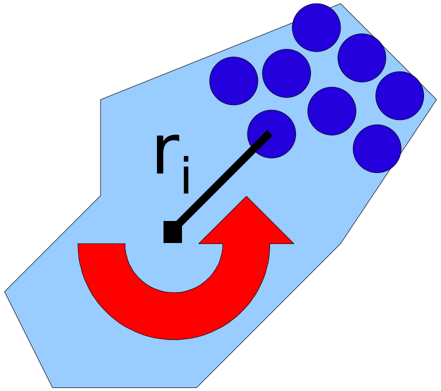
Correspondences:

- v <-> ω
- I <-> m

Rotation

Kinetic energy of a rotating body:

- Lets split up the body into a collection of particles



Rotational Inertia:

$$\mathbf{I = (\Sigma m_i\, r_i^2)}$$

- In a real body, the sum turns into an integral:

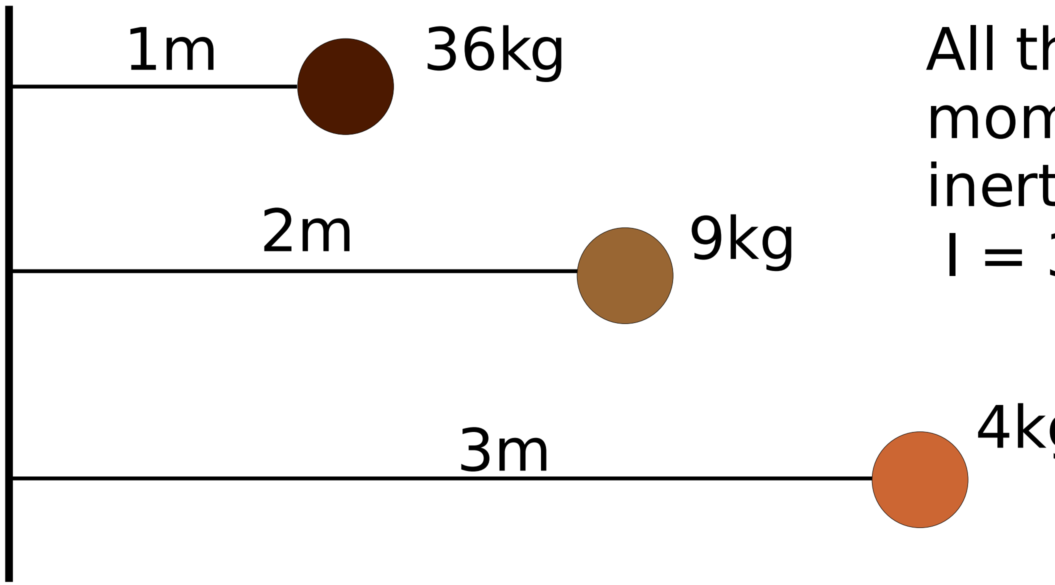
$$I = \int r^2 dm$$

Rotation

Moment of inertia of a rotating body:

$$I = \int r^2 dm$$

Examples:



All the same moment of inertia.
$I = 36\ kg\,m^2$

Apparently: Distance from the axis is more important than mass.

Rotation

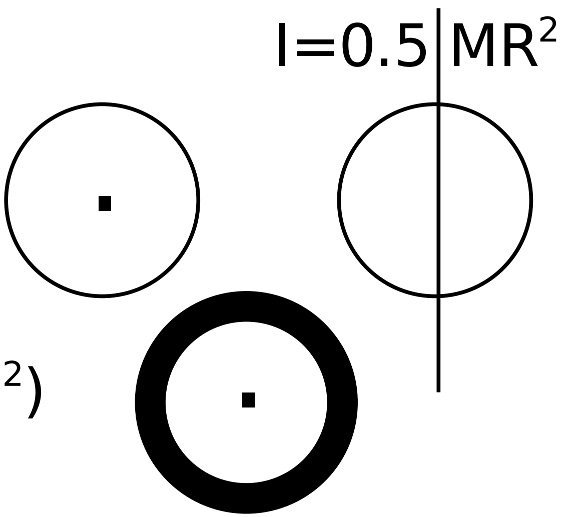
Moment of inertia of a rotating body:

$$I = \int r^2 dm$$

Table for rotational inertia of a number of different bodies is given in the book (Table 10-2, pg 253)

Examples (axis through center of mass):
Hoop around a central axis: $I=MR^2$
Hoop about any diameter: $I=0.5\ MR^2$

Ring around central axis: $I = 0.5\ M(R_1^{\ 2}+R_2^{\ 2})$



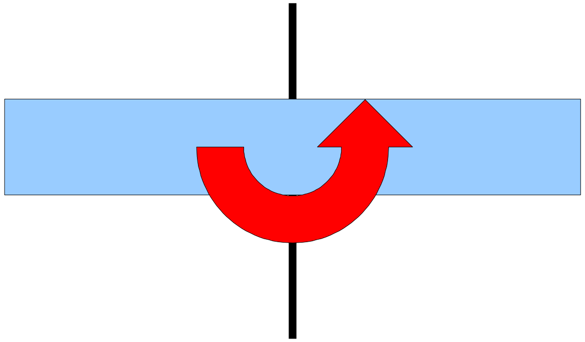
Solid cylinder: You should be able to guess that now!

