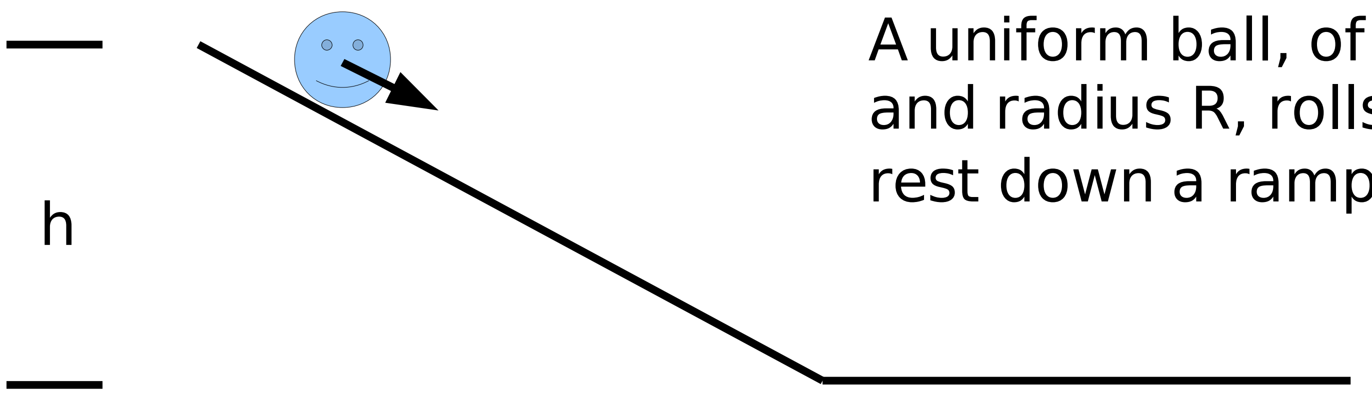
Example

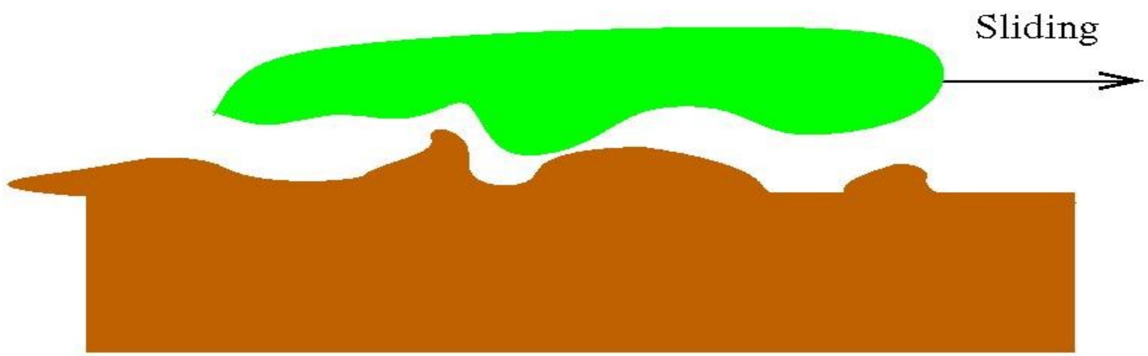


A uniform ball, of mass M = 6kg and radius R, rolls smoothly from rest down a ramp at angle $\theta=30^o$.

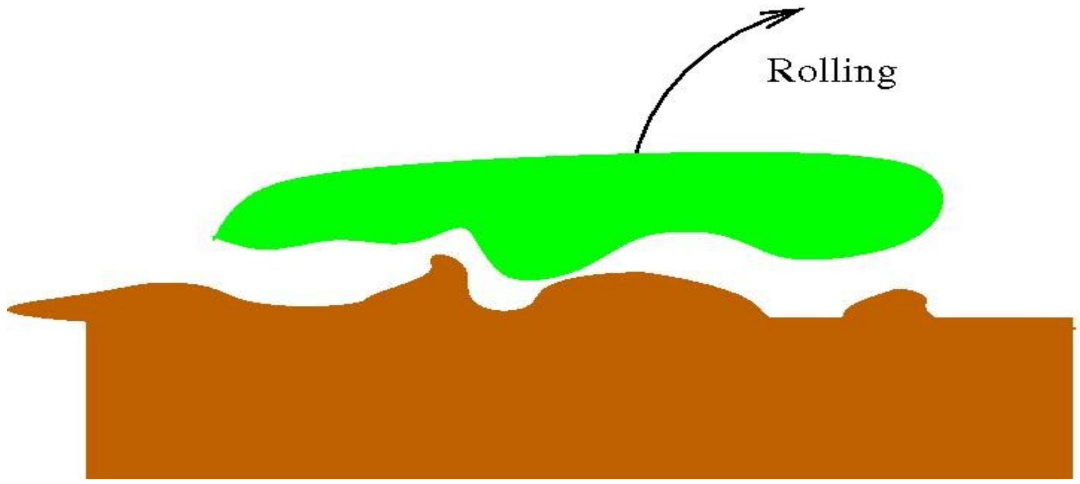
a) The ball descends a vertical height h=1.2m to reach the bottom of the ramp. What is the speed at the bottom?

The only force doing work is gravity, a conservative force.
The normal force is perpendicular to the path.
The frictional force does **not transfer** any energy to thermal because the ball does **not slide** (rolls smoothly).

Brief reminder about friction



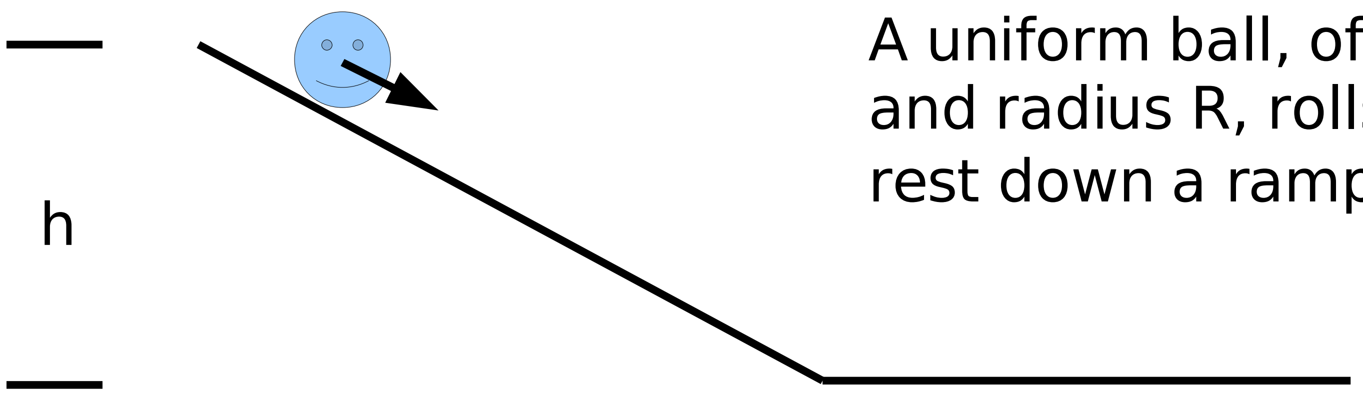
Sliding: Ripples in the surfaces get stuck and the force has to overcome this so the piece can move again -> generates heat



Smooth rolling: The 'Ripples' are moving in and out of each other without getting stuck -> Does not generate heat

Do you see the connection with static friction?

