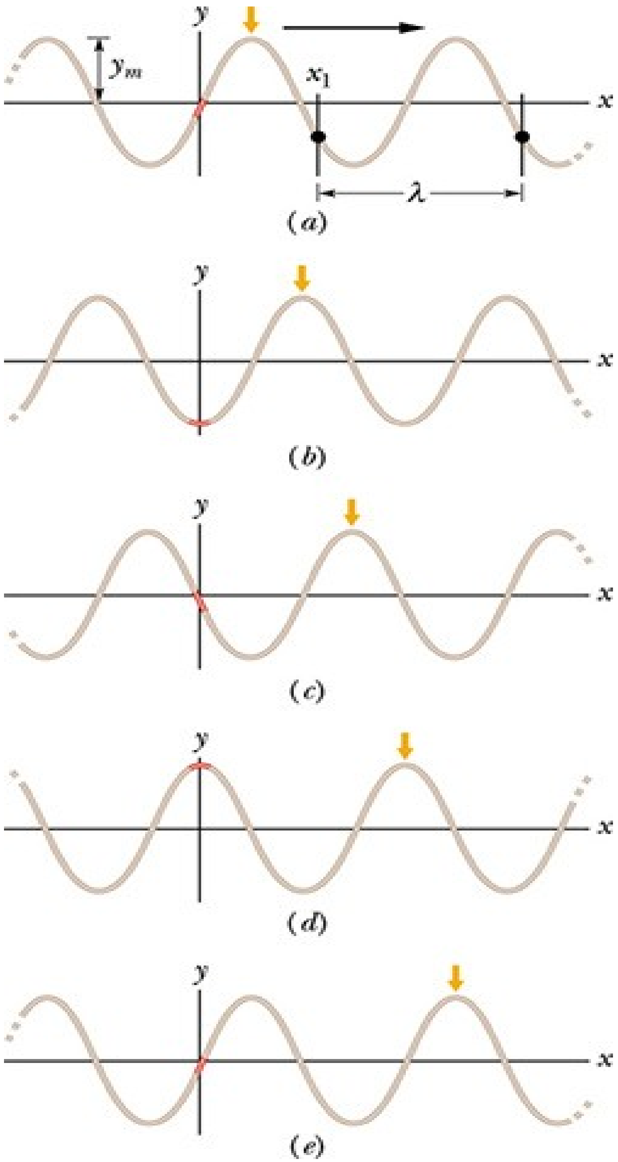
Announcements

Exam 3 Review:
Sunday, December 7th 4:00pm-6:00pm in 1001 NPB (here)

Teacher Evaluations:
Wednesday, December 10th in class. In addition to the evaluations the students will be asked 3 HITT questions about the class. If they come and do the evaluations and answer the 3 questions they will get 6 HITT points! (And all possible answers are 'correct')

Chapter 16: Waves I



sin(kx-ωt)

Periodic function in space and time:

sin(kx-ωt) = sin[k(x+λ)-ωt]

k=2π/λ

sin(kx-ωt) = sin[kx-ω(t+T)]

ω=2π/T

f = 1/T   f: Frequency

v=ω/k = fλ   Velocity of sin wave

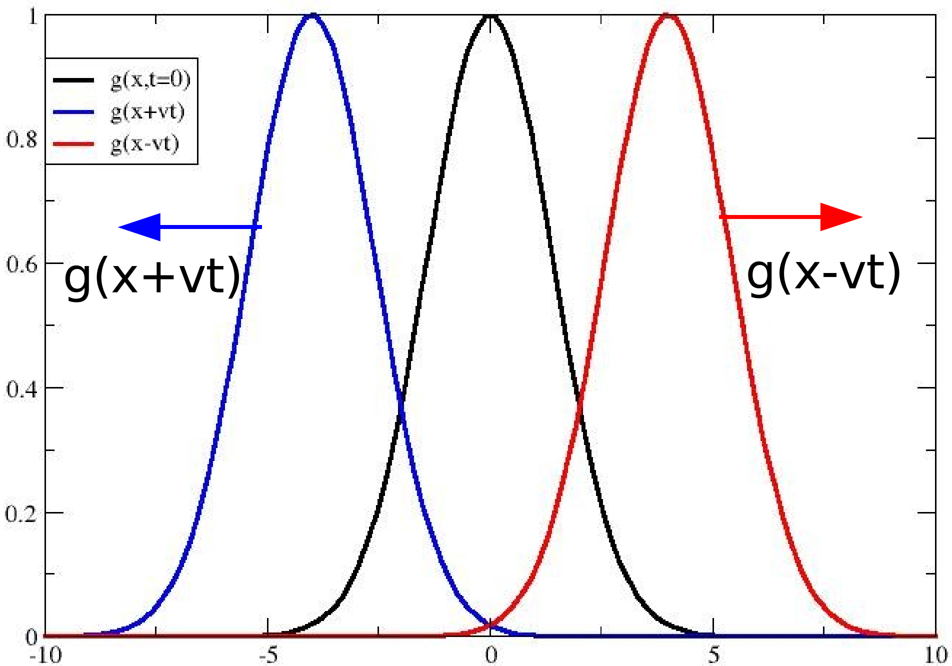
(more precise: phase velocity, but we only discuss one velocity)

