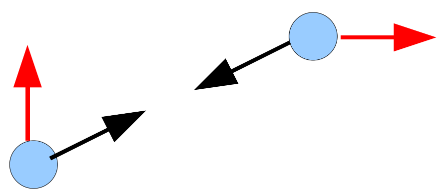
Chapter 9

How can we describe the motion of a real body?

Let's first look at:
Internal and external forces on a two particle system:



$$m_1\vec{a}_1=\vec{F}_{int,1}+\vec{F}_{ext,1}$$

$$m_1\vec{a}_2=\vec{F}_{int,2}+\vec{F}_{ext,2}$$

Use Newton's 3rd law:

$$\vec{F}_{int,1}=-\vec{F}_{int,1}$$

$$\Rightarrow \quad m_1\vec{a}_1+m_2\vec{a}_2=\vec{F}_{ext,1}+\vec{F}_{ext,2}=\vec{F}_{net}$$

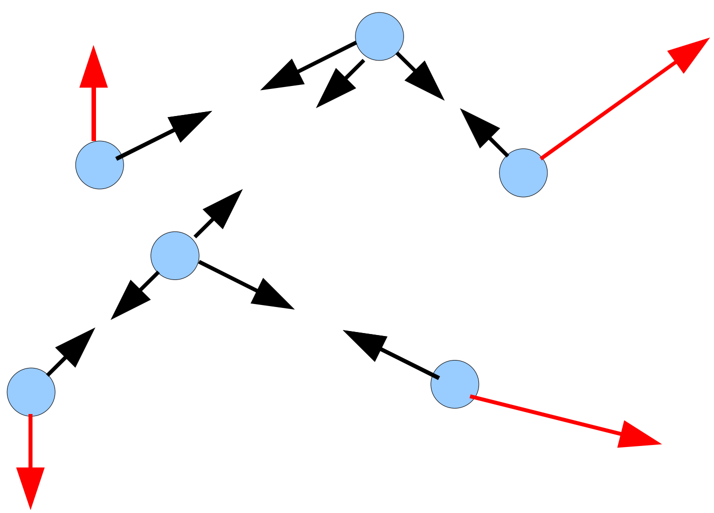
The sum over the accelerations of all particles forming the body weighted by their individual masses is equal to the sum over all external forces (= the net force).

Chapter 9

How can we describe the motion of a real body?

Multiple particles:

- Internal Forces between particles (atoms, molecules, ...) forming the body
- External forces acting on some or all particles



Recall Newton's 3rd law:
Actio=Reactio

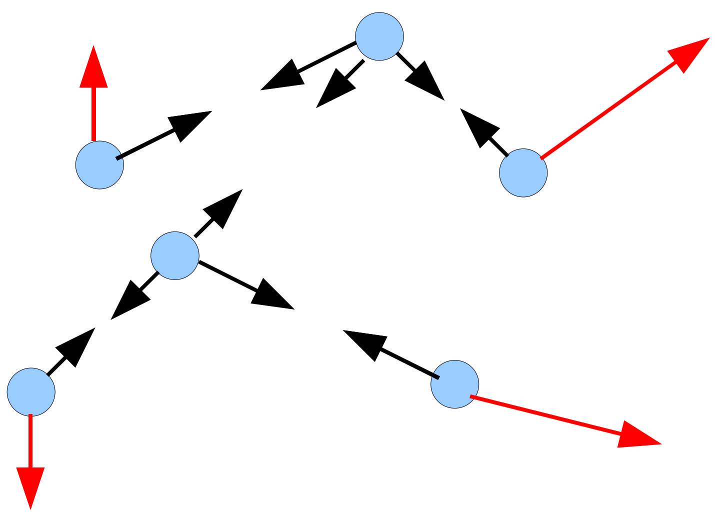
> Internal forces do **not** contribute to the net force
> The net force is the sum over all external forces

$$\vec{F}_{net} = \Sigma$$

Chapter 9

How can we describe the motion of a real body?

Can we find a specific location/point associated with the body or with the mass distribution that would allow us to describe the overall motion using Newton's 2nd law?



$$\vec{F}_{net} = M_{tot}\,\vec{a}_{com}$$

Does a point exist which would be accelerated by the net force like a point mass of mass M_{tot}?

