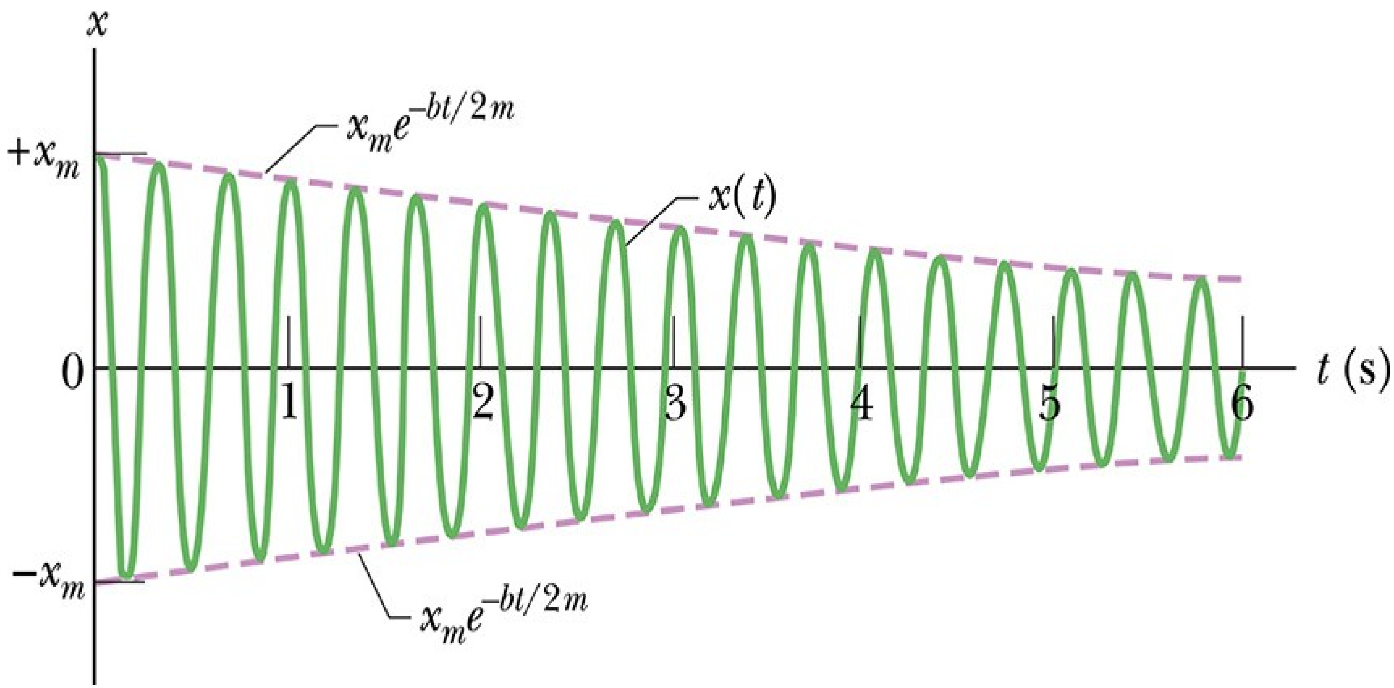
Damped Simple Harmonic Motion

$$x(t) = x_m e^{-\frac{bt}{2m}} \cos(\omega' t + \phi) \qquad \omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$



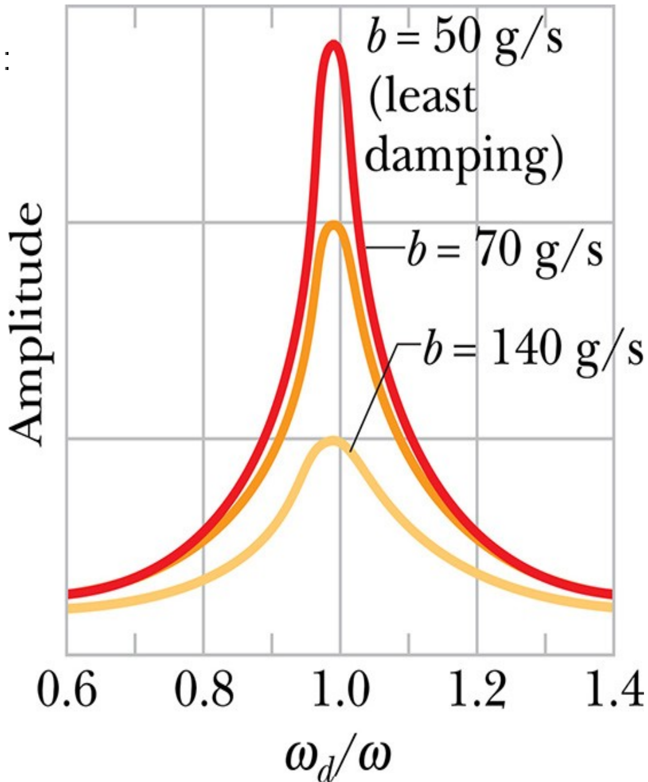
- Exponentially decreasing envelope of harmonic motion
- Shift in frequency

Forced Oscillations

$$x(t) = x_m \sin(\omega_d t + \phi_0) \qquad x_m = \text{Amplitude}$$

$$x_m = \frac{F_m}{m\sqrt{\left(\frac{b}{m}\omega_d\right)^2 + \left(\omega_d^2 - \omega_0^2\right)^2}}$$

Very large if damping is small and $\omega_d = \omega_0$



This is when bridges fail, buildings collapse, lasers oscillate, microwaves cook food, swings swing, radio stations transmit, radios receive, ...