Chapter 16: Waves I

General wave:

$$y(x,t) = h(kx \pm \omega t) = g(x \pm vt)$$

$x - vt$:	Propagates in the positive x-direction
$x + vt$:	Propagates in the negative x-direction



Examples for: $h(kx \mp \omega t) = g(x \mp vt)$

$$y(x,t) = 4x - 3t = 4(x - \frac{3}{4}t)$$

$$y(x,t) = 2e^{-(2x-4t)^2} = 2e^{-4(x-2t)^2}$$

$$y(x,t) = 3(x-2t)e^{-4(x-2t)^2}$$

Not a wave:

$$y(x,t) = 3(x-3t)e^{-4(x-2t)^2}$$

See plots in last lecture notes

Wave on a string:

The tension pulling to the left on a small element of length dx is equal in amplitude but has a slightly different direction than the tension pulling to the right (because of curvature, 2^{nd} derivative $\neq 0$).

$$y-\text{component:} \qquad T_{2y}+T_{1y}=a_y dm \qquad T_{2y},\,T_{1y}\,\text{have opposite signs}$$

$$\text{using}\; dm=\mu dx \qquad T_{2y}+T_{1y}=a_y\mu dm=\frac{d^2y}{dt^2}\mu dx$$

gives the acceleration of the small element in the y-direction.

The difference in direction is so small that the x-components of the two tensions are assumed to be equal (change with $\cos\theta$).

The direction of the tension is tangential to the string:

$$\frac{T_{2y}(x+dx)}{T_{2x}}=\frac{dy}{dx}=\frac{y(x+dx)-y(x)}{dx} \qquad \frac{T_{1y}(x-dx)}{T_{1x}}=-\frac{dy}{dx}=-\frac{y(x)-y(x-dx)}{dx}$$

$$\Rightarrow \qquad \frac{T_{2y}(x+dx)}{T_{2x}}+\frac{T_{1y}(x-dx)}{T_{1x}}=\frac{T_{2y}(x+dx)+T_{1y}(x-dx)}{T}=\frac{y(x+dx)-2y(x)+y(x-dx)}{dx}$$

$$\Rightarrow \qquad \mu\frac{d^2y}{dt^2}=T\frac{y(x+dx)-2y(x)+y(x-dx)}{dx^2}=T\frac{d^2y}{dx^2}$$

$$\Rightarrow \qquad \frac{d^2y}{dt^2}=\frac{T}{\mu}\frac{d^2y}{dx^2}=v^2\frac{d^2y}{dx^2} \qquad \text{Wave equation for small amplitudes}$$

$$v=\sqrt{\frac{T}{\mu}} \qquad \text{Wave velocity} \qquad \text{Solution:} \qquad y(x,t)=h(x\pm vt)$$

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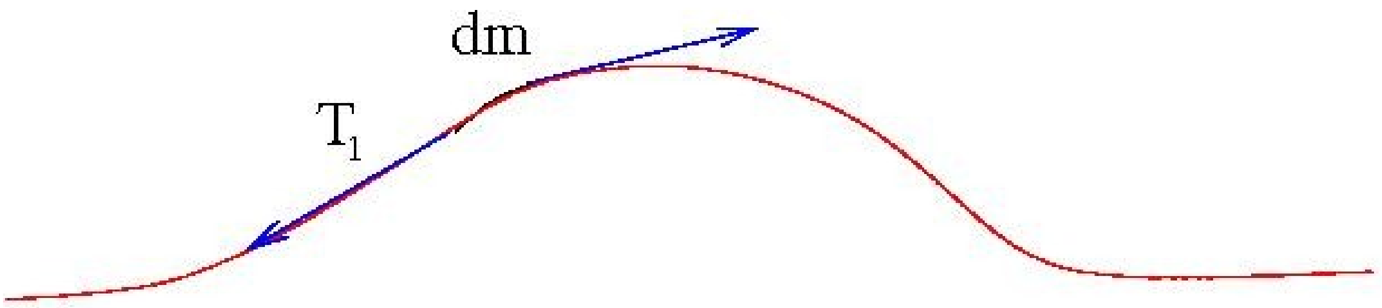