Example



A uniform ball, of mass M = 6kg and radius R, rolls smoothly from rest down a ramp at angle $\theta=30^o$.

a) The ball descends a vertical height h=1.2m to reach the bottom of the ramp. What is the speed at the bottom?

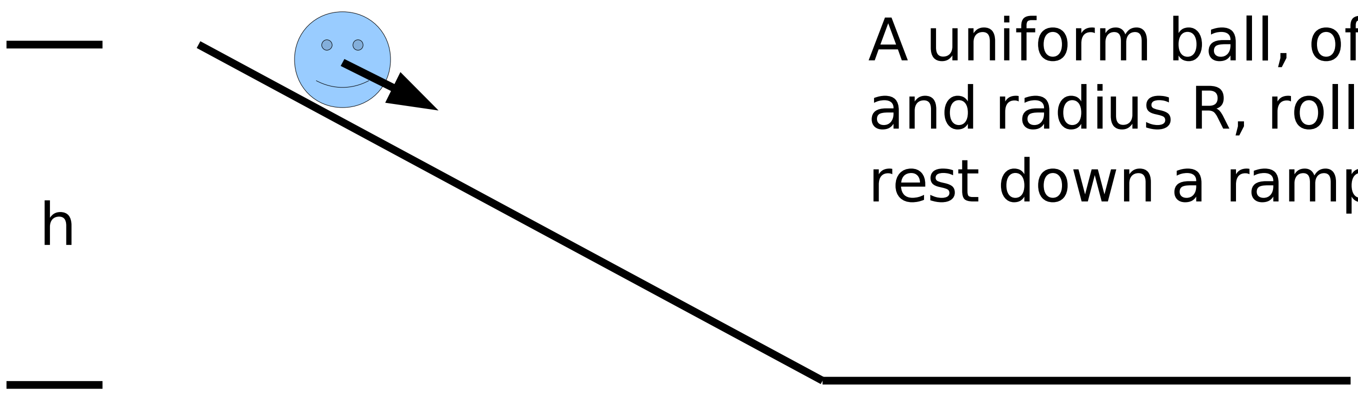
Energy is conserved:

initial: $U_i = mgh$, $K_i=0$ final: $U_f=0$, $K_f=0.5I_{com}\omega^2+ 0.5mv_{com}^2$

Use: $v_{com} = R\omega$ and $I_{com}= 2MR^2/5$ $v_{com} = (10gh/7)^{1/2}=4.1m/s$

$a_{com}= -g\sin\theta/(1+I_{com}/MR^2) = -g\sin\theta/(1+2/5) = -3.5m/s^2$

Example



A uniform ball, of mass M = 6kg and radius R, rolls smoothly from rest down a ramp at angle $\theta=30^o$.

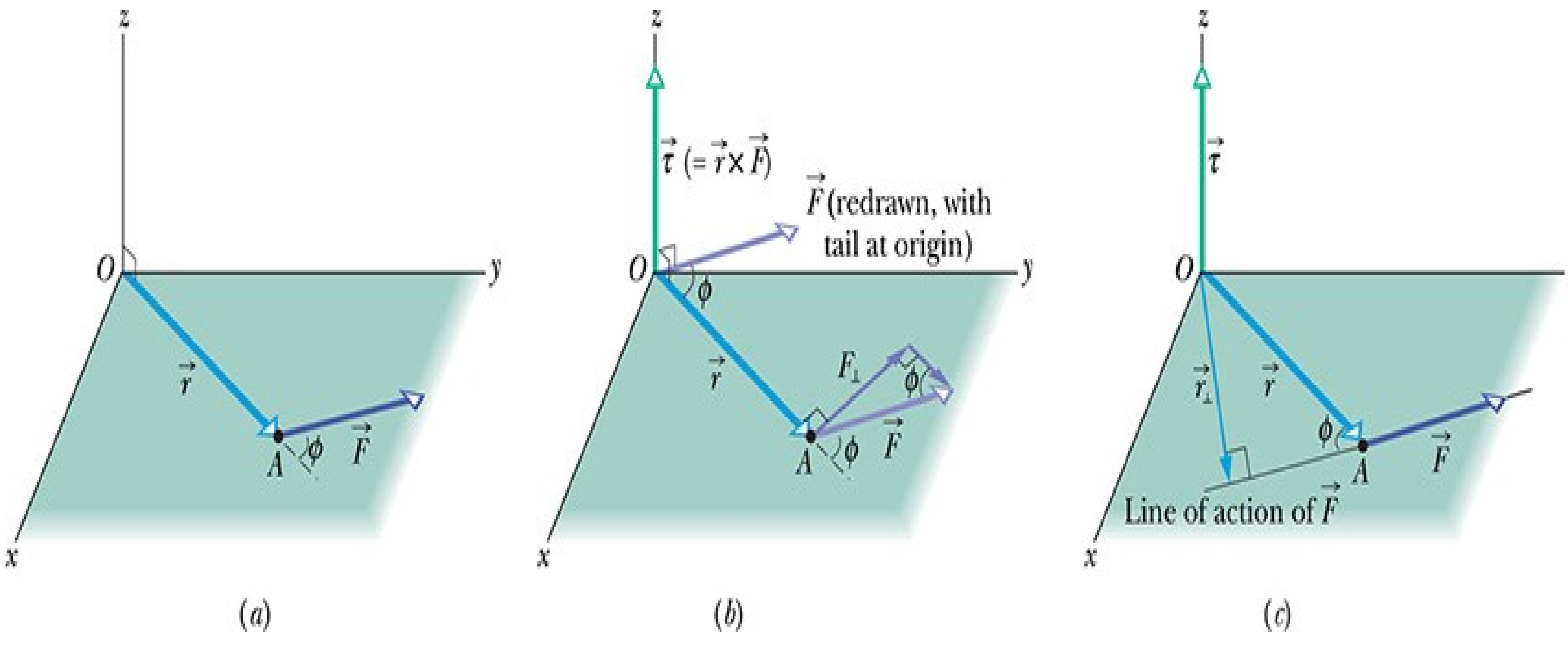
$$a_{com} = -g\sin\theta/(1+I_{com}/MR^2) = -g\sin\theta/(1+2/5) = -3.5 m/s^2$$

