Simple Harmonic Motion

Simple harmonic motion requires a force or torque proportional to a coordinate:

$F = -kx = ma$ $\tau = -\kappa\theta = I\alpha$

with

$\omega = (k/m)^{1/2}$ $\omega = (\kappa/I)^{1/2}$

Solutions:

$x(t) = x_m \cos(\omega t + \phi_0)$

$v(t) = -\omega x_m \sin(\omega t + \phi_0)$

$a(t) = -\omega^2 x_m \cos(\omega t + \phi_0)$

$\theta(t) = \theta_m \cos(\omega t + \phi_0)$

$\Omega(t) = -\omega\theta_m \sin(\omega t + \phi_0)$

$\alpha(t) = -\omega^2\theta_m \cos(\omega t + \phi_0)$

Such forces have these potentials:

$U = 0.5kx^2$ $U = 0.5\kappa\theta^2$

$F = -kx = ma$ $\tau = -\kappa\theta = I\alpha$

How to solve:

- **Identify all forces**

  -- Add them up to a net force

  -- This net Force will act on the center of mass

  -- Is this force proportional to $-(x-x_0)$?

  (Does the force pull/push the mass always back to a stable equilibrium point at $x=x_0$?)

--> SHM in one linear dimension (here along the x-axis)

- **Identify the axis of rotation**
- **Identify all torques**

  -- Add them up to a net torque

  -- Is this net torque proportional to $-(\theta-\theta_0)$?

  (Does the torque rotate the mass always back to a stable orientation, stable equilibrium, at $\theta=\theta_0$?)

--> SHM in one rotational dimension

Damped Simple Harmonic Motion

Approximations used up 'til now:

-- small angles

-- linear force response (~x,θ)

-- no friction or damping

-- no external force

Now with friction or damping:

-- Often a very complicated process

-- Useful approximation: ~ velocity (or angular velocity)

$$F_d = -bv$$

Damped Simple Harmonic Motion

Forces: $F = -bv - kx = ma$

$$\Rightarrow$$

Equation of motion: $m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$

Solution:

$$x(t) = x_m e^{-\frac{bt}{2m}} \cos(\omega' t + \phi) \qquad \omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

Exponentially damped simple harmonic oscillator

If you are interested in the math: ...

Differential Equation:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = 0$$

Ansatz:

$$x(t) = \Re\left\{x_m e^{-i\beta t}\right\}$$

Goal is to find the right β which solves the differential equation:

$$\frac{dx}{dt} = -i\beta x(t) \qquad \text{and} \qquad \frac{d^2x}{dt^2} = -\beta^2 x(t)$$

inserted in differential equation and divide by x(t):

$$-m\beta^2 - i\beta b + k = 0 \qquad \Rightarrow \qquad \beta^2 + i\frac{\beta b}{m} - \frac{k}{m} = 0$$

Quadratic equation:

$$\beta = -i\frac{b}{2m} \pm \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

provides a solution:

$$x(t) = x_m e^{-\frac{b}{2m}t} e^{\pm i\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}t}$$