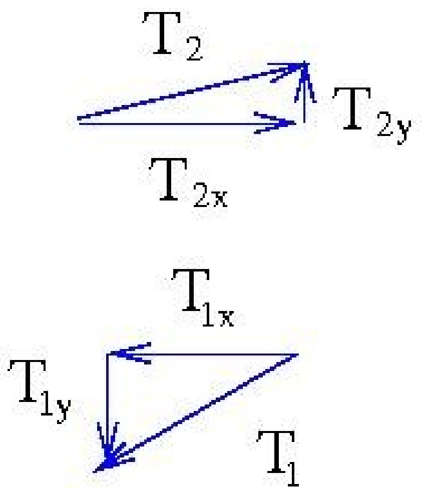

Key idea:
Tension is pulling in non-parallel directions
on curved string as the slope changes.
For ideal longitudinal wave:
$T=T_{1x}=-T_{2x}$ (small amplitude <=> small angles, difference will scale with $\cos\theta$),
compensate each other and no acceleration in x-direction (propagation direction)

The y-components are different in first order in θ and cause an acceleration in the y-direction: $T_{1y}+T_{2y}=a_y dm=a_y \mu dx$ (plus: T_{1y} will be negative in the above case)

The y-components of the tension are also proportional to the slope of the rope on both sides of the string segment: $T_{1y}=-T dy/dx$ at (x) $T_{2y}=T dy/dx$ at (x+dx)

Adding the two tensions and dividing by dx gives $(T_{1y}+T_{2y})/dx= T\, d^2y/dx^2$
Putting this together gives wave equation.

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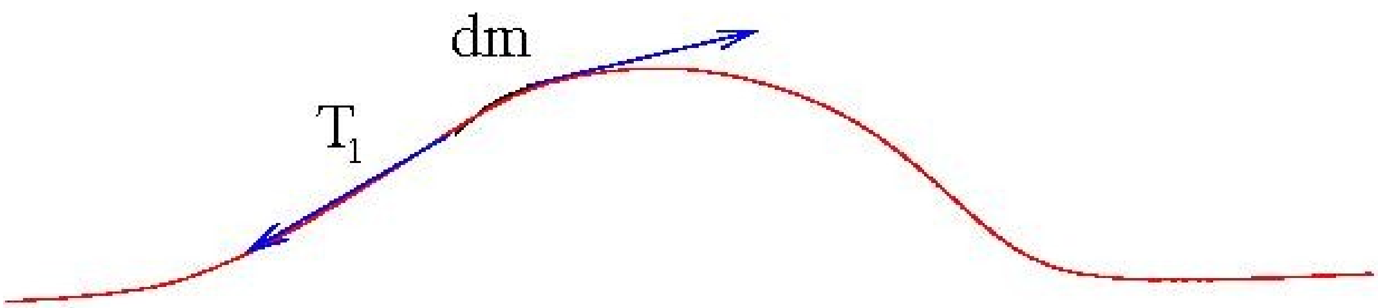
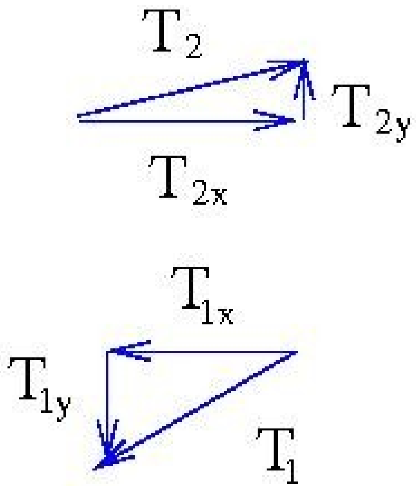