We would call this point the center of mass (COM)

Chapter 9

Lets start with:

$$\vec{F}_{net} = m_1\vec{a}_1+m_2\vec{a}_2+...= (m_1+m_2+...)\ \vec{a}_{COM}$$

Integrate all accelerations up:

$$\Rightarrow \quad m_1\vec{v}_1+m_2\ \vec{v}_2+...= (m_1+m_2+...)\ \vec{v}_{COM}$$

Integrate all velocities up:

$$\Rightarrow \quad m_1\vec{r}_1+ m_2\ \vec{r}_2+...= (m_1+m_2+...)\ \vec{r}_{COM}$$

$$\vec{r}_{COM}=\frac{m_1\vec{r}_1+m_2\ \vec{r}_2+...}{(m_1+m_2+...)}$$

would be a point which fulfills:

$$\vec{F}_{net}= M_{tot}\ \vec{a}_{com}$$

Chapter 9

The book starts with: $\vec{r}_{COM} = \dfrac{m_1 \vec{r}_1 + m_2 \vec{r}_2 + \ldots}{(m_1 + m_2 + \ldots)}$

$$\Rightarrow \quad m_1 \vec{r}_1 + m_2 \vec{r}_2 + \ldots = (m_1 + m_2 + \ldots)\, \vec{r}_{COM}$$

Then differentiates twice:

$$\Rightarrow \quad m_1 \vec{v}_1 + m_2 \vec{v}_2 + \ldots = (m_1 + m_2 + \ldots)\, \vec{v}_{COM}$$

$$\Rightarrow \quad m_1 \vec{a}_1 + m_2 \vec{a}_2 + \ldots = (m_1 + m_2 + \ldots)\, \vec{a}_{COM} = \vec{F}_{net}$$

Center of Mass

$$\vec{r}_{COM} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2 + \ldots}{(m_1 + m_2 + \ldots)} = \frac{1}{M}\Sigma m_i\vec{r}_i$$

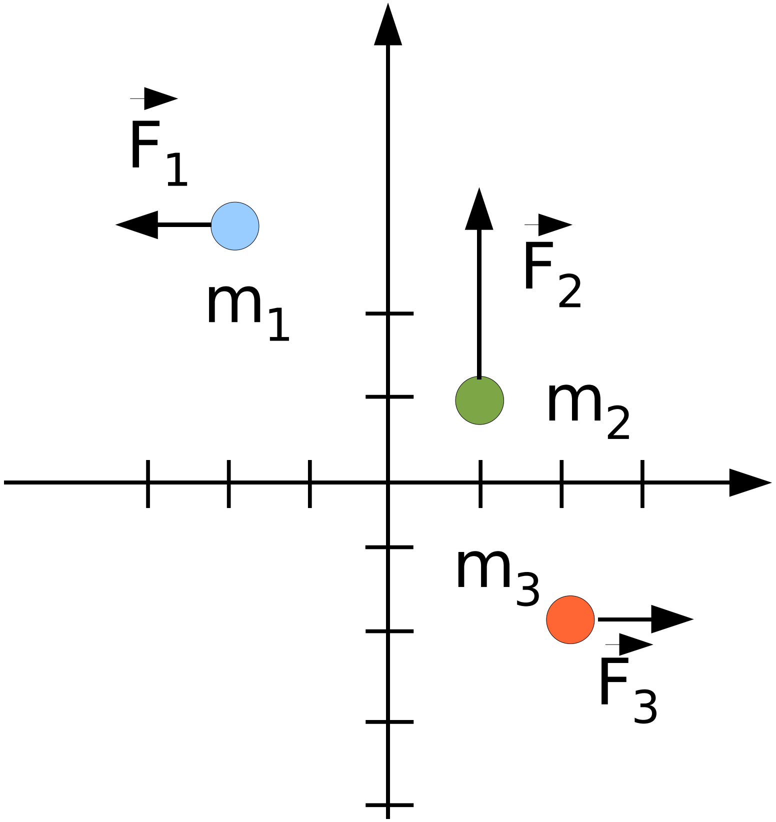
or written for the individual components and solid bodies:

$$x_{COM} = \frac{1}{M}\Sigma m_i x_i \quad \Rightarrow \quad x_{COM} = \frac{1}{M}\int x\, dm$$

$$y_{COM} = \frac{1}{M}\Sigma m_i y_i \quad \Rightarrow \quad y_{COM} = \frac{1}{M}\int y\, dm$$

$$z_{COM} = \frac{1}{M}\Sigma m_i z_i \quad \Rightarrow \quad z_{COM} = \frac{1}{M}\int z\, dm$$

