Damped Simple Harmonic Motion

\[ x(t) = x_m e^{-\frac{bt}{2m}} \cos(\omega't + \phi) \]

\[ \omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}} \]

- Exponentially decreasing envelope of harmonic motion
- Shift in frequency
Forced Oscillations

\[ x(t) = x_m \sin(\omega_d t + \phi_0) \quad x_m = \text{Amplitude} \]

\[ x_m = \frac{F_m}{m \sqrt{\left(\frac{b}{m} \omega_d\right)^2 + \left(\omega_d^2 - \omega_0^2\right)^2}} \]

Very large if damping is small and \( \omega_d = \omega_0 \).

This is when bridges fail, buildings collapse, lasers oscillate, microwaves cook food, swings swing, radio stations transmit, radios receive, ...
HITT Question:
If the force applied to a simple harmonic oscillator oscillates with frequency $\omega_d$ and the resonance frequency of the oscillator is $\omega_0 = (k/m)^{1/2}$, at what frequency does the harmonic oscillator oscillate?

A: $\omega_d$  B: $\omega_0$  C: $\omega_0 - \omega_d$  D: $(\omega_0 - \omega_d)/2$  E: $(\omega_0 \omega_d)^{1/2}$
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A: \( \omega_d \)

If we stop now applying a force, with which frequency will the oscillator continue to oscillate?

A: \( \omega_d \)  B: \( \omega_0 \)  C: \( \omega_0 - \omega_d \)  D: \( (\omega_0 - \omega_d)/2 \)  E: \( (\omega_0 \omega_d)^{1/2} \)
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A: \( \omega_d \)

If we stop now applying a force, with which frequency will the oscillator continue to oscillate?

B: \( \omega_0 \)
Chapter 16: Waves I

**Types of Waves:**
- **Mechanical Waves**
  - Water Waves
  - Acoustic Waves (Pressure waves)
  - Seismic Waves
  - ...
- **Electro-magnetic Waves:**
  - Visible
  - Infrared (IR)
  - Ultraviolet (UV)
  - Radio waves
  - X-rays
  - ...
- **Matter Waves:**
  - Electrons
  - Protons
  - ... everything ... but now we are deep in quantum mechanics...
Chapter 16: Waves I

Transverse and Longitudinal Waves:

**Transversal waves:**
Something moves perpendicular (transversal) to the propagation direction of the wave (or wave pulse)

**Longitudinal waves:**
Something moves parallel and anti-parallel (longitudinal) to the propagation direction of the wave (or wave pulse)
y(x, t=0) moves with velocity v to the right

\[ y(x, t) = y(x - vt) \]

describes the change of y as a function of x and t.

Similar to moving into a coordinate system where the pulse is stationary.
Chapter 16: Waves I

\[ y(x,t) = y(x - vt) \] describes the change of \( y \) as a function of \( x \) and \( t \).

One possible wave is a sinusoidal (harmonic) wave:

\[ y(x,t) = y_m \sin(kx - \omega t) \quad \Rightarrow \quad v = \frac{\omega}{k} \quad \text{(Phase velocity)} \]

\[ = y_m \sin[k(x-vt)] \]

- \( y_m \): Amplitude
- \( k \): Angular wave number
- \( x \): Position along the propagation axis
- \( \omega \): Angular frequency
- \( t \): Time
- \( \sin(kx-\omega t) \): Oscillating term
- \( kx-\omega t \): Phase

We will focus again on harmonic waves. Why?

Superposition:

We can express every pulse as a linear combination of harmonic waves.

Most of the physics problems become 'easily' solvable for harmonic waves.
Chapter 16: Waves I

\[ y(x,t) = y_m \sin(kx - \omega t) \]

\( y_m \): Amplitude

Examples:
- Displacement of an element in a string
- Displacement of water molecules in water wave
- Electric or magnetic field in an E-M wave
- ...