For you important is:

$$\frac{d^2y}{dt^2} = \frac{T}{\mu}\frac{d^2y}{dx^2} = v^2\frac{d^2y}{dx^2} \qquad \text{Wave equation}$$

$$v = \sqrt{\frac{T}{\mu}} \qquad \text{Wave velocity} \qquad \text{Solution:} \qquad y(x,t) = h(x \mp vt)$$

Check: Substitute: $x' = x \mp vt \qquad \frac{dx'}{dx} = 1, \ \frac{dx'}{dt} = \mp v$

$$\frac{d}{dt}\left[\frac{dh(x')}{dx'}\frac{dx'}{dt}\right] = \frac{d}{dt}\left[\mp v\frac{dh(x')}{dx'}\right] = \mp v\frac{d}{dx'}\left[\frac{dh(x')}{dt}\right] = \mp v\frac{d}{dx'}\left[\frac{dh(x')}{dx'}\frac{dx'}{dt}\right] = v^2\frac{d^2h(x')}{dx'^2}$$

$$v^2\frac{d}{dx}\frac{dh(x \mp vt)}{dx} = v^2\frac{d}{dx}\left[\frac{dh(x')}{dx'}\frac{dx'}{dx}\right] = v^2\frac{d^2h(x')}{dx'^2} \qquad \text{q.e.d.}$$

Chapter 16: Waves I

Problem:

Assume a string of length 200m, a mass of $m = 6\,\text{kg}$, stretched out by a tension of $T = 500\,\text{N}$. If this string is excited by an external force at one end with a frequency of $f = 100\,\text{Hz}$, what is the wavelength of the traveling wave?

Plan:

To get the wavelength: $\lambda = \frac{v}{f}$ we need the speed

To get the speed: $v = \sqrt{\frac{T}{\mu}}$ we need μ

General velocity
for all types of waves
(not only harmonic)

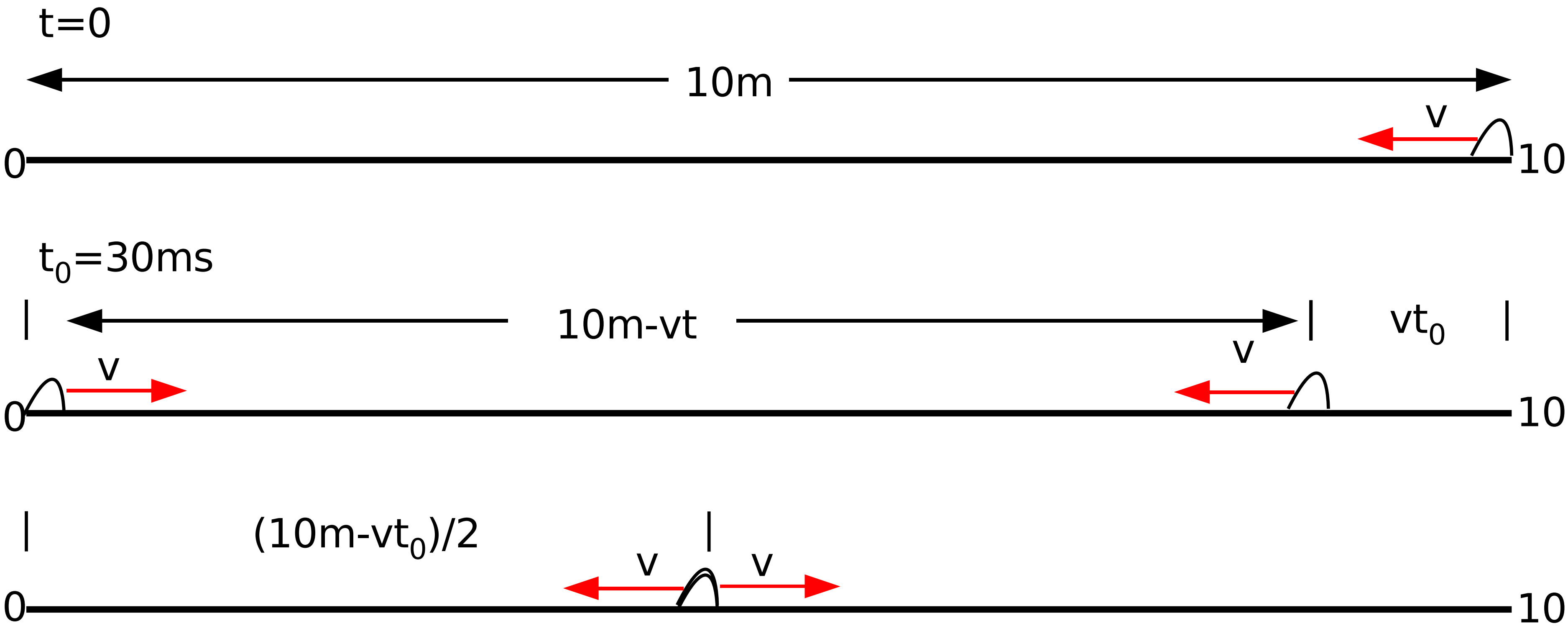
$$\mu = \frac{m}{L} = \frac{6\,\text{kg}}{200\,\text{m}} = 3 \times 10^{-2} \frac{\text{kg}}{\text{m}} \qquad v = \sqrt{\frac{500\,\text{Nm}}{3 \times 10^{-2}\text{kg}}} = 129 \frac{\text{m}}{\text{s}}$$