Example



At t=0 all velocities = 0

m_1=2kg m_2=1kg m_3=3kg

$|F_1|$=1N $|F_2|$=2N $|F_3|$=1N

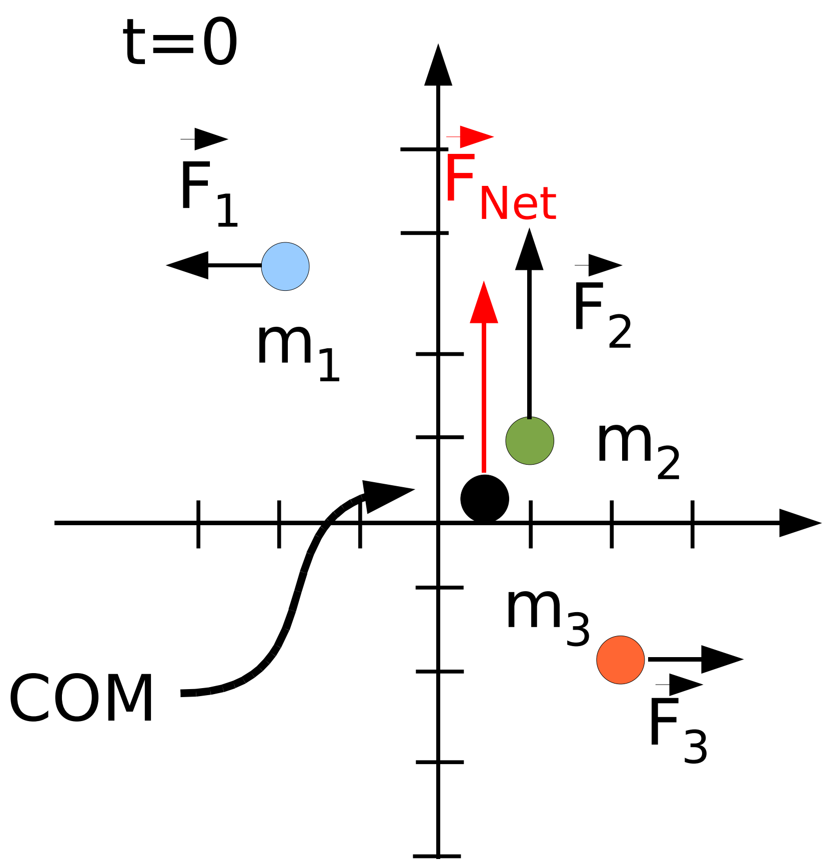
$$X_{com} = \frac{2*(-2)+1*1+3*2}{2+1+3} m = 0.5m$$