Forced Oscillations

HITT Question:
If the force applied to a simple harmonic oscillator oscillates with frequency ω_d and the resonance frequency of the oscillator is $\omega_0 = (k/m)^{1/2}$, at what frequency does the harmonic oscillator oscillate?

A: ω_d B: ω_0 C: $\omega_0 - \omega_d$ D=$(\omega_0 - \omega_d)/2$ E=$(\omega_0 \omega_d)^{1/2}$

Forced Oscillations

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If the force applied to a simple harmonic oscillator oscillates with frequency ω_d and the resonance frequency of the oscillator is $\omega_0 = (k/m)^{1/2}$, at what frequency does the harmonic oscillator oscillate?

A: ω_d

If we stop now applying a force, with which frequency will the oscillator continue to oscillate?

A: ω_d B: ω_0 C: $\omega_0 - \omega_d$ D=$(\omega_0 - \omega_d)/2$ E=$(\omega_0 \omega_d)^{1/2}$

Forced Oscillations

HITT Question:
If the force applied to a simple harmonic oscillator oscillates with frequency ω_d and the resonance frequency of the oscillator is $\omega_0 = (k/m)^{1/2}$, at what frequency does the harmonic oscillator oscillate?

A: ω_d

If we stop now applying a force, with which frequency will the oscillator continue to oscillate?

B: ω_0

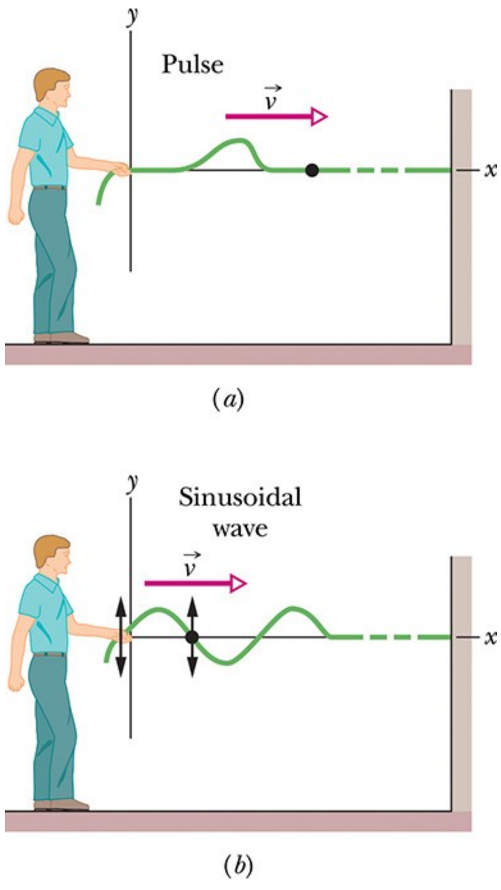
Chapter 16: Waves I

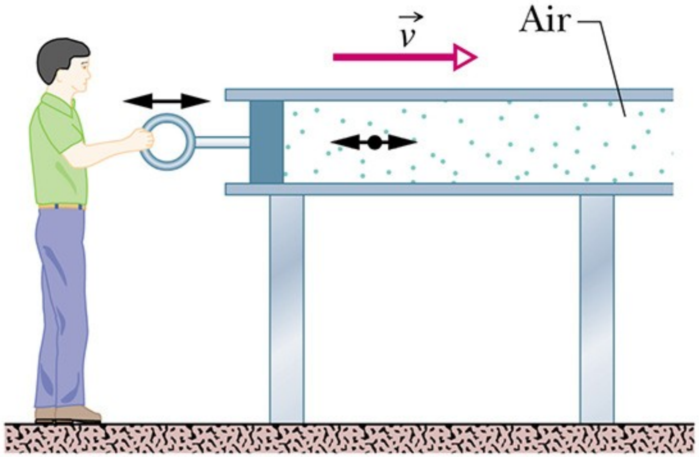
Types of Waves:

- Mechanical Waves
  - Water Waves
  - Acoustic Waves (Pressure waves)
  - Seismic Waves
  - ...
- Electro-magnetic Waves:
  - Visible
  - Infrared (IR)
  - Ultraviolet (UV)
  - Radio waves
  - X-rays
  - ...
- Matter Waves:
  - Electrons
  - Protons
  - ... everything ... but now we are deep in quantum mechanics...

Chapter 16: Waves I

Transverse and longitudinal Waves:





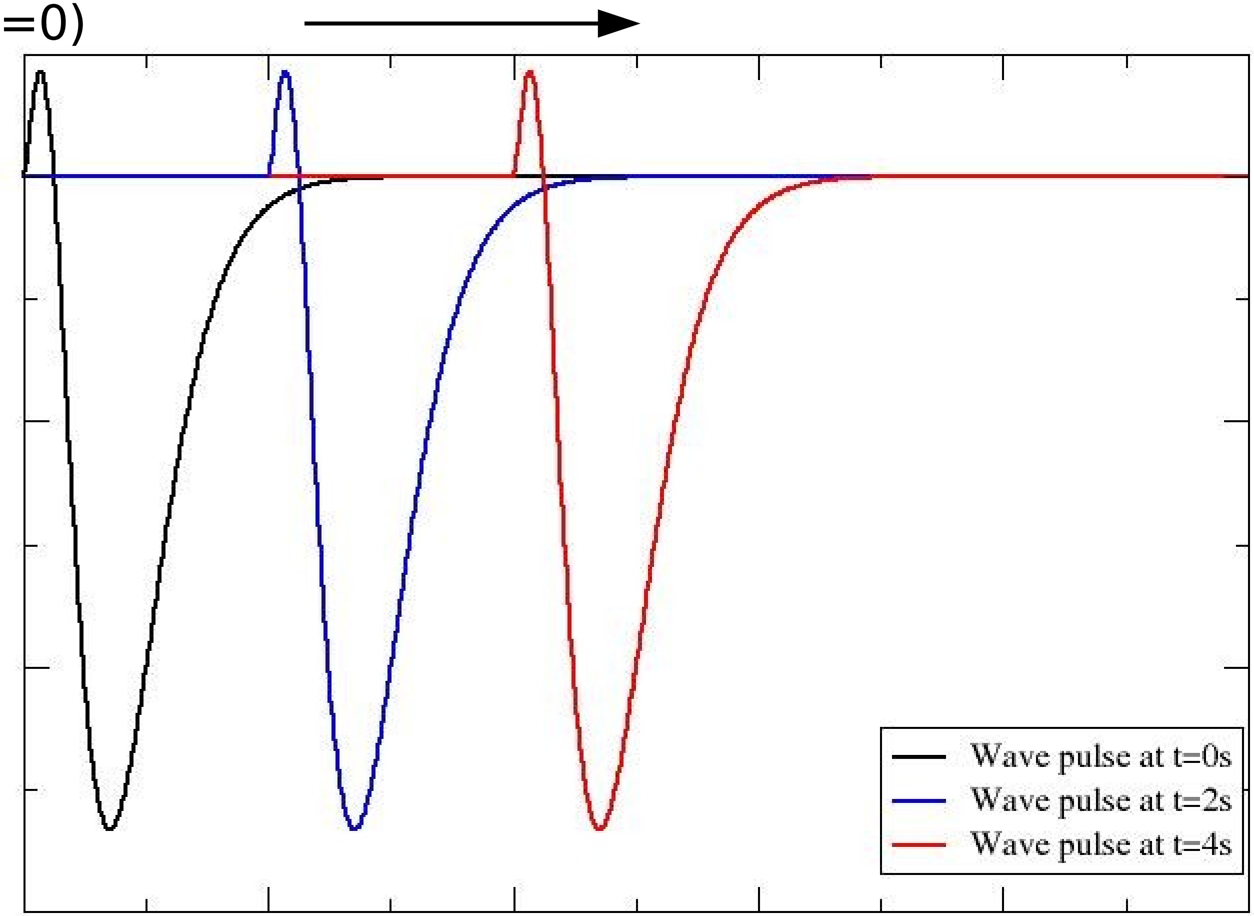
Transversal waves:
Something moves perpendicular (transversal) to the propagation direction of the wave (or wave pulse)

Longitudinal waves:
Something moves parallel and anti-parallel (longitudinal) to the propagation direction of the wave (or wave pulse)

Chapter 16: Waves I

moves with velocity v to the right

y(x,t=0)



2s 4s 6s x

$y(x,t)=y(x-vt)$

describes the change of y as a function of x and t.

Similar to moving into a coordinate system where the pulse is stationary.

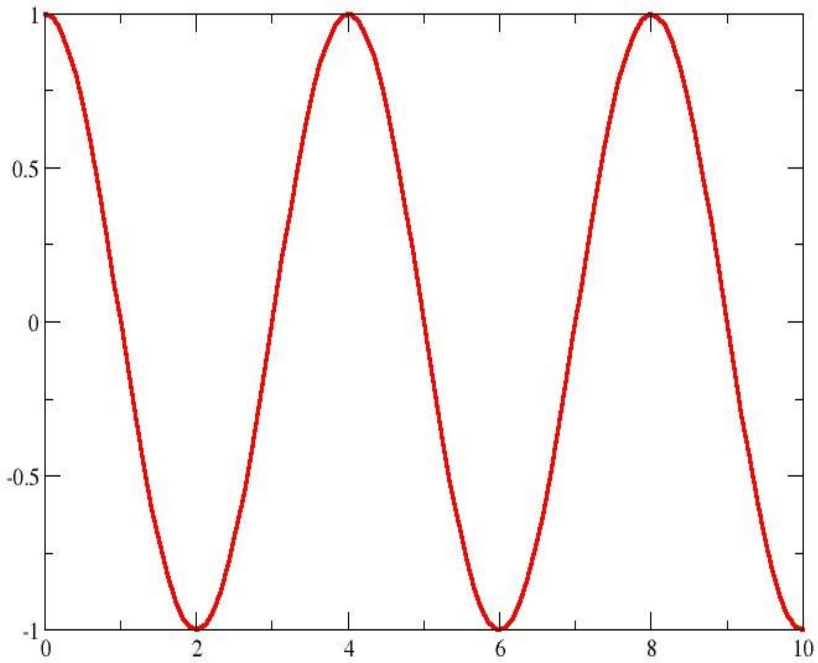
Chapter 16: Waves I

$y(x,t)=y(x-vt)$ describes the change of y as a function of x and t.

One possible wave is a sinusoidal (harmonic) wave:

$$y(x,t)=y_m\sin(kx-\omega t) \quad \Rightarrow \quad v=\omega/k \quad \textbf{(Phase velocity)}$$

$$=y_m\sin[k(x-vt)]$$



y_m : Amplitude
k : Angular wave number
x : Position along the propagation axis
ω : Angular frequency
t : Time
$\sin(kx-\omega t)$: Oscillating term
$kx-\omega t$: Phase

We will focus again on harmonic waves.
Why?
Superposition:
We can express every pulse as a linear combination of harmonic waves
Most of the physics problems become 'easily' solvable for harmonic waves.