Always independent of mass and radius if I_{com} = Constant times MR^2

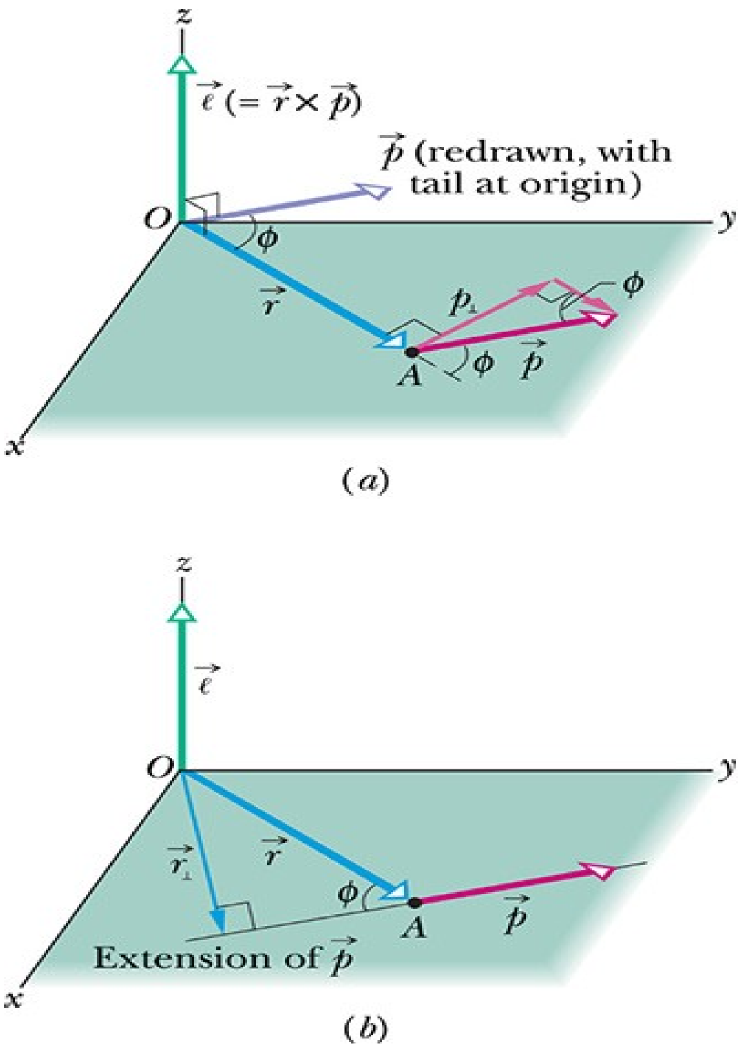
Torque Revisited

$\vec{\tau} = \vec{r} \times \vec{F}$

$\vec{\tau}$ starts at the origin and is perpendicular to the $\vec{r}$-$\vec{F}$ plane. (Right-hand rule)



Angular Momentum



Angular Momentum

The angular momentum for a single particle is:

$$\vec{l} = \vec{r} \times \vec{p} = m\,(\vec{r} \times \vec{v})$$

$$\frac{d}{dt}\vec{l} = \vec{\tau}_{net}$$

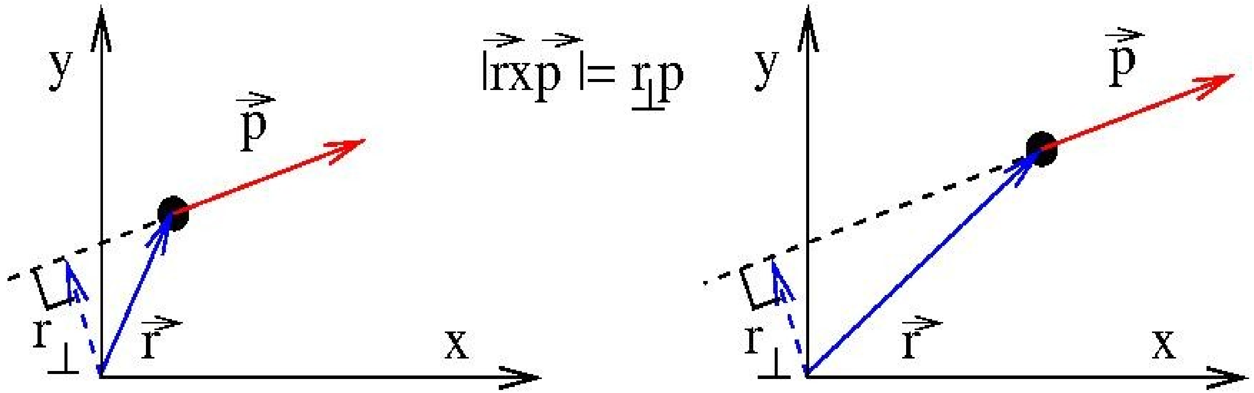
Angular Momentum

$$\frac{d}{dt}\vec{l} = \vec{\tau}_{net}$$

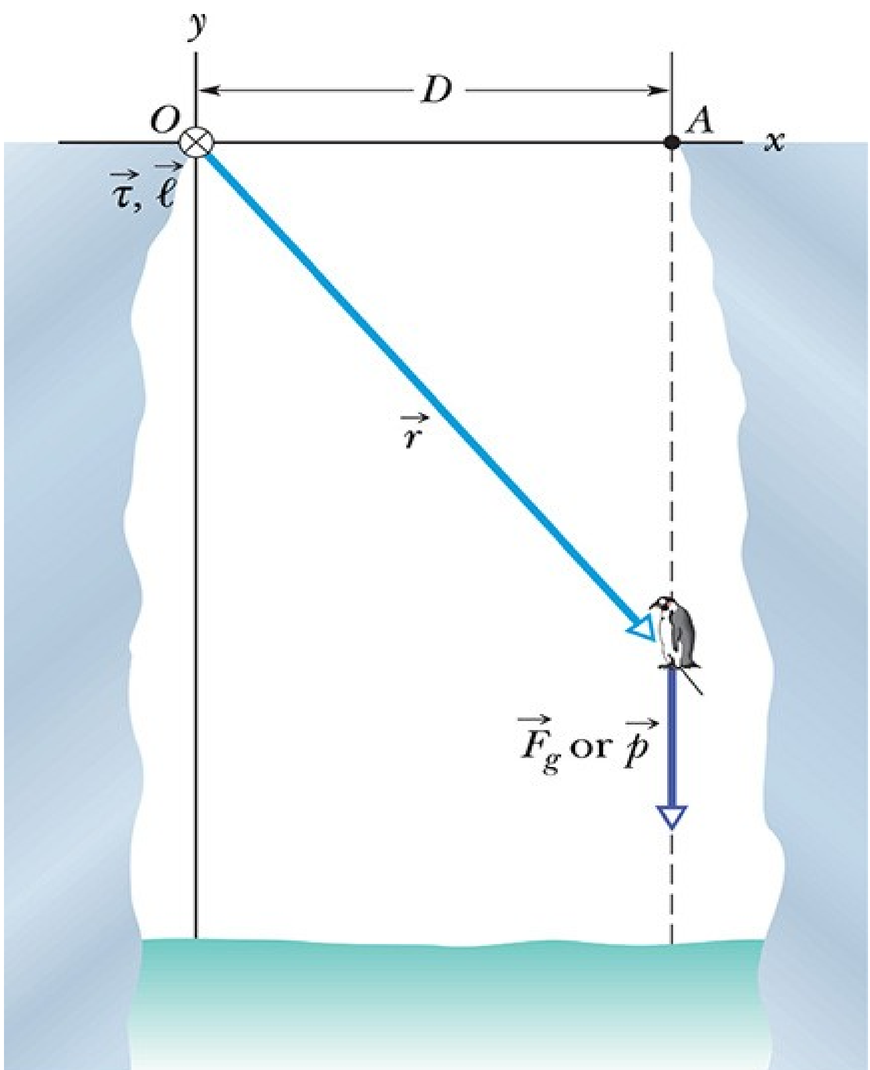
1. The vector sum of all torques acting on a particle is equal to the time rate of change of the angular momentum of that particle.