Chapter 16: Waves I

\[ y(x,t) = y_m \sin(kx - \omega t) \]

- **\( y_m \)**: Amplitude
  - Examples: - Displacement of an element in a string
    - Displacement of water molecules in water wave
    - Electric or magnetic field in an E-M wave
    - ...

- **\( k \)**: Angular wave number, Units: rad/m
  - If you take a snapshot of a wave (say at \( t=0 \)), the angular wavenumber tells you by how much the phase measured in radian changes when you move 1m along the x-axis.
Chapter 16: Waves I

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  - If you take a snapshot of a wave (say at \( t=0 \)), the angular wavenumber tells you by how much the phase measured in radian changes when you move 1m along the x-axis.

- \( \omega \): Angular frequency, Units: \( \text{rad/s} \)
  - If you observe the wave passing by at a certain location (say at \( x=0 \)), the angular frequency tells you by how many radians the wave changes in 1s.
Chapter 16: Waves I

\[ y(x, t) = y_m \sin(kx - \omega t) = y_m \sin[k(x - vt)] \]

**Velocity of constant phase:**

\[ kx - \omega t = \text{const.} \] (Phase velocity)

\[ \frac{d}{dt}: kv - \omega = 0 \quad \leftrightarrow \quad v = \frac{\omega}{k} \]

This is the phase velocity of a wave; the only velocity we are dealing with in this class.

In cases where the
- amplitude becomes time dependent
- the pulse contains multiple frequency components
we can also define other velocities (group velocity, energy velocity, ...)

Beyond this class
Chapter 16: Waves I

$\sin(kx-\omega t)$

Periodic function in space and time:

$\sin(kx-\omega t) = \sin[k(x+\lambda)-\omega t]$  

$k = \frac{2\pi}{\lambda}$

$\lambda$: wave length

Take a snapshot at some time and measure the distance between similar points on a wave (say betw. local maxima)
Chapter 16: Waves I

\[ \sin(kx - \omega t) \]

Periodic function in space and time:

\[ \sin(kx - \omega t) = \sin[k(x + \lambda) - \omega t] \]

\[ k = \frac{2\pi}{\lambda} \]

\[ \sin(kx - \omega t) = \sin[kx - \omega(t + T)] \]

\[ \omega = \frac{2\pi}{T} \]

T: Period, Unit: s
Observe the wave passing by at specific location and measure the time it takes before the wave repeats itself, or the time between local maxima.
Chapter 16: Waves I

**sin(kx - ωt)**

*Periodic function in space and time:*

\[ \sin(kx - ωt) = \sin[k(x + \lambda) - ωt] \]

\[ k = \frac{2π}{\lambda} \]

\[ \sin(kx - ωt) = \sin[kx - ω(t + T)] \]

\[ ω = \frac{2π}{T} \]

\[ f = \frac{1}{T} \quad f: \text{ Frequency} \]

\[ v = \frac{ω}{k} = fλ \quad \text{Velocity of sin wave} \]

*(more precise: phase velocity, but we only discuss one velocity)*
Chapter 16: Waves I

Different ways to express a harmonic (sin) wave:

Traveling into the \textbf{positive} x-direction:

\[ y(x,t) = y_m \sin(kx - \omega t) \quad k: \text{angular wave number}, \quad \omega: \text{angular frequency} \]
\[ = y_m \sin[k(x-\nu t)] \quad k: \text{angular wave number}, \quad \nu: \text{phase velocity} \]
\[ = y_m \sin[2\pi(x/\lambda - t/T)] \quad \lambda: \text{wavelength}, \quad T: \text{period} \]

Same for cosine, might also add an additional initial phase

Traveling into the \textbf{negative} x-direction:

\[ y(x,t) = y_m \sin(kx + \omega t) \quad k: \text{angular wave number}, \quad \omega: \text{angular frequency} \]
\[ = y_m \sin[k(x+\nu t)] \quad k: \text{angular wave number}, \quad \nu: \text{phase velocity} \]
\[ = y_m \sin[2\pi(x/\lambda + t/T)] \quad \lambda: \text{wavelength}, \quad T: \text{period} \]