$$\lambda = \frac{129 \frac{\text{m}}{\text{s}}}{100\text{Hz}} = 1.29\text{m}$$

In sinusoidal (harmonic)
waves: λ,f,v are coupled
(In non-sinusoidal waves:
we don't have a wavelength)

Chapter 16: Waves I

A 100g wire is held under a tension of 250N with one end at x=0 and the other at x=10.0m. At time t=0, pulse 1 is sent along the wire from the end at x=10.0m. At time t=30ms, pulse 2 is sent along the wire from the end at x=0. Where do the pulses meet?



$$v = \sqrt{\frac{T}{\mu}} = \sqrt{\frac{TL}{m}} = \sqrt{\frac{250\text{N}\,10.0\text{m}}{0.1\text{kg}}} = 158\frac{\text{m}}{\text{s}}$$

$$vt_0 = 158\frac{\text{m}}{\text{s}}0.03\text{s} = 4.74\text{m}$$

$$d = \frac{10\text{m} - 4.74\text{m}}{2} = 2.63\text{m}$$

Chapter 16: Waves I

Energy of a wave traveling along a string:

➢ Kinetic energy

- as the mass elements dm move up and down
- max when string element passes through y=0
- zero when mass element reaches the max displacement $y=y_m$

➢ Elastic potential energy

- as the wave stretches the string
- max at y=0 where the string is stretched the most
- zero when mass element reached the max displacement $y=y_m$

➢ Energy transport

- both energy forms are max when y=0
- no energy at locations where $y=y_m$
- the tension in the string where y=0 pulls the parts where y~ym back, causing them to accelerate.
- the parts where y=0 continue to move up (or down) with their initial velocity until the tension which is now building up in the other areas stops them.

Chapter 16: Waves I

To get the potential energy of string element, we start with:

$$U = \frac{1}{2}ks^2 \qquad \frac{dU}{ds} = ks = -T \qquad s: \text{additional length after stretching string}$$

$$\Rightarrow dU = -T\,ds \qquad ds = \sqrt{[y(x+dx) - y(x)]^2 + dx^2} - dx \qquad \text{additional length of string}$$

$$\sqrt{[y(x+dx) - y(x)]^2 + dx^2} = dx\sqrt{1 - \left(\frac{dy}{dx}\right)^2} \approx dx\left[1 - \frac{1}{2}\left(\frac{dy}{dx}\right)^2\right] \qquad \text{for small derivatives}$$

$$\Rightarrow ds = -\frac{1}{2}\left(\frac{dy}{dx}\right)^2 dx \qquad \Rightarrow dU = T\frac{1}{2}\left(\frac{dy}{dx}\right)^2 dx$$

Recall: $T = v^2\mu \qquad \Rightarrow \qquad \dfrac{dU}{dt} = \dfrac{1}{2}v^3\mu\left(\dfrac{dy}{dx}\right)^2 \left[= \dfrac{1}{2}\mu v\omega^2 y_m^2 \cos^2(kx - \omega t) \text{ for harmonic w.}\right]$

Chapter 16: Waves I

kinetic energy of line element: $dK = \frac{1}{2}u^2 dm = \frac{1}{2}u^2 \mu dx$ u: transverse speed