2. If the net torque is zero, the angular momentum is conserved. Not that this is a vector quantity and works for each component individually.

Example:



Angular Momentum

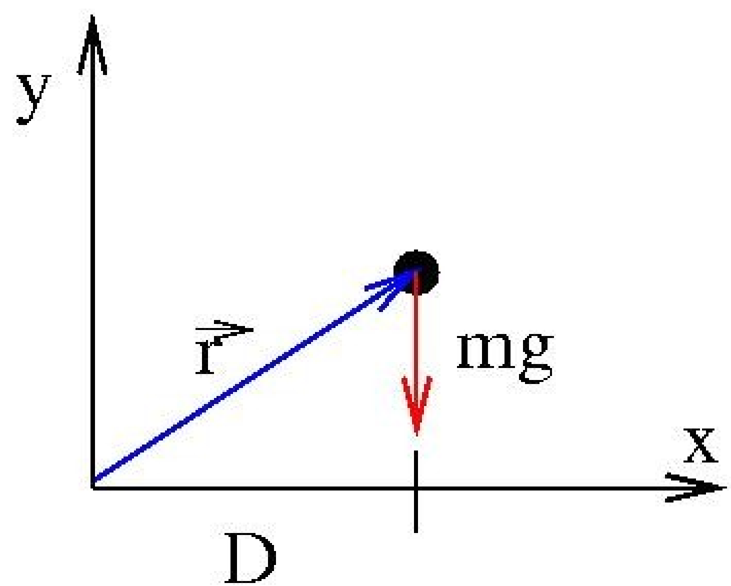


v=gt

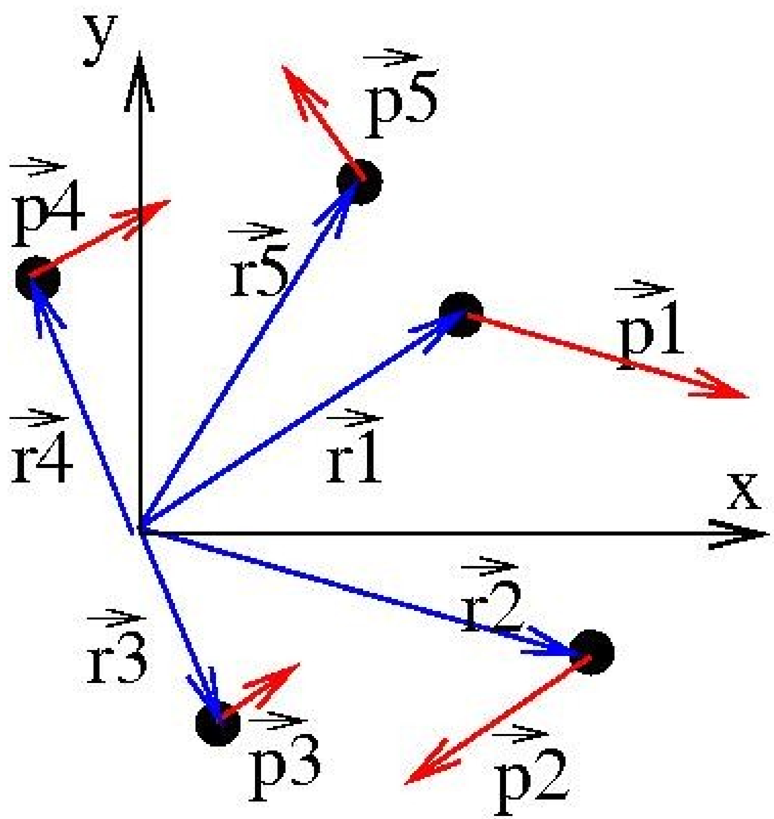
$l = r_{\perp} mv$ = Dmv = Dmgt

τ = Dmg

Direction: going 'clockwise'
-> negative z-direction



Angular Momentum of a system of particles



$$\vec{L} = \vec{l}_1 + \vec{l}_2 + \vec{l}_3 + \vec{l}_4 + \vec{l}_5 + \ldots$$

Time derivative

$$\frac{d}{dt} L = \frac{d}{dt} l_1 + \frac{d}{dt} l_2 + \ldots$$

$$\frac{d}{dt} L = \Sigma \tau_i$$

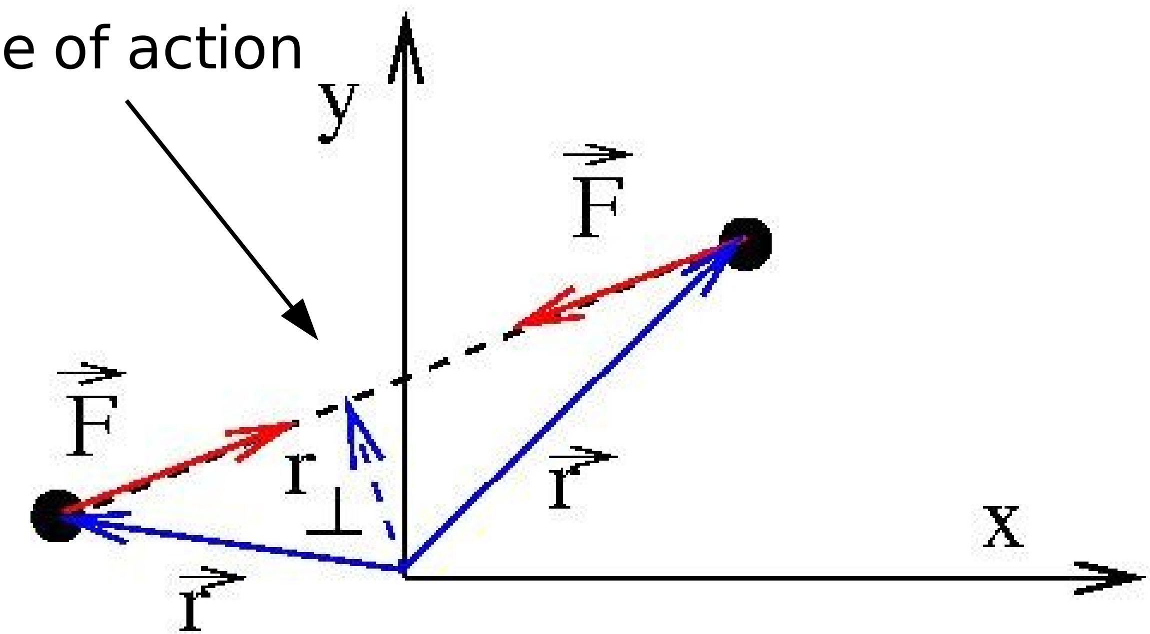
Is the sum over all torques.

Linear momentum: All internal forces compensate each other
Time derivative of linear momentum = sum over all net Forces
Can we ignore the internal torques here too?