$$y_{com} = \frac{2*3+1*1+3*(-2)}{2+1+3} m = 0.165m$$

F_{net}=2N in +y direction

$F_{net} = Ma_{com}$

Example



$m_1=2kg$ $m_2=1kg$ $m_3=3kg$

$|F_1|=1N$ $|F_2|=2N$ $|F_3|=1N$

$$X_{com}=\frac{2*(-2)+1*1+3*2}{2+1+3}m=0.5m$$

$$y_{com}=\frac{2*3+1*1+3*(-2)}{2+1+3}m=0.165m$$

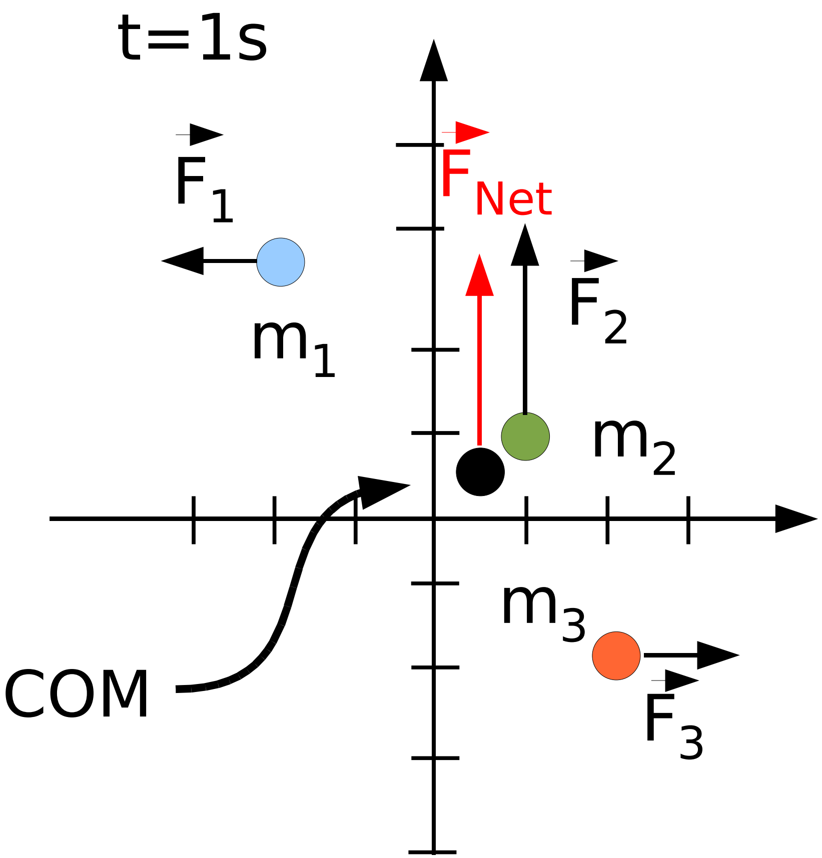
$F_{net}=2N\,\hat{j}$ use $F_{net}=Ma_{com}$

$\Rightarrow \vec{a}_{com}=\vec{F}_{net}/M \Rightarrow \vec{v}_{com}=\vec{a}_{com}t \Rightarrow \vec{r}_{com}=\vec{r}_{com,0}+0.5\vec{a}_{com}t^2$

at t=1s $\Rightarrow a_{com}=0.33\ m/s^2 \Rightarrow v_{com}=0.33m/s$

$\Rightarrow \vec{r}_{com}=0.5m\,\hat{i}+(0.165m+0.165m)\,\hat{j}=0.5m\,\hat{i}+0.33m\,\hat{j}$

Example



m_1=2kg m_2=1kg m_3=3kg

$|F_1|$=1N $|F_2|$=2N $|F_3|$=1N

$$X_{com}= \frac{2*(-2)+1*1+3*2}{2+1+3} m=0.5m$$

$$y_{com}= \frac{2*3+1*1+3*(-2)}{2+1+3} m=0.165m$$

$F_{net}=2N\ \hat{j}$ use $F_{net}=Ma_{com}$

➡ $\vec{a}_{com}=\vec{F}_{net}/M$ ➡ $\vec{v}_{com}=\vec{a}_{com}t$ ➡ $\vec{r}_{com}=\vec{r}_{com,0}+0.5\vec{a}_{com}t^2$

at t=1s ➡ $a_{com}=0.33\ m/s^2$ ➡ $v_{com}=0.33m/s$

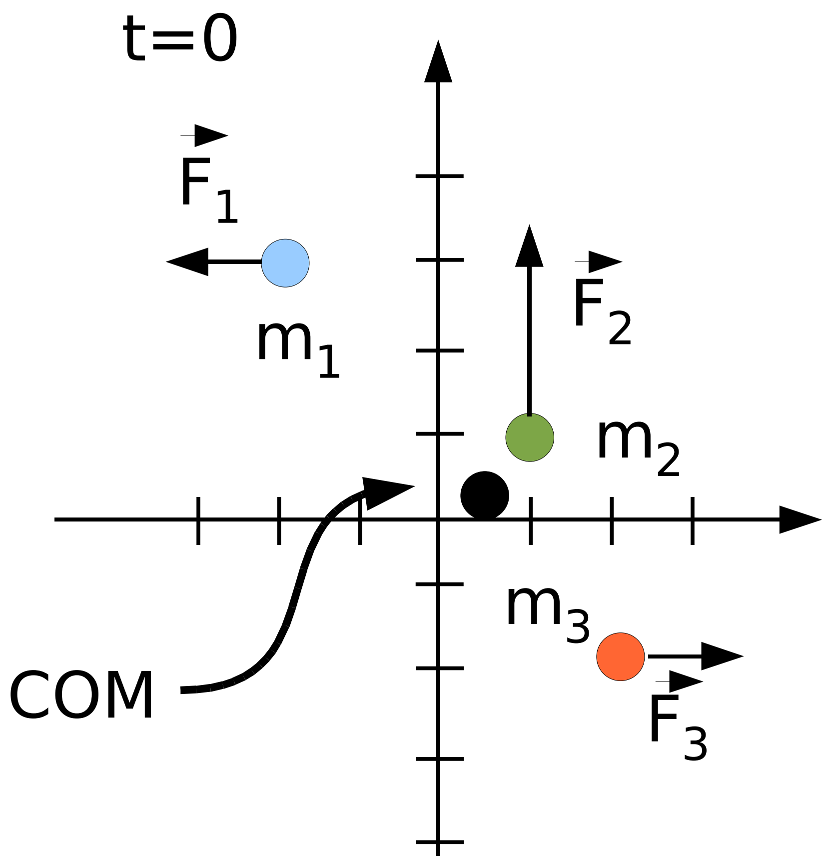
➡ $\vec{r}_{com}= 0.5m\ \hat{i} + (0.165m+0.165m)\ \hat{j} = 0.5m\ \hat{i} + 0.33m\ \hat{j}$