General: $u = \frac{dy(x-vt)}{dt} = -v\frac{dy}{dx}$ $[= -\omega y_m \cos(kx - \omega t)$ for harmonic waves$]$

transverse velocity=wave speed x slope of pulse form

$$dK = \frac{1}{2}v^2\left(\frac{dy}{dx}\right)^2 \mu dx \left[= \frac{1}{2}\omega^2 y_m^2 \cos^2(kx-\omega t)\,\mu dx \quad \text{for harmonic waves}\right]$$

The rate at which kinetic energy passes through a line element:

$$\frac{dK}{dt} = \frac{1}{2}v^2\left(\frac{dy}{dx}\right)^2 \mu\frac{dx}{dt} = \frac{1}{2}v^3\mu\left(\frac{dy}{dx}\right)^2 \left[= \frac{1}{2}\mu v\omega^2 y_m^2 \cos^2(kx-\omega t) \text{ for harmonic w.}\right]$$

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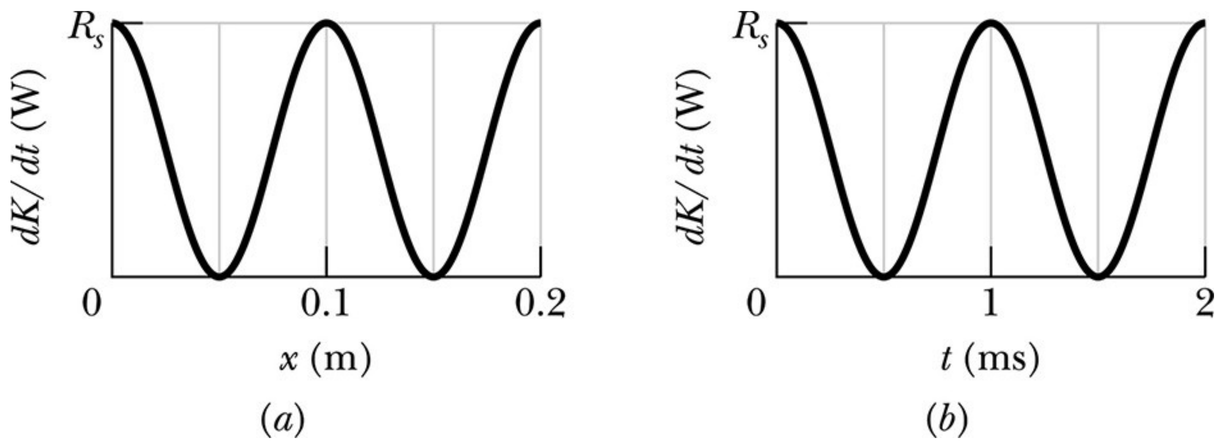
$$\frac{dK}{dt}+\frac{dU}{dt}=v^3\mu\left(\frac{dy}{dx}\right)^2\left[=\mu v\omega^2 y_m^2\cos^2(kx-\omega t)\text{ for harmonic w.}\right]$$

The average power (averaged over several wave lengths) is the average rate with which energy propagates through the string

$$P_{avg}=v^3\mu\left(\frac{dy}{dx}\right)^2\left[=\frac{1}{2}\mu v\omega^2 y_m^2\text{ for harmonic waves}\right]$$

As the book only discusses energy in harmonic waves, you will only need the expressions for harmonic waves for exams and homeworks.

Chapter 16: Waves I



The graphs show the rate dK/dt at which kinetic energy passes through string elements:

a) at a particular time as a function of distance x

b) at a particular location as a function of time

The amplitude Rs=10W in both graphs. The linear mass density is μ=2g/m

What is the amplitude of the wave?

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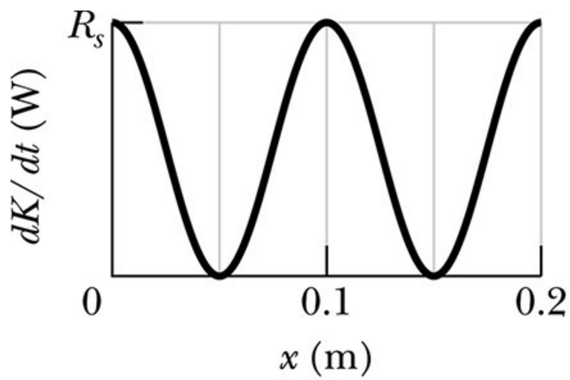
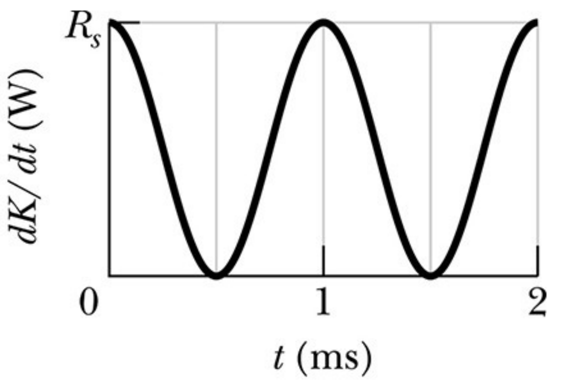