Rotation

Moment of inertia of a rotating body:

$$I = \int r^2 dm$$

Examples (axis through center of mass):
Solid cylinder about central diameter: $I=0.25MR^2+ML^2/12$



Solid sphere: $I = 2MR^2/5$

Thin shell: $I=2MR^2/3$

Rotation

Moment of inertia of a rotating body:

$$I = \int r^2 dm$$

Usually reasonably easy to calculate when
- Body has symmetries
- Rotation axis goes through Center of mass

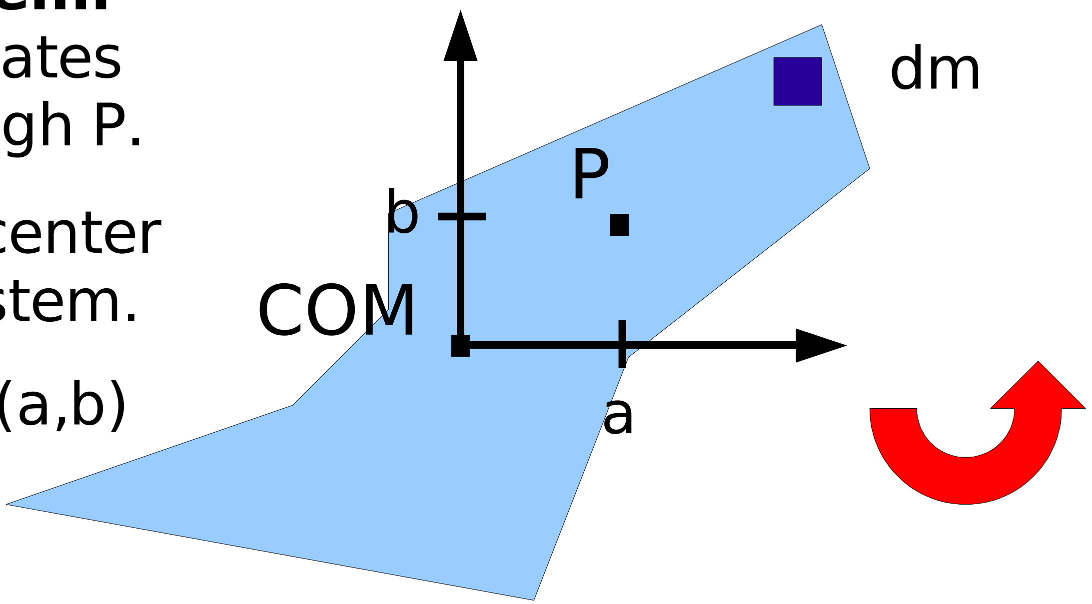
Exams: All moment of inertia will be given!
No need to copy the table from the book.

Rotation

Parallel axis theorem:
Assume the body rotates around an axis through P.

Let the COM be the center of our coordinate system.

P has the coordinates (a,b)



$$I = \int r^2 dm = \int (x-a)^2 + (y-b)^2 dm$$

$$= \int (x^2+y^2) dm - 2a\int x dm - 2b\int y dm + (a^2+b^2)\int dm$$

$$= I_{COM} \quad - \quad 0 \quad - \quad 0 \quad + \; h^2 M$$

$$= I_{COM} + h^2 M$$

This is something you might need

Rotation

Parallel axis theorem:
Assume the body rotates around an axis through P.

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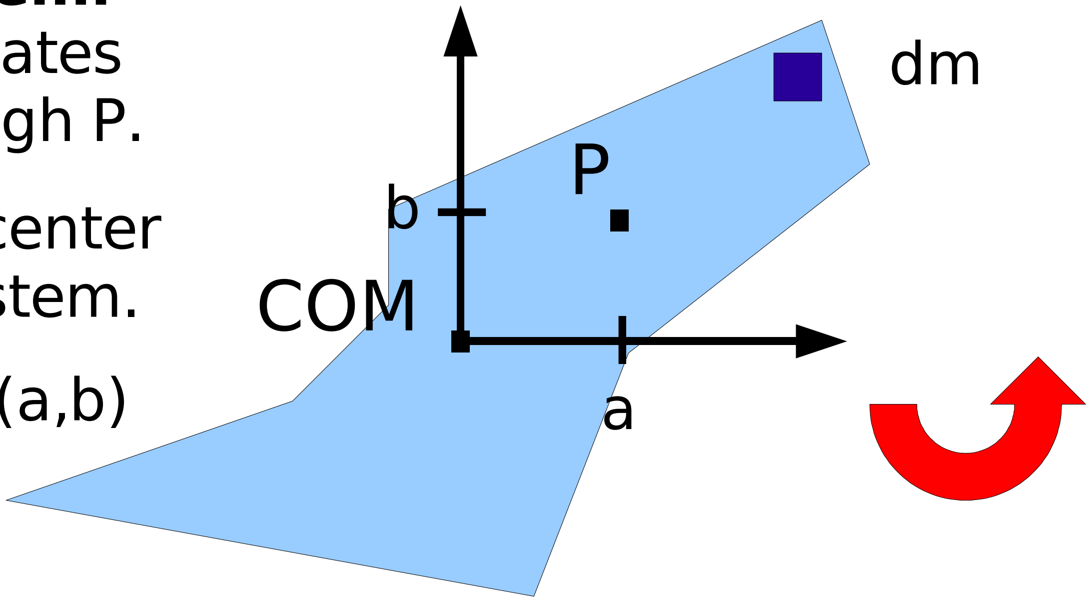

$I = I_{COM} + Mh^2$

The moment of inertia of a body rotating around an arbitrary axis is equal to the moment of inertia of a body rotating around a parallel axis through the center of mass plus the mass times the perpendicular distance between the axes squared.