Angular Momentum of a system of particles

Linear momentum: All internal forces compensate each other
Time derivative of linear momentum = sum over all net Forces
Can we ignore the internal torques here too?

Line of action



$\tau_{net} = \Sigma \, \tau_{ext}$

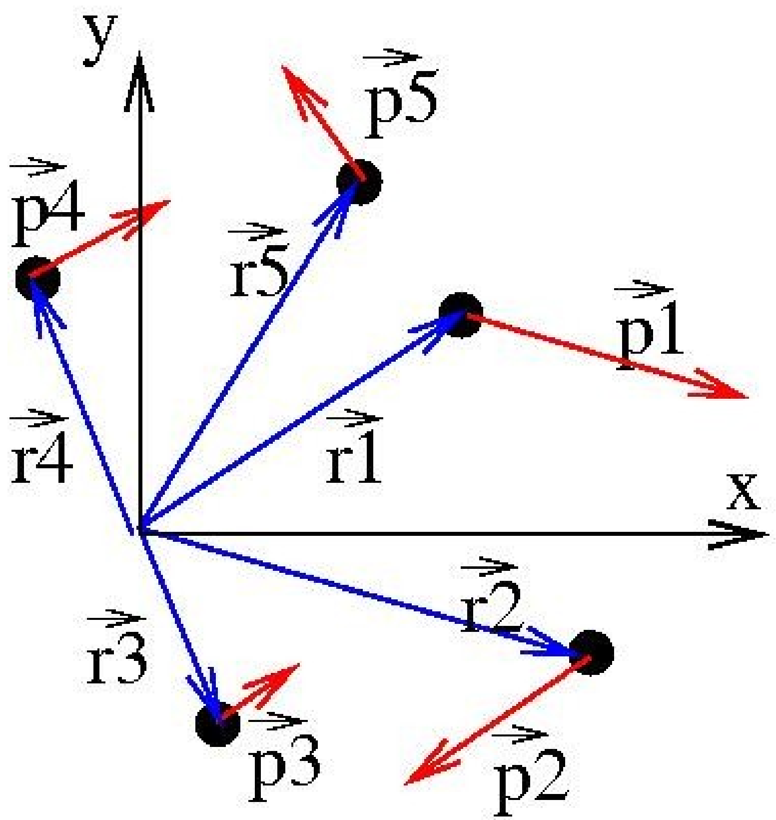
Actio = Reactio

Internal Forces
are equal and opposite
in sign.

Lever arm $r_\perp$ = equal

Torques are equal
and opposite in sign
(composite each other
when summed up)

Angular Momentum of a system of particles



$$\vec{L} = \vec{l}_1 + \vec{l}_2 + \vec{l}_3 + \vec{l}_4 + \vec{l}_5 + \ldots$$

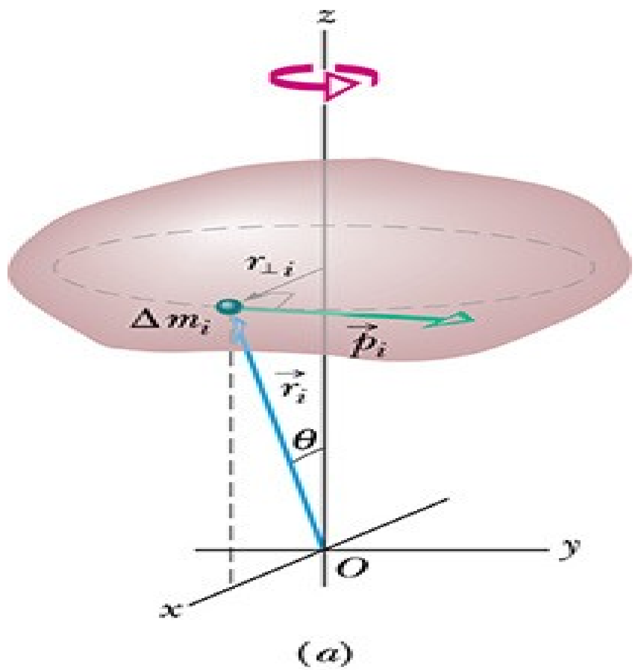
Time derivative

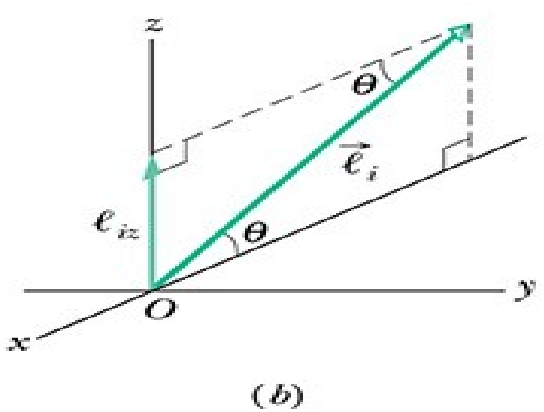
$$\frac{d}{dt} L = \frac{d}{dt} l_1 + \frac{d}{dt} l_2 + \ldots$$

$$\frac{d}{dt} L = \Sigma \tau_i = \tau_{net} = \Sigma \tau_{ext}$$

The net torque acting on a system of particles is equal to the time rate of change of the system's total angular momentum.

Angular Momentum of a Rigid body rotating around a fixed axis





Angular moment of mass element:

$$dl_i = r_i\, v_i\, dm_i = \omega\, r_i^2 dm$$

r_i is the distance to dm and is perpendicular to v_i

$$L = \int dl_i = \omega \int r_i^2 dm = I\omega$$

Rigid Body, Fixed Axis!

and obviously: $\frac{d}{dt} L = \tau_{net}$

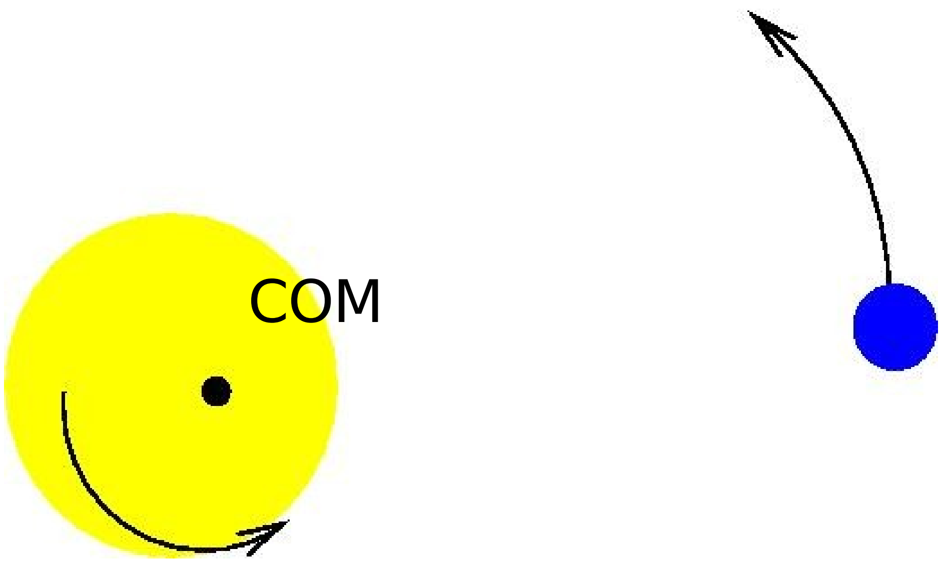
Conservation of Angular Momentum

If the **component** of the **net external torque** on a system along a certain axis **is zero**, then the **component** of the **angular momentum** along that axis is **conserved**.