Chapter 16: Waves I

$$y(x,t)=y_m \sin(kx-\omega t)$$

y_m : Amplitude

Examples:
- Displacement of an element in a string
- Displacement of water molecules in water wave
- Electric or magnetic field in an E-M wave
- ...

Chapter 16: Waves I

$$y(x,t)=y_m \sin(kx-\omega t)$$

y_m : Amplitude

Examples:
- Displacement of an element in a string
- Displacement of water molecules in water wave
- Electric or magnetic field in an E-M wave
- ...

k : Angular wave number, Units: **rad/m**

If you take a snapshot of a wave (say at t=0), the angular wavenumber tells you by how much the phase measured in radian changes when you move 1m along the x-axis.

Chapter 16: Waves I

$$y(x,t)=y_m \sin(kx-\omega t)$$

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Examples:
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- ...

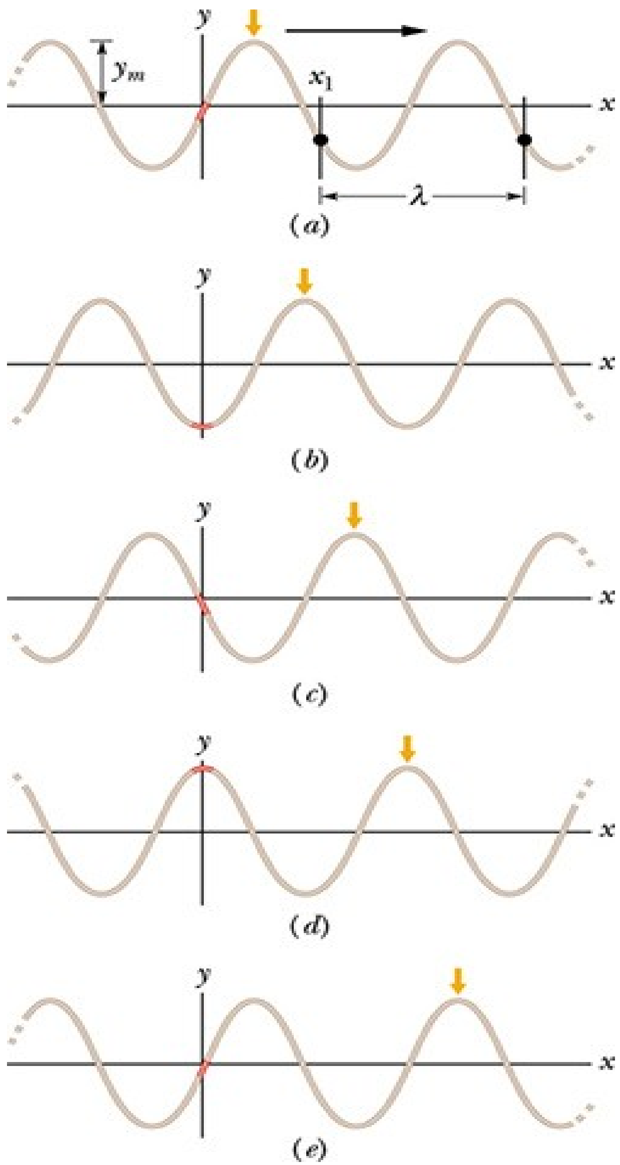
k : Angular wave number, Units: **rad/m**

If you take a snapshot of a wave (say at t=0), the angular wavenumber tells you by how much the phase measured in radian changes when you move 1m along the x-axis.

ω : Angular frequency, Units: **rad/s**

If you observe the wave passing by at a certain location (say at x=0), the angular frequency tells you by how many radians the wave changes in 1s.

Chapter 16: Waves I



$$y(x,t) = y_m \sin(kx - \omega t)$$

$$= y_m \sin[k(x - vt)]$$

Velocity of constant phase:

$kx - \omega t = \text{const.}$ (Phase velocity)

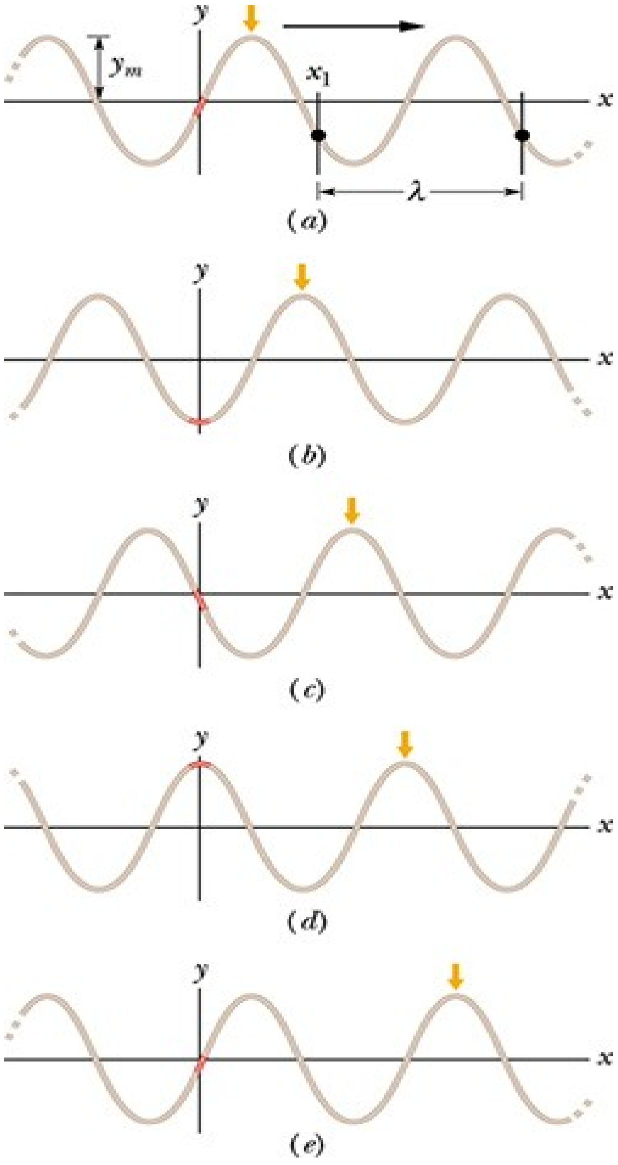
d/dt: $kv - \omega = 0$ ⟺ $v = \omega/k$