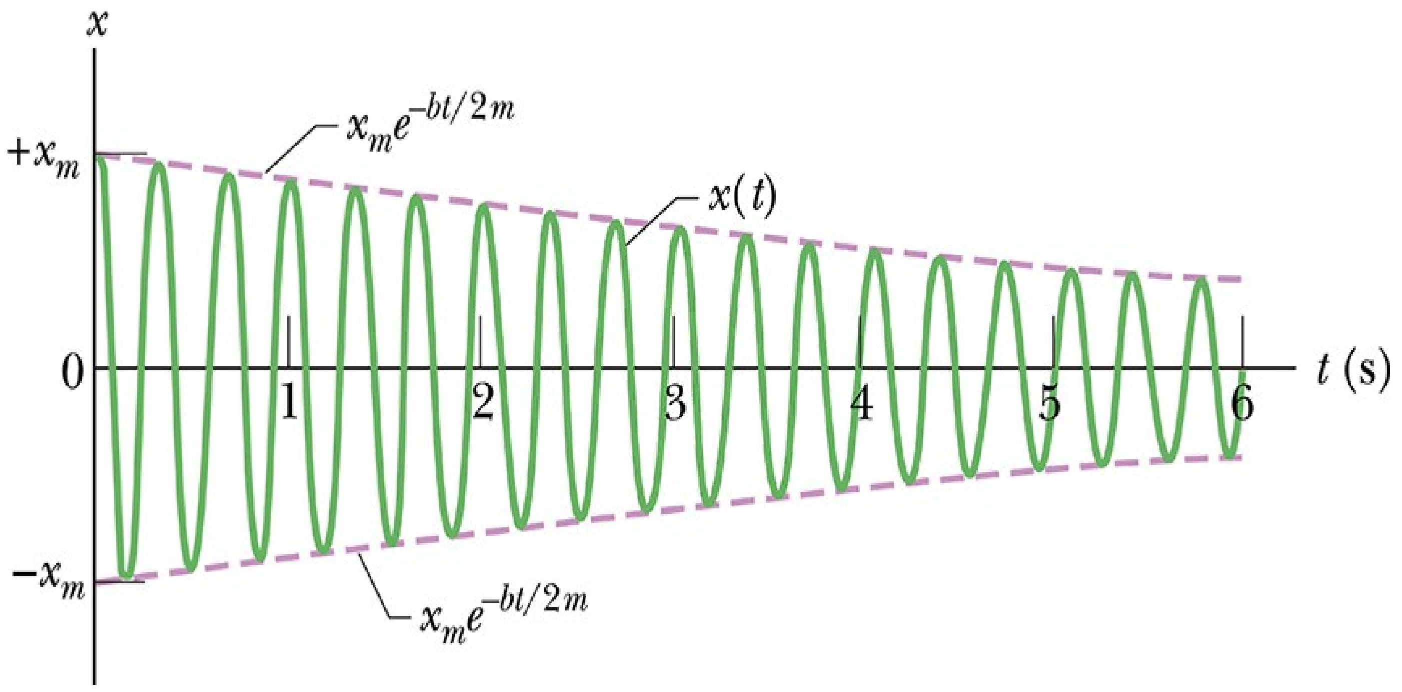
Now: Physics is real, so we only take the real part

$$x(t) = x_m e^{-\frac{bt}{2m}} \cos(\omega' t) \qquad \omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

and both (±) solutions are actually identical.

Damped Simple Harmonic Motion

$$x(t) = x_m e^{-\frac{bt}{2m}} \cos(\omega' t + \phi) \qquad \omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$



- Exponentially decreasing envelope of harmonic motion
- Shift in frequency

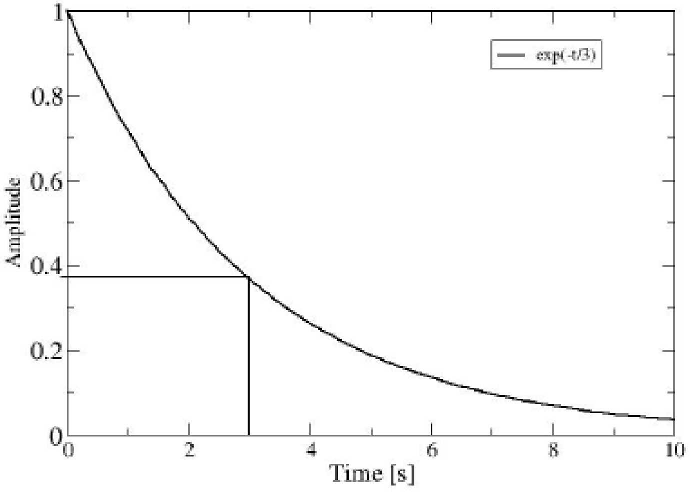
Damped Simple Harmonic Motion

For large damping

$$\frac{b^2}{4m^2} > \frac{k}{m} \qquad \Rightarrow \qquad \omega' = i\sqrt{\frac{b^2}{4m^2} - \frac{k}{m}}$$

Only exponentially damped, no oscillation

$$x(t) = x_m e^{-\left(\frac{b}{2m} + \sqrt{\frac{b^2}{4m^2} - \frac{k}{m}}\right)t} = x_m e^{-\alpha t}$$