- It is a conservation law for each individual component of the angular momentum, not the magnitude.
- Similar to the conservation law for the components of the linear momentum.

Conservation of Angular Momentum

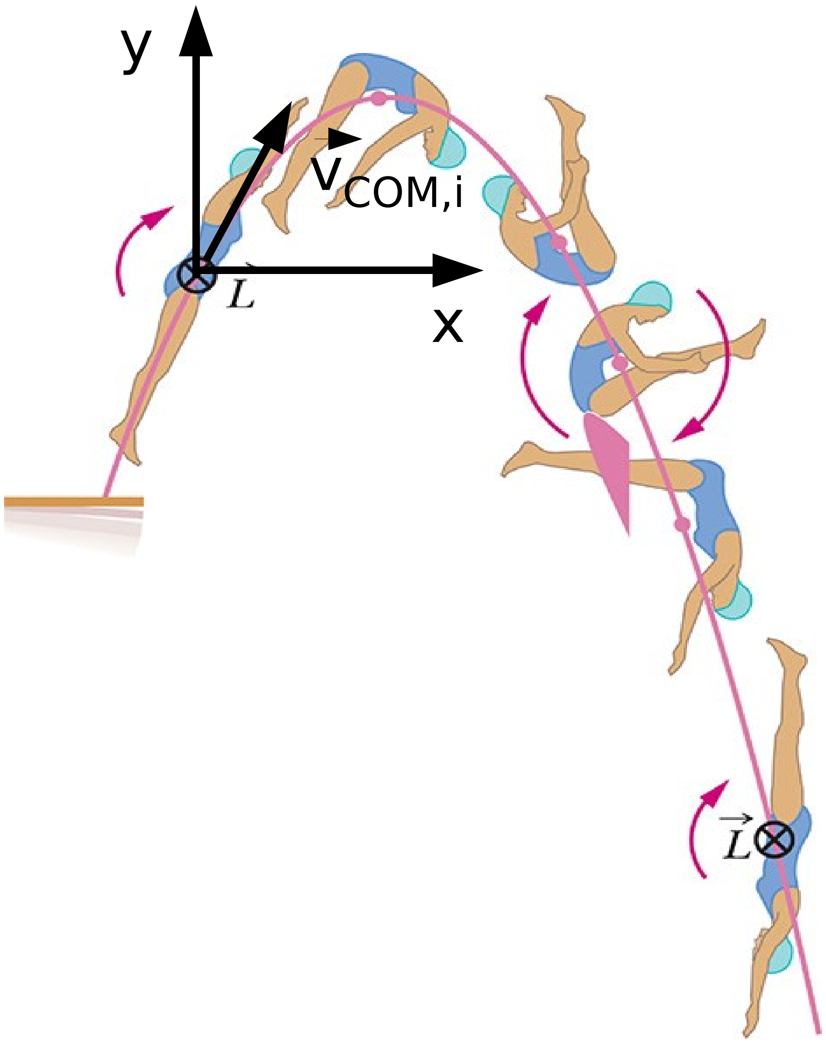
Example:



Sun-Earth system rotating around their Center of mass.

No torques:
Angular moment is conserved

Conservation of Angular Momentum



Motion can be split into:

- Center of mass motion
- Rotation around center of mass

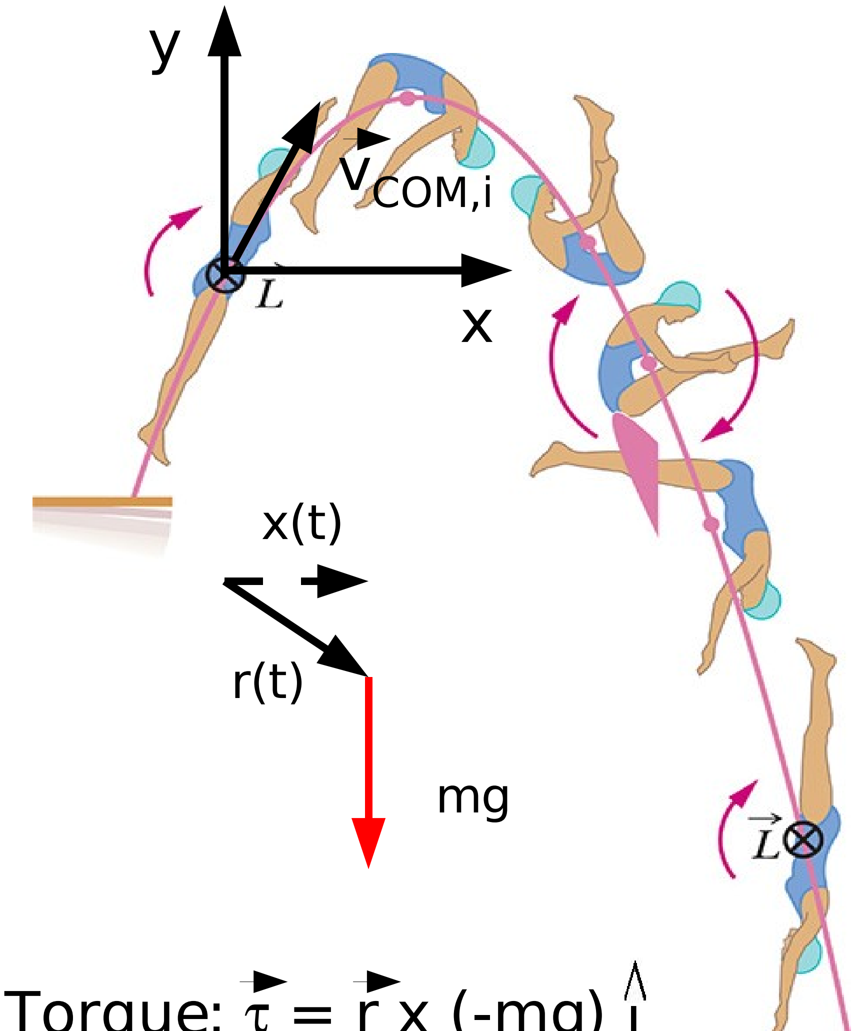
Initial conditions:

- Initial velocity $\vec{v}_{COM,i}$
- Initial rotation ω_i

Initial momenta:

- Linear momenta p_x, p_y
- Angular momenta (in z-axis):
  - Rotation around COM L_{int}
  - COM motion L_{ext}