This is the phase velocity of a wave; the only velocity we are dealing with in this class.

In cases where the
- amplitude becomes time dependent
- the pulse contains multiple frequency components

we can also define other velocities (group velocity, energy velocity, ...)
Beyond this class

Chapter 16: Waves I



sin(kx-ωt)

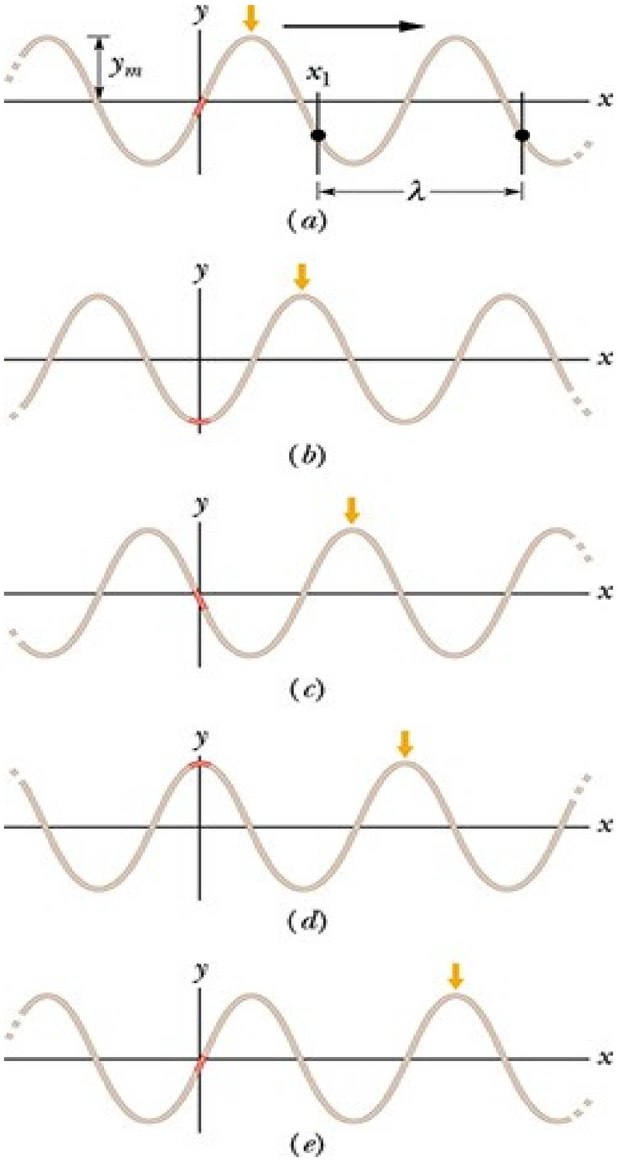
Periodic function in space and time:

sin(kx-ωt) = sin[k(x+λ)-ωt]

→ **k=2π/λ**

λ: wave length
Take a snapshot at some time and measure the distance between similar points on a wave (say betw. local maxima)

Chapter 16: Waves I



sin(kx-ωt)

Periodic function in space and time:

sin(kx-ωt) = sin[k(x+λ)-ωt]