Same for cosine, might also add an additional initial phase
Chapter 16: Waves I

General wave:
\[ y(x, t) = h(kx \pm \omega t) = g(x \pm vt) \]

- \( x - vt \): Propagates in the positive x-direction
- \( x + vt \): Propagates in the negative x-direction

In this context:
- \( v \): Magnitude of velocity (always positive)
  - depends on 'material' in which wave propagates
- If you carry signs for velocities (allow for negative velocities when traveling to the left (lower x-values))
  - it is always \( h(kx - \omega t) = g(x - vt) \)
Chapter 16: Waves I

\[ h(kx - \omega t) = g(x - vt) \]

or

\[ h(kx + \omega t) = g(x + vt) \]

Examples for general waves \( h(kx \mp \omega t) = g(x \mp vt) \)

\[ y(x, t) = 4x - 3t \quad \text{velocity: } y(x, t) = 4(x - \frac{3}{4}t) \Rightarrow v = \frac{3}{4} \frac{m}{s} \]

\[ y(x, t) = 2e^{-(2x-4t)^2} \quad \text{velocity: } y(x, t) = 2e^{-4(x-2t)^2} \Rightarrow v = 2 \frac{m}{s} \]

\[ y(x, t) = (3x - 6t)e^{-4(x-2t)^2} = 3(x - 2t)e^{-4(x-2t)^2} \quad \text{also a wave with } v = 2 \frac{m}{s} \]

But

\[ y(x, t) = 3(x - 3t)e^{-4(x-2t)^2} \quad \text{not a wave as } v \text{ changes in the two expressions} \]
\[ y(x, t) = (3x - 6t) e^{-4(x-2t)^2} = 3(x - 2t) e^{-4(x-2t)^2} \] also a wave with \( v = \frac{2 \text{m}}{\text{s}} \)

Nice wave: Pulse maintains shape, just moves to the right.
$y(x,t) = 3(x - 3t)e^{-4(x-2t)^2}$

not a wave as $v$ changes in the two expressions

Form changes completely after 3s.

This is not a wave (but can in principle be described by a linear combination of waves having different velocities $\rightarrow$ Dispersion)
Note:
General waves are better described by \( g(x-vt) \) or \( g(x+vt) \)
as the meaning of \( k \) and \( \omega \) in a non-harmonic function is not clear.

Example:
\[
y(x,t) = 4 \sin(0.4x-0.2t) \quad k=0.4, \ \omega=0.2 \quad \text{(Obvious)}
\]

But:
\[
y(x,t) = 4 (0.2x-0.4t) = 0.8x-1.6t = 2 (0.4x-0.8t)
\]

what is \( k \), what is \( \omega \) in this case? Still a wave!
\[
y(x,t) = 0.8(x-2t) \quad \text{the velocity is obviously } v=2\text{m/s}
\]
Which of the following expressions does **not** describe a wave?

\[
A(x, t) = (x - t)^2 \quad B(x, t) = x^2 + 4xt + 4t^2 \quad C(x, t) = \frac{\sin(2x - 4t)}{(x - 2t)}
\]

\[
D(x, t) = (x - 2t)e^{-(x^2 - 2xt + t^2)} \quad E(x, t) = \sqrt{\frac{\cos(2x - t)}{(6x - 3t)}}
\]
Which of the following expressions does not describe a wave?