~exp(-t/3s)

damped to 1/e=0.368 of the initial amplitude after t=3s

Damped Simple Harmonic Motion

$$x(t) = \Re\left\{x_m e^{-i\beta t}\right\} = x_m e^{-\frac{b}{2m}t}\cos\omega' t \qquad \beta = -i\frac{b}{2m} \pm \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

$$v(t) = \Re\left\{-i\beta x_m e^{-i\beta t}\right\} = -\frac{b}{2m}x_m e^{-\frac{b}{2m}t}\cos(\omega' t) - \omega' x_m e^{-\frac{b}{2m}t}\sin(\omega' t)$$

$$a(t) = \Re\left\{-\beta^2 x_m e^{-i\beta t}\right\} = -x_m\left[\omega'^2 + \left(\frac{b}{2m}\right)^2\right]e^{-\frac{b}{2m}t}\cos\omega' t - x_m\omega'\frac{b}{2m}e^{-\frac{b}{2m}t}\sin\omega' t$$

Significant Damping changes the phase relation between displacement, velocity, and acceleration.

For

$$\frac{b^2}{4m^2} \ll \frac{k}{m} \qquad \text{low damping} \qquad \omega = \sqrt{\frac{k}{m}} \approx \omega'$$

$$v(t) \approx -\omega x_m e^{-\frac{b}{2m}t}\sin(\omega t) \qquad a(t) \approx -x_m\omega^2 e^{-\frac{b}{2m}t}\cos\omega t$$

Damped Simple Harmonic Motion

Energy in damped simple harmonic motion (small damping only):

$$U(t) = \frac{1}{2}kx^2(t) = \frac{1}{2}kx_m^2 e^{-\frac{b}{m}t}\cos^2\omega t$$

$$K(t) = \frac{1}{2}mv^2(t) = \frac{1}{2}m\omega^2 x_m^2 e^{-\frac{b}{m}t}\sin^2\omega t$$

$$E(t) = K(t) + U(t) = \frac{1}{2}kx_m^2 e^{-\frac{b}{m}t} \qquad \text{exponentially damped}$$

Equations become a little more complicated when we have medium damping. That's where computers/theorists are for, not in this class.

Damped Simple Harmonic Motion

Approximations used up 'til now:

-- small angles

-- linear force response ($\sim x, \theta$)

-- no friction or damping

-- no external force

Forced oscillations with friction or damping:

-- Apply an oscillating force to the oscillator

-- Frequency of oscillating force is usually not identical to resonance frequency (eigenfrequency) of oscillator

Forced Oscillations

Forced oscillations are everywhere
in physics and engineering:

- Person on a swing
- Car suspensions driven by bumps in the road
- Buildings shaken by an earthquake
- Bridge shaken by wind (Tacoma bridge)
  or by marching soldiers
- Electrical circuit driven by an oscillating voltage
- Atomic resonances driven by electro-magnetic waves
- ...

Forced Oscillations

$$m\frac{d^2x}{dt^2}+b\frac{dx}{dt}+kx=F_0\sin\omega_d t$$

ω_d : angular frequency of driving force

Solution:

$$x(t)=\frac{F_m}{m\sqrt{\left(\frac{b}{m}\omega_d\right)^2+\left(\omega_d^2-\omega_0^2\right)^2}}\sin(\omega_d t+\phi_0)$$

Very large if damping is small and $\omega_d=\omega_0=\sqrt{k/m}$!