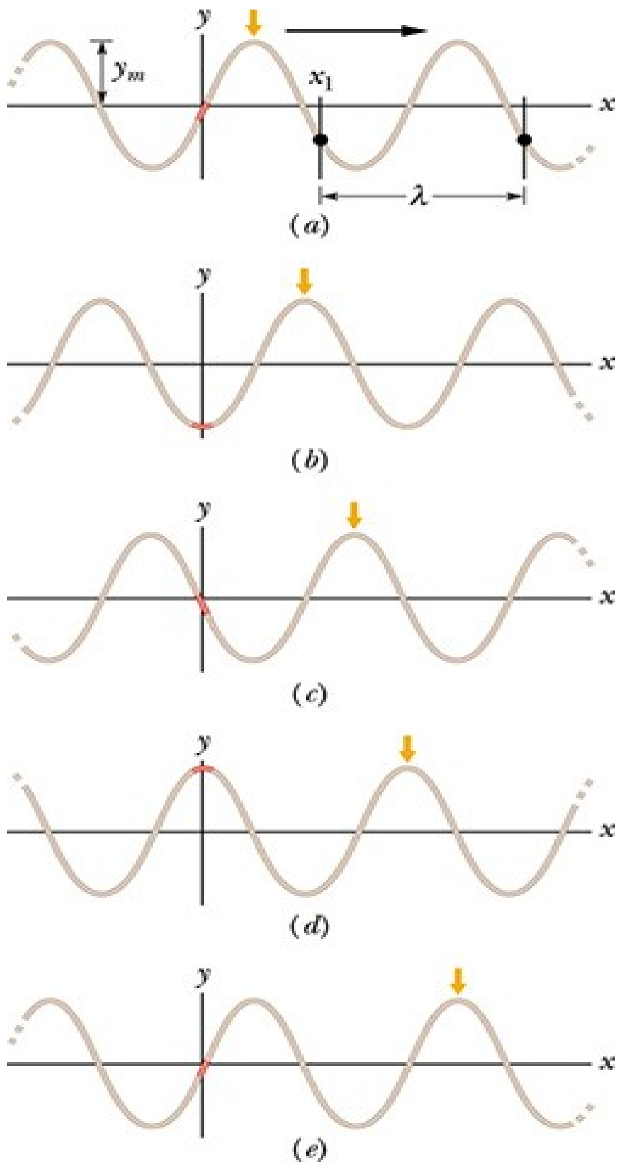
k=2π/λ

sin(kx-ωt) = sin[kx-ω(t+T)]

→ **ω=2π/T**

T: Period, Unit: s
Observe the wave passing by at specific location and measure the time it takes before the wave repeats itself, or the time between local maxima.

Chapter 16: Waves I



$\sin(kx-\omega t)$

Periodic function in space and time:

$\sin(kx-\omega t) = \sin[k(x+\lambda)-\omega t]$

$k=2\pi/\lambda$

$\sin(kx-\omega t) = \sin[kx-\omega(t+T)]$

$\omega=2\pi/T$

$f = 1/T$ f: Frequency

$v=\omega/k = f\lambda$ Velocity of sin wave

(more precise: phase velocity, but we only discuss one velocity)

Chapter 16: Waves I

Different ways to express a harmonic (sin) wave:

Traveling into the positive x-direction:

$y(x,t) = y_m \sin(kx - \omega t)$ **k: angular wave number, ω: angular frequency**

$= y_m \sin[k(x - vt)]$ **k: angular wave number, v: phase velocity**

$= y_m \sin[2\pi(x/\lambda - t/T)]$ **λ: wavelength, T: period**

Same for cosine, might also add an additional initial phase

Traveling into the negative x-direction:

$y(x,t) = y_m \sin(kx + \omega t)$ **k: angular wave number, ω: angular frequency**

$= y_m \sin[k(x + vt)]$ **k: angular wave number, v: phase velocity**

$= y_m \sin[2\pi(x/\lambda + t/T)]$ **λ: wavelength, T: period**

Same for cosine, might also add an additional initial phase

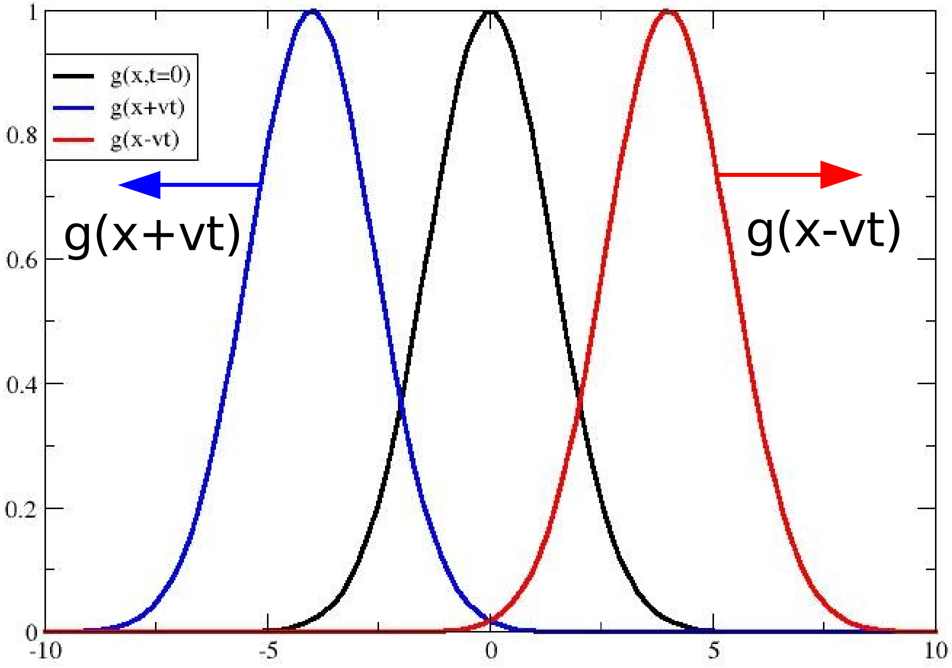
Chapter 16: Waves I

General wave:

$$y(x,t) = h(kx \pm \omega t) = g(x \pm vt)$$

$x - vt$: Propagates in the positive x-direction

$x + vt$: Propagates in the negative x-direction



In this context:
v: Magnitude of velocity
(always positive)
depends on 'material' in which wave propagates

If you carry signs for velocities (allow for negative velocities when traveling to the left (lower x-values)) it is always

$$h(kx-\omega t)=g(x-vt)$$

Chapter 16: Waves I

h(kx-ωt)=g(x-vt)

or

h(kx+ωt)=g(x+vt)