Conservation of Angular Momentum



The center of mass travels along a parabola

$$x(t) = v_{ix} t$$
$$y(t) = v_{iy} t - 0.5 g t^2$$

$$\vec{v}(t) = v_{ix}\,\hat{i} + (v_{iy} - gt)\,\hat{j}$$

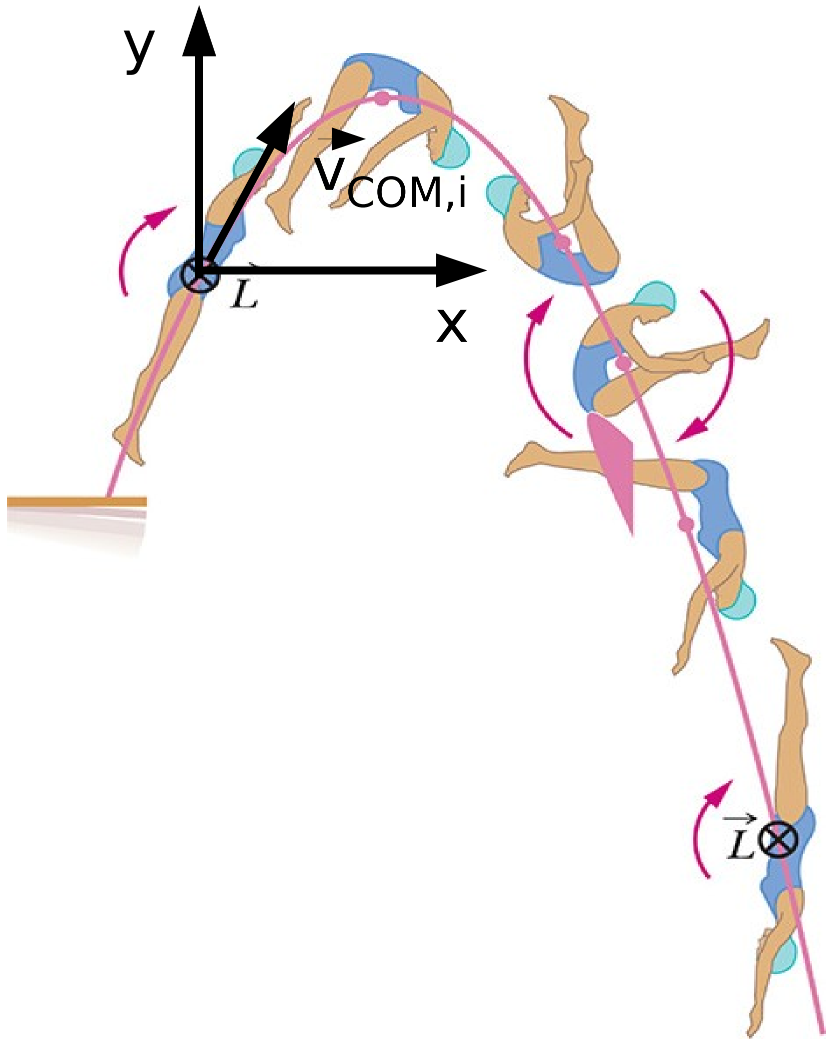
$$\vec{L}_{ext} = m(\vec{r} \times \vec{v}) \quad = -0.5 m g t^2 v_{ix} \hat{k}$$

Torque: $\vec{\tau} = \vec{r} \times (-mg)\,\hat{j}$

$$\tau = -\, mg\, x(t) \hat{k} = -\, m g v_{ix} t\, \hat{k} = \frac{d}{dt} \vec{L}_{ext}$$

The external force (gravity) changes the external angular momentum

Conservation of Angular Momentum



In addition: The jumper is rotating around its center of mass.
-> L_{int} (t=0) < 0 (clock wise)

Only force (gravity) works on center of mass.
-> No lever arm to change internal rotation

-> L_{int} conserved

-> Reduce I, ω has to go up.

Hitt

A beetle rides the rim of a small rotating disk.
If the beetle crawls towards the center of the disk, do the following decrease, increase, or stay the same:
rotational inertia, angular momentum, angular speed

A: same, increase, increase

B: decrease, same, increase

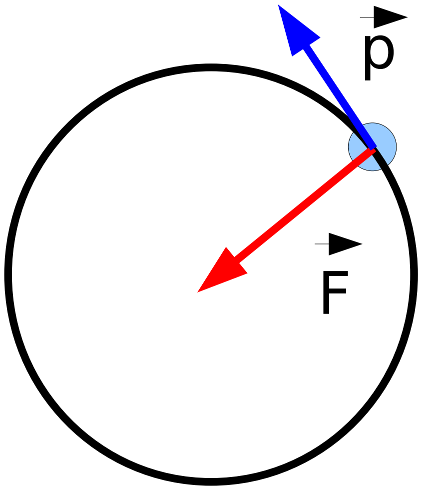
C: decrease, increase, same

D: same, decrease, increase

E: increase, same, decrease

Precession of a Gyroscope

Recall:



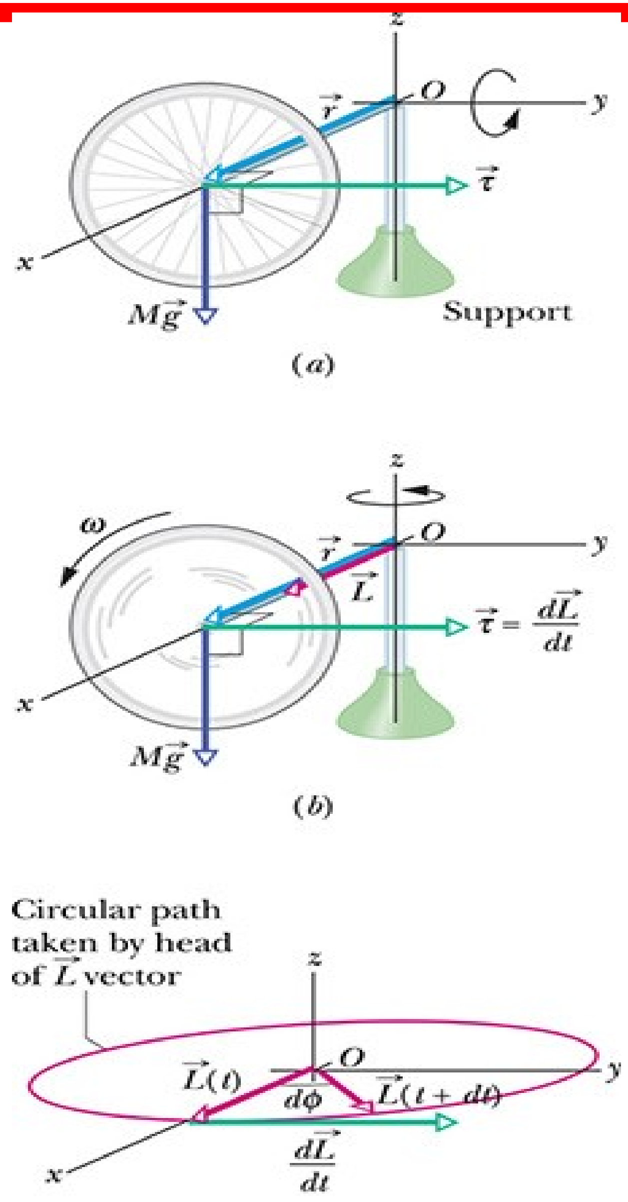
$$\frac{d}{dt}\vec{p} = \vec{F}_{net}$$

when $\vec{F}$ is orthogonal to $\vec{p}$
it only changes the direction
of p and not it's magnitude

$$\frac{d}{dt}\vec{L} = \vec{\tau}_{net}$$

Something similar should happen
when $\vec{\tau}_{net}$ is perpendicular to $\vec{L}$