Example

Lets see if that is correct:

- Calculate the new position of each mass after 1s
- Calculate the new center of mass
- Has to be the same!

Example



Use: $\vec{r}_k(t)=\vec{r}_{k0}+\vec{v}_{k0}t+0.5\vec{a}_k t^2$

$m_1=2kg$ $m_2=1kg$ $m_3=3kg$

$|F_1|=1N$ $|F_2|=2N$ $|F_3|=1N$

$|a_1|=0.5$ $|a_2|=2$ $|a_3|=0.33$

all in m/s^2

at t=1s

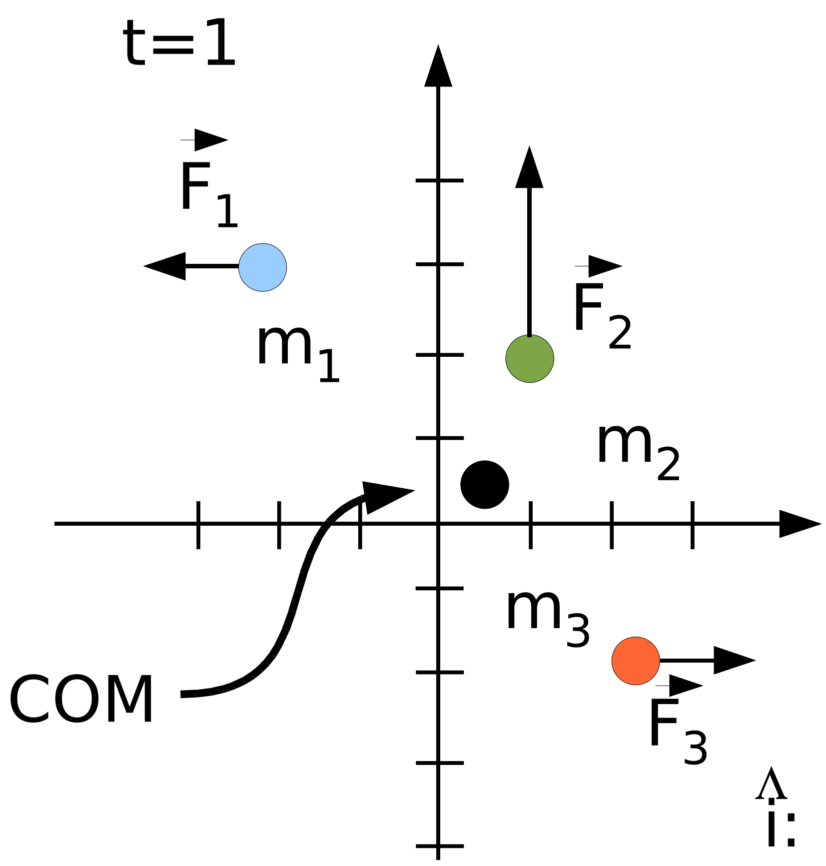
$\vec{r}_1=(-2-0.5*0.5)\,\hat{i}+3\,\hat{j} = -2.25\,\hat{i}+3\,\hat{j}$