The rate of change in kinetic energy of a harmonic wave is:

$$\frac{dK}{dt}_{max} = \frac{1}{2}\mu v \omega^2 y_m^2 = 10W \qquad \text{with: } \mu = 2 \times 10^{-3}\frac{\text{kg}}{\text{m}}$$

From the graphs we see:

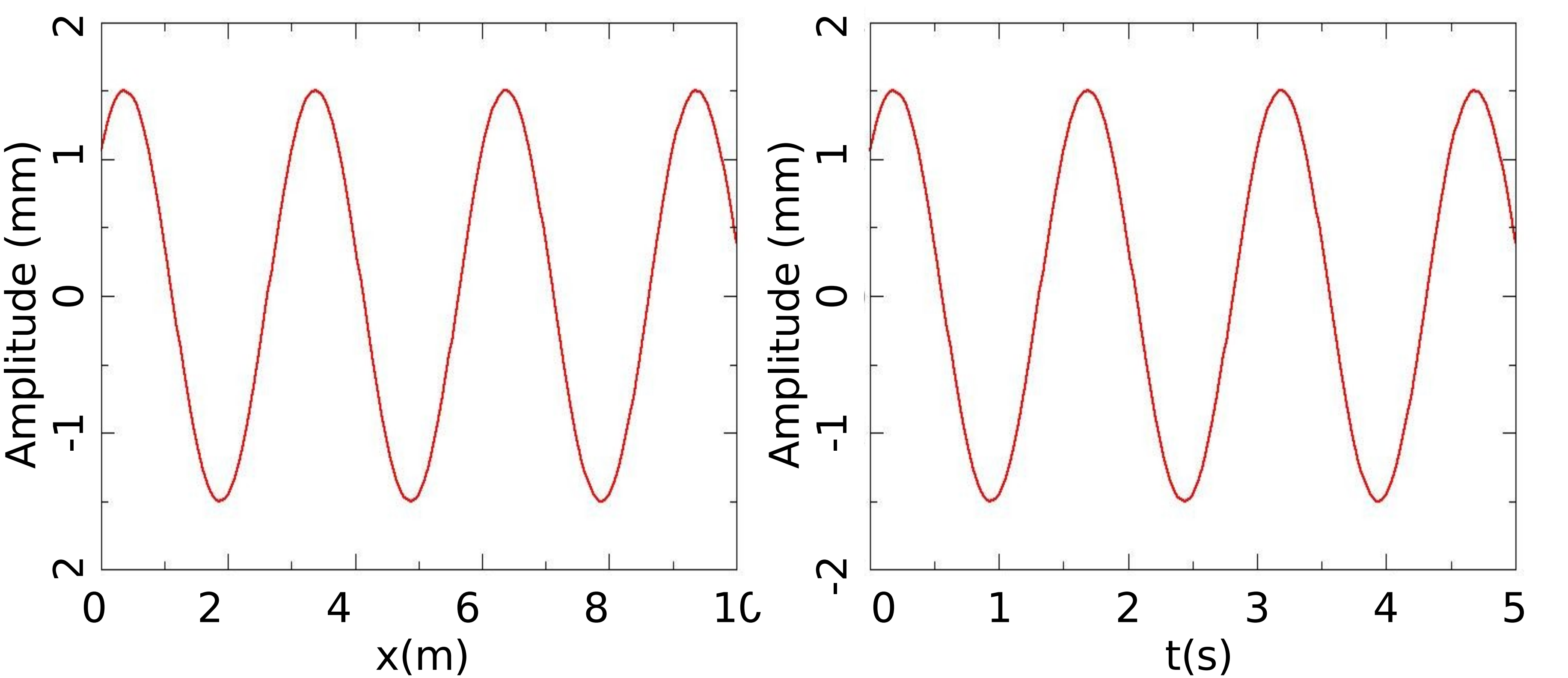
$$\lambda = 0.2\,\text{m} \qquad \text{This is a } \cos^2-\text{graph!} \qquad f = \frac{1}{T} = \frac{1}{2\text{ms}} = 500\text{Hz}$$