Examples for general waves $h(kx \mp \omega t) = g(x \mp vt)$

$$y(x,t) = 4x - 3t \qquad \text{velocity:}\, y(x,t) = 4(x - \frac{3}{4}t) \Rightarrow v = \frac{3}{4}\frac{\text{m}}{\text{s}}$$

$$y(x,t) = 2e^{-(2x-4t)^2} \qquad \text{velocity:}\, y(x,t) = 2e^{-4(x-2t)^2} \Rightarrow v = 2\frac{\text{m}}{\text{s}}$$

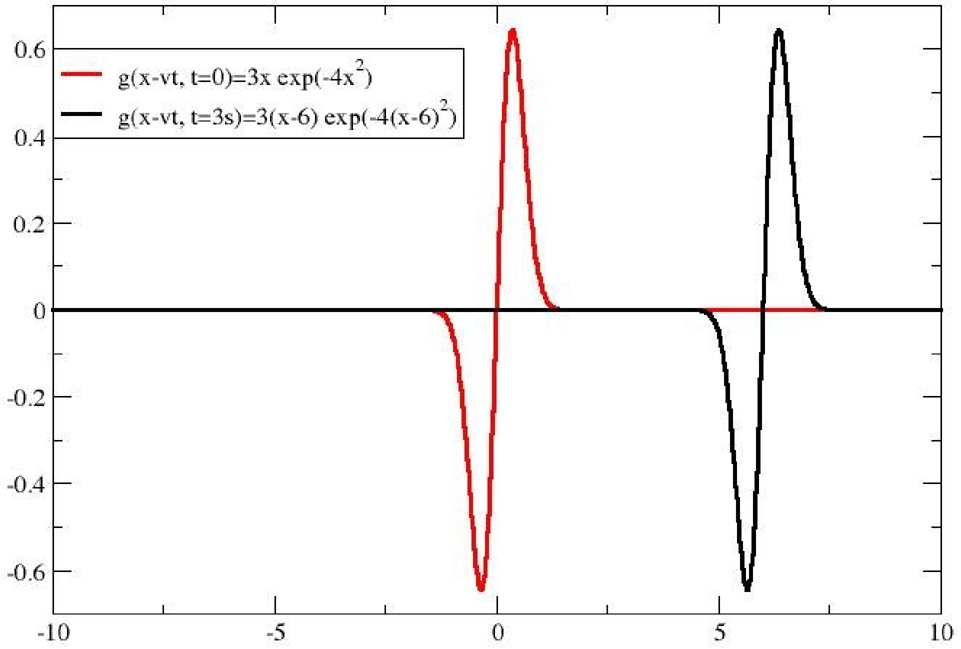
$$y(x,t) = (3x-6t)e^{-4(x-2t)^2} = 3(x-2t)e^{-4(x-2t)^2} \qquad \text{also a wave with}\, v = 2\frac{\text{m}}{\text{s}}$$

But

$$y(x,t) = 3(x-3t)e^{-4(x-2t)^2} \qquad \text{not a wave as v changes in the two expressions}$$

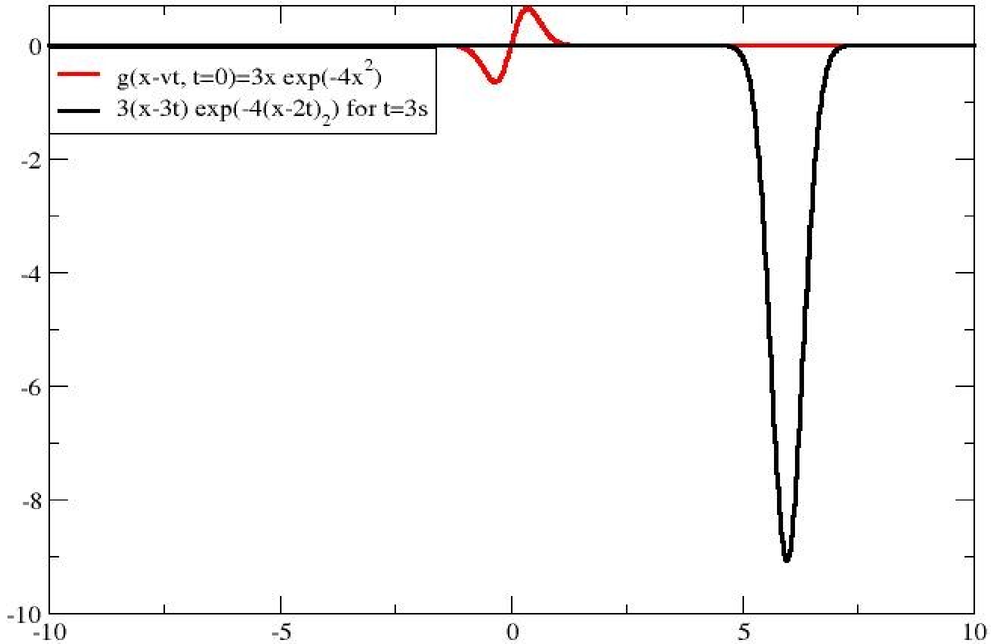
$$y(x,t) = (3x - 6t)e^{-4(x-2t)^2} = 3(x-2t)e^{-4(x-2t)^2}$$

also a wave with $v = 2\frac{\text{m}}{\text{s}}$



Nice wave:
Pulse maintains shape, just moves to the right.

$$y(x,t) = 3(x-3t)e^{-4(x-2t)^2}$$ not a wave as v changes in the two expressions



Form
changes completely
after 3s.