\[ A(x,t) = (x-t)^2 \quad B(x,t) = x^2 + 4xt + 4t^2 \quad C(x,t) = \frac{\sin(2x - 4t)}{(x - 2t)} \]

\[ D(x,t) = (x - 2t)e^{-(x^2 - 2xt + t^2)} \quad E(x,t) = \sqrt{\frac{\cos(2x - t)}{(6x - 3t)}} \]

\[ A(x,t) = (x-t)^2 \text{ Wave} \quad B(x,t) = (x+2t)^2 \text{ Wave} \quad C(x,t) = \frac{\sin(2(x-2t))}{(x-2t)} \text{ Wave} \]

\[ D(x,t) = (x-2t)e^{-(x-t)^2} \text{ Not a wave} \quad E(x,t) = \sqrt{\frac{\cos(2(x-0.5t))}{6(x-0.5t)}} \text{ Wave} \]
Which of the following expressions does not describe a wave?

\[ A(x,t) = (x - 3t)e^{-(x^2 - 2xt + t^2)} \quad B(x,t) = (x - t)^{1/2} \quad C(x,t) = \frac{\cos(3x - t)}{(9x - 3t)} \]

\[ D(x,t) = x^2 + 4xt + 4t^2 \quad E(x,t) = \sqrt{\frac{\sin(x - 2t)}{(2x - 4t)}} \]
Which of the following expressions does **not** describe a wave?

\[ A(x,t) = (x - 3t)e^{-(x^2 - 2xt + t^2)} \quad B(x,t) = (x - t)^{1/2} \quad C(x,t) = \frac{\cos(3x - t)}{(9x - 3t)} \]

\[ D(x,t) = x^2 + 4xt + 4t^2 \quad E(x,t) = \sqrt{\frac{\sin(x - 2t)}{(2x - 4t)}} \]

A\((x,t) = (x - 3t)e^{-(x-t)^2}\) **Not a Wave** \quad B\((x,t) = \sqrt{(x-t)}\) **Wave** \quad C\((x,t) = \frac{\cos(3(x - \frac{1}{3}t))}{9(x - \frac{1}{3}t)}\) **Wave**

\[ D(x,t) = (x + 2t)^2 \text{ wave} \quad E(x,t) = \sqrt{\frac{\sin(x - 2t)}{2(x - 2t)}} \text{ Wave} \]
Wave on a string:
The tension pulling to the left on a small element of length $dx$ is equal in amplitude but has a slightly different direction than the tension pulling to the right (because of curvature, $2^{nd}$ derivative $\neq 0$).

\[ y - \text{component: } T_{2y} + T_{1y} = a_y dm \quad T_{2y}, T_{1y} \text{ have opposite signs} \]

using \( dm = \mu dx \)

\[ T_{2y} + T_{1y} = a_y \mu dm = \frac{d^2 y}{dt^2} \mu dx \]

gives the acceleration of the small element in the y-direction.

The difference in direction is so small that the x-components of the two tensions are assumed to be equal (change with \( \cos \theta \)).

The direction of the tension is tangential to the string:

\[
\frac{T_{2y}(x + dx)}{T_{2x}} = \frac{dy}{dx} = \frac{y(x + dx) - y(x)}{dx} \quad \frac{T_{1y}(x - dx)}{T_{1x}} = -\frac{dy}{dx} = -\frac{y(x) - y(x - dx)}{dx}
\]

\[
\Rightarrow \quad \frac{T_{2y}(x + dx)}{T_{2x}} + \frac{T_{1y}(x - dx)}{T_{1x}} = \frac{T_{2y}(x + dx) + T_{1y}(x - dx)}{T} = \frac{y(x + dx) - 2y(x) + y(x - dx)}{dx}
\]

\[
\Rightarrow \quad \mu \frac{d^2 y}{dt^2} = T \frac{y(x + dx) - 2y(x) + y(x - dx)}{dx^2} = T \frac{d^2 y}{dx^2}
\]

\[
\Rightarrow \quad \frac{d^2 y}{dt^2} = \frac{T}{\mu} \frac{d^2 y}{dx^2} = v^2 \frac{d^2 y}{dx^2} \quad \text{Wave equation for small amplitudes}
\]

\[
v = \sqrt{\frac{T}{\mu}} \quad \text{Wave velocity} \quad \text{Solution:} \quad y(x, t) = h(x \pm vt)
\]