$\vec{r}_2= 1\,\hat{i}+(1+0.5*2)\,\hat{j} = 1\,\hat{i}+2\,\hat{j}$

$\vec{r}_3= (2+0.5*0.33)\,\hat{i}-2\,\hat{j} = 2.16\,\hat{i}-2\,\hat{j}$

Example

Use: $\vec{r}_k(t)=\vec{r}_{k0}+\vec{v}_{k0}t+0.5\vec{a}_k t^2$



$m_1=2kg \quad m_2=1kg \quad m_3=3kg$

$\vec{r}_1 = -2.25\,\hat{i}+3\,\hat{j}$

$\vec{r}_2 = 1\,\hat{i}+2\,\hat{j}$

$\vec{r}_3 = 2.16\,\hat{i}-2\,\hat{j}$

$$\vec{r}_{COM}=\frac{1}{M}\Sigma m_i \vec{r}_i$$

$\hat{i}$: (-2.25*2+1*1+2.16*3)/M= 3/6=0.5m

$\hat{j}$: (3*2+2*1-2*3)/M = 2/6 = 0.33m

Works

Center of Mass

Two ways to calculate the trajectory of the center of mass:

- Complicated way:
  1. Calculate the position in time (trajectory) of each piece using the net force on each piece.
  2. Use these new positions to calculate the new position of the COM in time.
- Easy way:
  1. Calculate the center of mass for the initial mass distribution.
  2. Calculate the net force working on the system and use that to calculate the new position of the COM in time.

Center of Mass

$$x_{COM} = \frac{1}{M} \sum m_i x_i \quad \Rightarrow \quad x_{COM} = \frac{1}{M} \int x \, dm$$

$$y_{COM} = \frac{1}{M} \sum m_i y_i \quad \Rightarrow \quad y_{COM} = \frac{1}{M} \int y \, dm$$

$$z_{COM} = \frac{1}{M} \sum m_i z_i \quad \Rightarrow \quad z_{COM} = \frac{1}{M} \int z \, dm$$

Calculating the COM of an inhomogeneous, non-symmetric body can be very complicated.

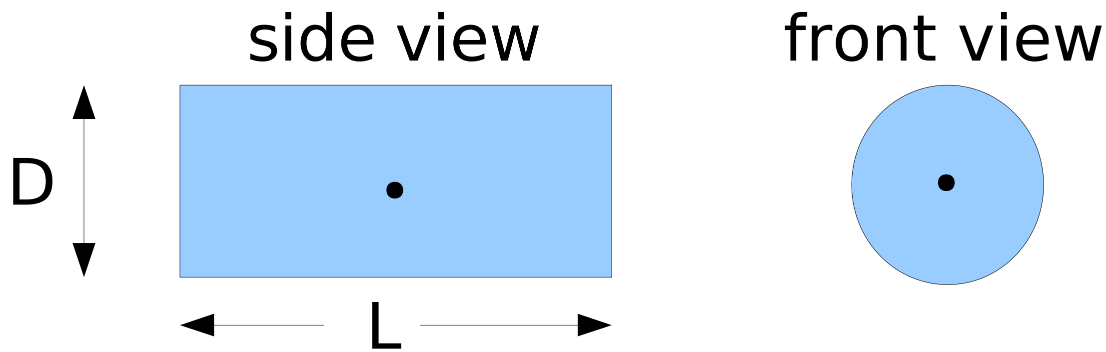
Typical exam and quiz problems use usually homogeneous bodies with one or more symmetries.

'The art of being smart'

Center of Mass

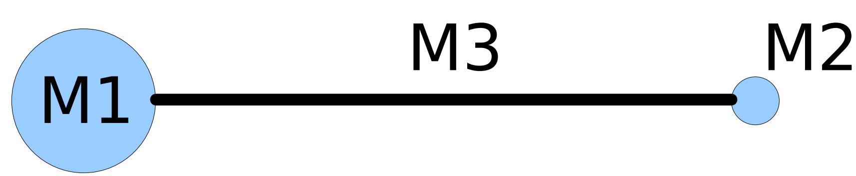
Examples:

- Homogeneous cyclinder



COM is certainly in the center

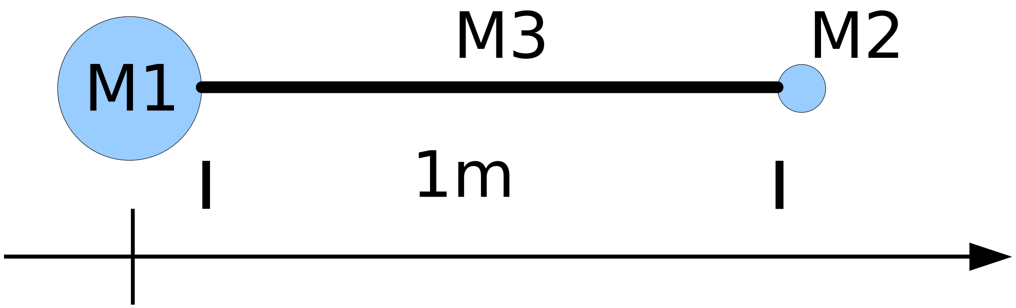
- Assembly of 2 spheres connected by a rod



General trick: Split the body up into symmetric pieces, calculate their COMs. The COM of the entire body is then identical to the COM of a distribution of point masses at the COMs of the pieces.

Center of Mass

- Assembly of 2 spheres connected by a rod



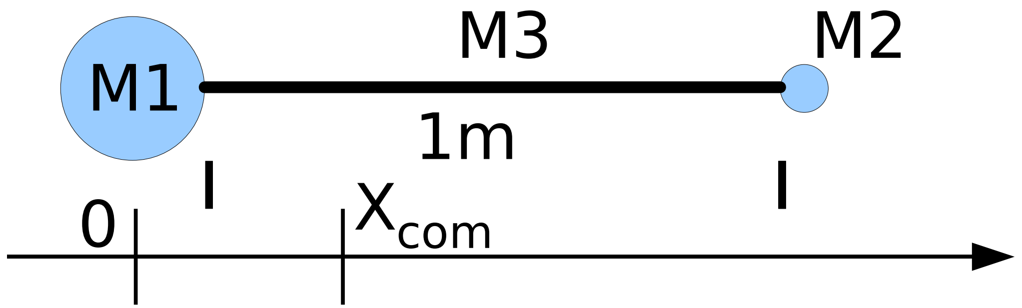
$$r_{com} = \frac{(M_1 r_{C1} + M_2 r_{C2} + M_3 r_{C3})}{M_1 + M_2 + M_3}$$

Parameters:
Sphere 1: Mass: 2kg, Radius: 0.1m (homogeneous)
Sphere 2: Mass: 1kg, Radius: 0.05m (hollow)
Rod: Mass: 0.1kg, Length: 1m

Choose coordinate system:
0 in center of sphere 1, x-axis parallel to rod.
-> COM in y and z dimension at y=0=z (Symmetry)

Center of Mass

- Assembly of 2 spheres connected by a rod



$$r_{com} = \frac{(M_1 r_{C1} + M_2 r_{C2} + M_3 r_{C3})}{M_1 + M_2 + M_3}$$

$M_1 = 2kg$, $R_1 = 0.1m$, $M_2 = 1kg$, $R_2 = 0.05m$, $M_3 = 0.1kg$, $L = 1m$

COM of Sphere 1: x=0
COM of Sphere 2: x=0.1m+1m+0.05m=1.15m
COM of Rod: x=0.5m+0.1m=0.6m