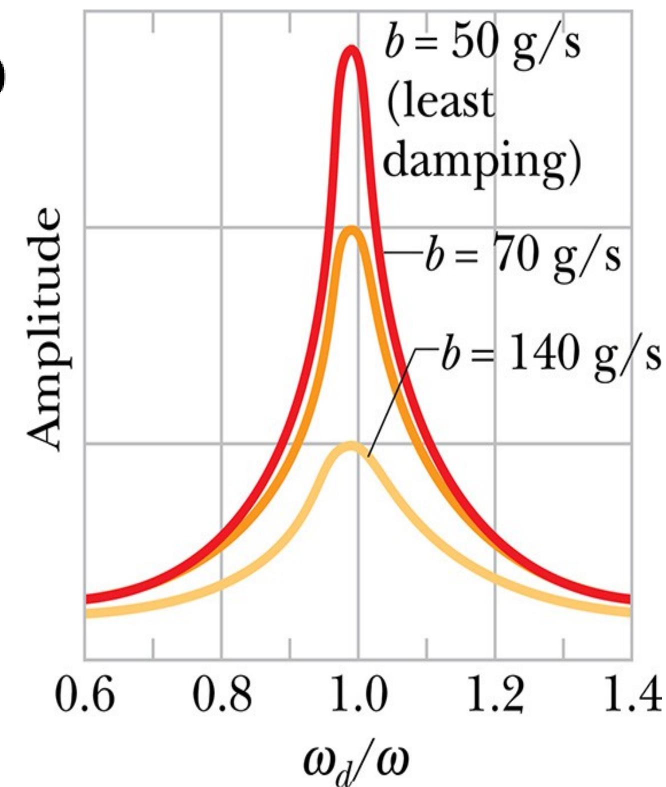
Forced Oscillations

$$x(t) = x_m \sin(\omega_d t + \phi_0) \qquad x_m = \text{Amplitude}$$

$$x_m = \frac{F_m}{m\sqrt{\left(\frac{b}{m}\omega_d\right)^2 + \left(\omega_d^2 - \omega_0^2\right)^2}}$$

Very large if damping is small and $\omega_d = \omega_0$

This is when bridges fail, buildings collapse,
lasers oscillate, microwaves cook food, swings swing,
radio stations transmit, radios receive, ...



Forced Oscillations

$$x_m = \frac{F_m}{m\sqrt{\left(\frac{b}{m}\omega_d\right)^2 + \left(\omega_d^2 - \omega_0^2\right)^2}}$$

For small damping:

small frequencies: $\omega_d \ll \omega_0$ $\quad x_m \approx \frac{F_m}{m}\frac{1}{\omega_0^2}$ Independent of driving frequency

(In phase with force)

On resonance: $\omega_d \approx \omega_0$ $\quad x_m \approx \frac{F_m}{b\omega_d}$ only limited by damping

(90deg out of phase with force)

large freqencies: $\omega_d \gg \omega$ $\quad x_m \approx \frac{F_m}{m}\frac{1}{\omega_d^2}$ proportional to $1/f_d^2$

(180deg out of phase with force)

large frequencies: Mechanical low pass filter.

Forced Oscillations

Where is this important? Everywhere!

Any mechanical system (such as cars, trains, airplanes, space shuttles, your body, bikes, buildings, ...) are always exposed to many forces.
These forces can be described as a combination of oscillating forces:

$$F(\omega) = \sqrt{\frac{1}{2\pi}} \int F(t) e^{-i\omega t} dt$$

$$x(t) = \sqrt{\frac{1}{2\pi}} \int \frac{F(\omega) e^{i\omega t} \; e^{i\phi(\omega)}}{m\sqrt{\left(\frac{b}{m}\omega\right)^2 + \left(\omega^2 - \omega_0^2\right)^2}} d\omega$$

The resulting motion is large and can result in damage if $F(\omega_0)$ is large

HITT

An ideal pendulum of length L is excited by an external force which moves the suspension point with a frequency far below (above) the resonance. If we now increase the frequency by a factor of two, by what factor does the resulting amplitude change? First number for below, second number for above.

$x_m(2\omega \ll \omega_0) = C_1\, x_m(\omega \ll \omega_0)$

$x_m(2\omega \gg \omega_0) = C_2\, x_m(\omega \gg \omega_0)$

A: $C_1=0.5$, $C_2=0.5$

B: $C_1=1$, $C_2=2$

C: $C_1=0.5$, $C_2=4$

D: $C_1=1$, $C_2=4$

E: $C_1=0.5$, $C_2=2$

HITT

An ideal pendulum of length L is excited by an external force which moves the suspension point with a frequency far below (above) the resonance. If we now increase the frequency by a factor of three, by what factor does the resulting amplitude change? First number for below, second number for above.

$x_m(3\omega << \omega_0) = C_1\, x_m(\omega << \omega_0)$ $x_m(3\omega >> \omega_0) = C_2\, x_m(\omega >> \omega_0)$

A: $C_1=1/3$, $C_2=1/3$

B: $C_1=1$, $C_2=9$

C: $C_1=1/3$, $C_2=3$

D: $C_1=1$, $C_2=1$

E: $C_1=1/9$, $C_2=9$