Precession of a Gyroscope

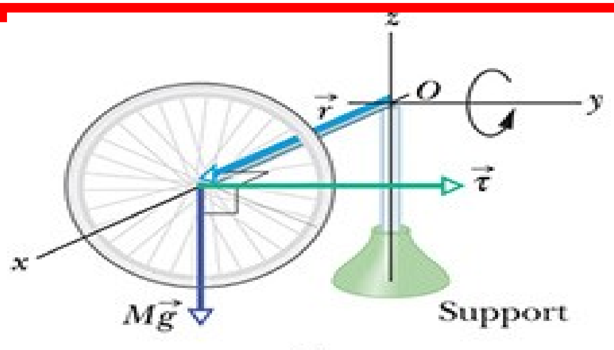


$$\frac{d}{dt}\vec{L} = \vec{\tau}_{net}$$

Spinning wheel supported at the end of a stick

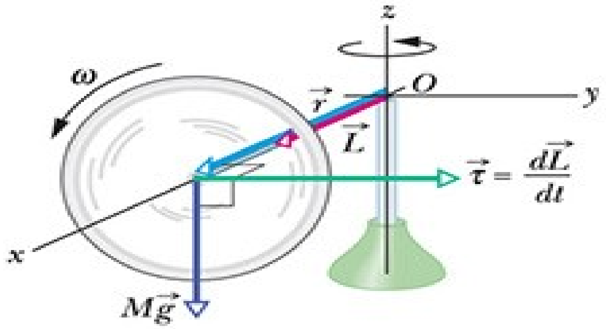
Gravity points down
Torque is going to the side perpendicular to the angular momentum
 -> Precession

Precession of a Gyroscope



(a)

$$\frac{d}{dt}\vec{L} = \vec{\tau}_{net}$$

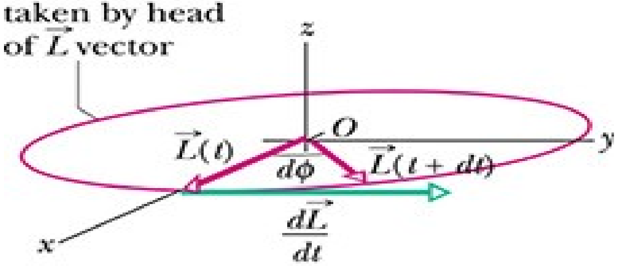


(b)

$$\tau = Mgr$$

$$dL = \tau \, dt = L d\phi$$

$$d\phi = dL/L = Mgr \, dt \, / \, I\omega$$



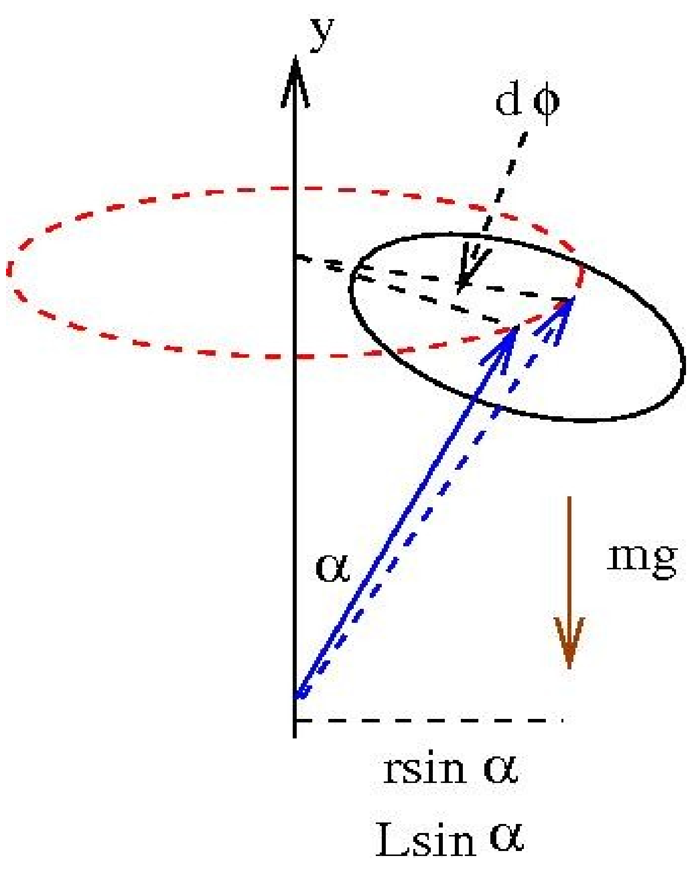
(c)

$$\frac{d}{dt}\phi = \Omega = \frac{Mgr}{I\omega}$$

precession rate

Precession of a Gyroscope

What happens if $\vec{L}$ is not horizontal?



The lever arm is now only r sinα :

$\tau = Mg\ r \sin\alpha$

But the change dL is also only scaling with L sinα :

$dL = \tau\ dt = L\sin\alpha\ d\phi$

$$\frac{d}{dt}\phi = \Omega = \frac{Mgr \sin\alpha}{I\omega \sin\alpha}$$

precession rate

Independent of α

Example

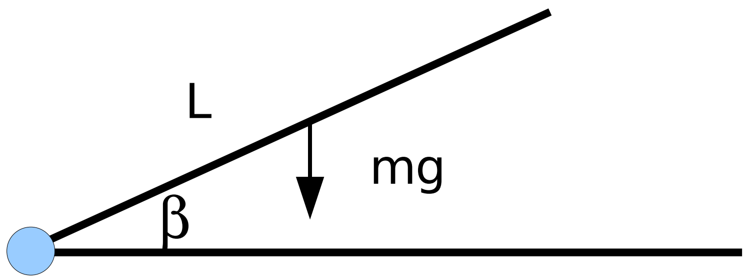
Problem 68: A top spins at 30rev/s about an axis that makes an angle of 30 deg with the vertical. The mass of the top is 0.5kg, the rotational inertia is 0.0005kgm2, and its center of mass is 4cm from the pivot point.
If the spin is clockwise from an overhead view, what are the
a) precession rate?
b) direction of the precession as viewed from oberhead?

$$\Omega = \frac{Mgr}{I\omega}$$

$$\omega = (2\pi)30 \text{ rad/s}$$

Source: www.smilemakers.com

Example Problem



Two boards are connected such that they can rotate around each a common pivot point.

One board lies flat on the ground. What is the acceleration of the end of the second board as a function of β ?

$$\tau = 0.5Lmg \cos\beta$$

$$\tau = I\alpha \qquad I = I_{com} + ML^2/4 = ML^2/3$$

$$\alpha = 3MgL\cos\beta \,/\, 2ML^2 = 1.5\, g \cos\beta \,/\, L$$

$$a = \alpha L = 1.5\, g \cos\beta$$ Larger than g for small β