COM of Assembly:
X_{com}=(2kg*0+1kg*1.15m+0.1kg*1m)/(3.1kg) = 0.403m

Center of Mass

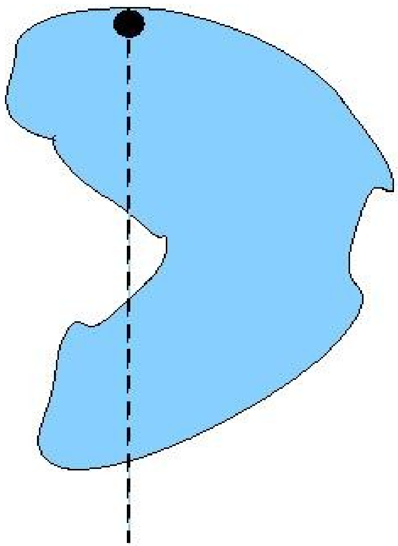
Experimental determination

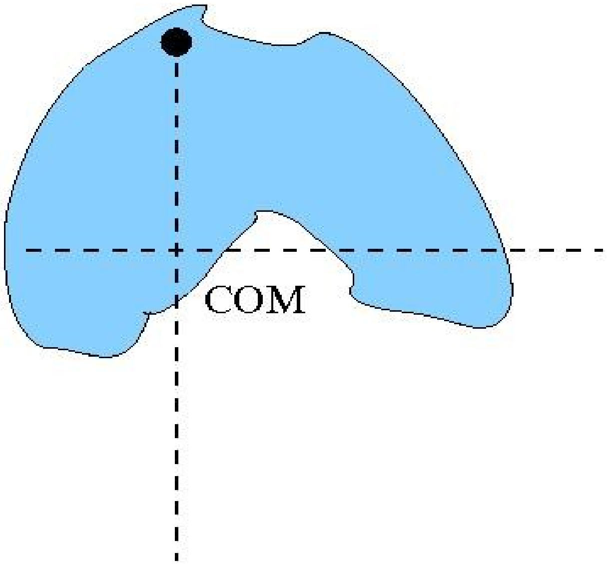
Idea: COM always below the suspension point when K=0.

Suspend at one point as a pendulum.

Draw a line downwards from suspension point.

Repeat with second suspension point.



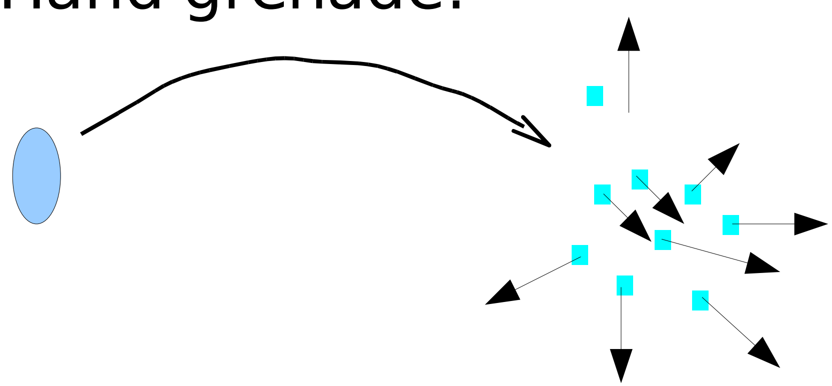


Center of Mass

What does $\vec{F}_{net} = M_{tot}\,\vec{a}_{com}$ actually means?

The COM will be accelerated by the net force acting on the body or mass distribution independent of any internal forces or what happens internally.

Example:
Hand grenade:



Although the grenade splits up into 100's of pieces, the COM continues to move as if the grenade never exploded

Center of Mass

Where is $\vec{F}_{net} = M_{tot}\,\vec{a}_{com}$ used?

Sports:
Basketball: Players who appear to stay afloat for a while
Long jump
Ballet
Gymnastics
...

Next time you watch sports in the O'Dome or on TV, think about it.

Linear Momentum

Linear momentum of a single particle:

$$\vec{p} = m\vec{v}$$

Newton's 2nd law in different form: $\vec{F}_{net} = \frac{d\vec{p}}{dt}$

Obvious: $\vec{F} = m\vec{a} = m\frac{d\vec{v}}{dt} = \frac{d(m\vec{v})}{dt} = \frac{d\vec{p}}{dt}$

The rate of change of the momentum of a particle is equal to the net force acting on the particle.

Motivation for using momentum:
Conservation law (next time)

Linear Momentum

Linear momentum of a system of particles:

$$\vec{P} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 + \ldots$$

$$\vec{P} = m_1\vec{v}_1 + m_2\vec{v}_2 + m_3\vec{v}_3 + \ldots = M\vec{v}_{com}$$

The linear momentum of a system of particles is equal to the product of the total mass M and the velocity of the center of mass.

Take the time derivative: $\vec{F}_{net} = \frac{d\vec{P}}{dt} = M\vec{a}_{com}$

The change in the linear momentum of a system of particles in time is equal to the applied net force. Note that this works for each component individually.

Linear Momentum

Take the time derivative: $\vec{F}_{net} = \frac{d\vec{P}}{dt} = M\vec{a}_{com}$

This leads directly to the conservation laws for linear momentum:

- If the component of the net (external) force is zero along an axis, then this component of the linear momentum is conserved.

- Alternative formulation: The component of the linear momentum which is perpendicular to the net force is conserved.