This is not a wave
(but can in principle
be described by a
linear combination of
waves having different
velocities -> Dispersion)

Chapter 16: Waves I

Note:
General waves are better described by $g(x-vt)$ or $g(x+vt)$
as the meaning of k and ω in a non-harmonic function is not clear.

Example:

$y(x,t) = 4 \sin(0.4x-0.2t)$ $k=0.4$, $\omega=0.2$ (Obvious)

But:

$y(x,t) = 4 (0.2x-0.4t) = 0.8x-1.6t = 2 (0.4x-0.8t)$

what is k, what is ω in this case? Still a wave!

$y(x,t) = 0.8(x-2t)$ the velocity is obviously $v=2m/s$

Hitt 1

Which of the following expressions does **not** describe a wave?

$$A(x,t) = (x-t)^2 \qquad B(x,t) = x^2 + 4xt + 4t^2 \qquad C(x,t) = \frac{\sin(2x-4t)}{(x-2t)}$$

$$D(x,t) = (x-2t)e^{-(x^2-2xt+t^2)} \qquad E(x,t) = \sqrt{\frac{\cos(2x-t)}{(6x-3t)}}$$

Hitt 1

Which of the following expressions does **not** describe a wave?

$$A(x,t) = (x-t)^2 \qquad B(x,t) = x^2 + 4xt + 4t^2 \qquad C(x,t) = \frac{\sin(2x-4t)}{(x-2t)}$$

$$D(x,t) = (x-2t)e^{-(x^2-2xt+t^2)} \qquad E(x,t) = \sqrt{\frac{\cos(2x-t)}{(6x-3t)}}$$

$$A(x,t) = (x-t)^2 \text{ Wave} \qquad B(x,t) = (x+2t)^2 \text{ Wave} \qquad C(x,t) = \frac{\sin(2(x-2t))}{(x-2t)} \text{ Wave}$$

$$D(x,t) = (x-2t)e^{-(x-t)^2} \text{ Not a wave} \qquad E(x,t) = \sqrt{\frac{\cos(2(x-0.5t))}{6(x-0.5t)}} \text{ Wave}$$

Hitt 2

Which of the following expressions does **not** describe a wave?

$$A(x,t) = (x-3t)e^{-(x^2-2xt+t^2)} \qquad B(x,t) = (x-t)^{1/2} \qquad C(x,t) = \frac{\cos(3x-t)}{(9x-3t)}$$

$$D(x,t) = x^2 + 4xt + 4t^2 \qquad E(x,t) = \sqrt{\frac{\sin(x-2t)}{(2x-4t)}}$$

Hitt 2

Which of the following expressions does **not** describe a wave?

$A(x,t) = (x-3t)e^{-(x^2-2xt+t^2)}$ $\quad B(x,t) = (x-t)^{1/2}$ $\quad C(x,t) = \dfrac{\cos(3x-t)}{(9x-3t)}$

$D(x,t) = x^2 + 4xt + 4t^2$ $\quad E(x,t) = \sqrt{\dfrac{\sin(x-2t)}{(2x-4t)}}$

$A(x,t) = (x-3t)e^{-(x-t)^2}$ Not a Wave $\quad B(x,t) = \sqrt{(x-t)}$ Wave $\quad C(x,t) = \dfrac{\cos(3(x-\frac{1}{3}t))}{9(x-\frac{1}{3}t)}$ Wave

$D(x,t) = (x+2t)^2$ wave $\quad E(x,t) = \sqrt{\dfrac{\sin(x-2t)}{2(x-2t)}}$ Wave

Wave on a string:

The tension pulling to the left on a small element of length dx is equal in amplitude but has a slightly different direction than the tension pulling to the right (because of curvature, 2^{nd} derivative $\neq 0$).

$$y-\text{component:} \qquad T_{2y}+T_{1y}=a_y dm \qquad T_{2y},\, T_{1y}\,\text{have opposite signs}$$

$$\text{using}\, dm=\mu dx \qquad T_{2y}+T_{1y}=a_y\mu dm=\frac{d^2y}{dt^2}\mu dx$$

gives the acceleration of the small element in the y-direction.

The difference in direction is so small that the x-components of the two tensions are assumed to be equal (change with $\cos\theta$).

The direction of the tension is tangential to the string:

$$\frac{T_{2y}(x+dx)}{T_{2x}}=\frac{dy}{dx}=\frac{y(x+dx)-y(x)}{dx} \qquad \frac{T_{1y}(x-dx)}{T_{1x}}=-\frac{dy}{dx}=-\frac{y(x)-y(x-dx)}{dx}$$

$$\Rightarrow \qquad \frac{T_{2y}(x+dx)}{T_{2x}}+\frac{T_{1y}(x-dx)}{T_{1x}}=\frac{T_{2y}(x+dx)+T_{1y}(x-dx)}{T}=\frac{y(x+dx)-2y(x)+y(x-dx)}{dx}$$

$$\Rightarrow \qquad \mu\frac{d^2y}{dt^2}=T\frac{y(x+dx)-2y(x)+y(x-dx)}{dx^2}=T\frac{d^2y}{dx^2}$$

$$\Rightarrow \qquad \frac{d^2y}{dt^2}=\frac{T}{\mu}\frac{d^2y}{dx^2}=v^2\frac{d^2y}{dx^2} \qquad \text{Wave equation for small amplitudes}$$

$$v=\sqrt{\frac{T}{\mu}} \qquad \text{Wave velocity} \qquad \text{Solution:} \qquad y(x,t)=h(x\pm vt)$$