Use:

$$v = \lambda f \qquad \omega = 2\pi f \qquad \Rightarrow \qquad 10W = 2\pi^2 \mu \lambda f^3 y_m^2$$

$$\Rightarrow \qquad y_m = \sqrt{\frac{10W}{2\pi^2 \mu \lambda f^3}} = 3.2\,\text{mm}$$

HITT 1



The two graphs show a harmonic wave at t=0 in space and at x=0 in time.
What is the speed of the wave?

Graph 1: Wavelength $\lambda=3$m Graph 2: Period T=1.5s -> Frequency f=2/3Hz

Velocity: $v=\lambda f=3*2/3$ m/s=2m/s

Superposition principle

Formal: Let $y_1(x,t)$ and $y_2(x,t)$ be solutions of the wave equation. Then

$$y'(x,t) = c_1 y_1(x,t) + c_2 y_2(x,t)$$

is also a solution.

Proof:

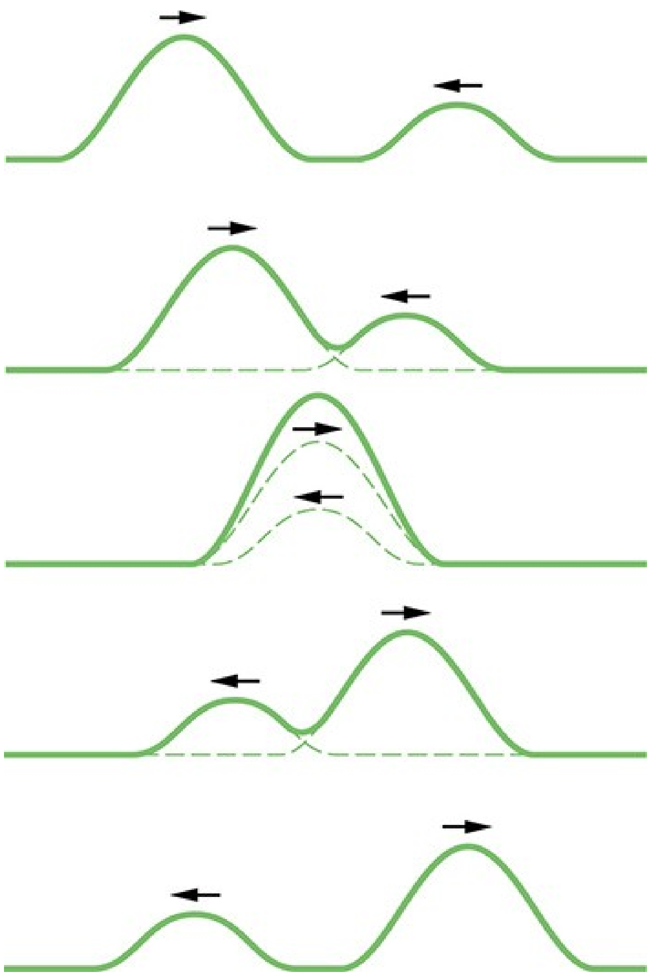
$$\frac{d^2y'}{dx^2} = c_1 \frac{d^2y_1}{dx^2} + c_2 \frac{d^2y_2}{dx^2} = \frac{c_1}{v^2}\frac{d^2y_1}{dt^2} + \frac{c_2}{v^2}\frac{d^2y_2}{dt^2} = \frac{1}{v_2}\frac{d^2(c_1y_1 + c_2y_2)}{dt^2} = \frac{1}{v^2}\frac{d^2y'}{dt^2}$$

What are the consequences?

If two waves travel in the same medium (along the same string), they add algebraically to a resultant net wave.

They travel independent of each other, don't influence each. In other words: The can be treated completely independent from each other (assuming that all approximations we used to derive the wave equation are still valid).

Superposition principle



Example:
Two waves propagating in opposite directions:

$$y_1(x,t) = y_{1m}h(x - vt)$$

$$y_2(x,t) = y_{2m}h(x + vt)$$

$$y'(x,t) = y_{1m}h(x - vt) + y_{2m}h(x + vt)$$

Note: This is not a single wave because it has two different velocities