# Review for Exam 2

## Chapter 7

**Kinetic Energy:**

$$K = \frac{1}{2}mv^2$$

**Work:** Energy transferred to (positive) or from (negative) an object via a force acting on the object.

$$W = \int \vec{F} d\vec{s} \overbrace{=}^{\vec{F}\text{ constant}} \vec{F}\vec{d}$$

**Work and kinetic energy:**

$$\Delta K = K_f - K_i = W$$

This is the net work done by **all** forces combined (net force) applied to the particle (including gravity and friction).

Example:

A mass m=1kg is pushed by an applied force of 10N up a slope with angle $30^o$. The frictional coefficient is $\mu_K = 0.2$. How much net work is done on the mass after it travelled by 1m?

Forces:

1. Applied Force $F_A = 10\,\text{N}$

2. Gravity $F_G = -mg$

   (a) The component in the direction of the motion: $F_G \sin 30^0$

3. Frictional Force: $F_F = -\mu_K mg \cos\theta$

Gravity and Friction point against the displacement or against the applied force. So the net force is:

$$F_{net} = 10N - mg\sin 30^0 - \mu_K mg\cos 30^0 = 3.4\,\text{N}$$

The net work:

$$W_{net} = F_{net} d = 3.4\,\text{J}$$

By how much did the kinetic energy change?

$$\Delta K = K_f - K_i = W_{net} = 3.4\,\text{J}$$

What is the velocity after 1m if the initial velocity was $1\text{m/s}$?

$$\Delta K = \frac{1}{2}m(v_f^2 - v_i^2) \Rightarrow v_f = \sqrt{\frac{2\Delta K}{m} + v_i^2} = 2.8\,\text{m/s}$$

We could have calculated this also via Newton:

$$F_{net} = ma \qquad \Rightarrow \qquad a = \frac{F_{net}}{m} = 3.4\frac{\text{m}}{\text{s}^2}$$

and

$$v_f^2 = v_i^2 + 2a(x_f - x_i) = 1\frac{\text{m}^2}{\text{s}^2} + 2 \times 3.4\frac{\text{m}}{\text{s}^2}1\text{m} = 7.8\frac{\text{m}^2}{\text{s}^2}$$

$$v_f = \sqrt{v_f^2} = 2.8\text{m/s}$$

How much work was done by the applied force?

$$W_a = F_a d = 10\text{N} \cdot 1\text{m} = 10\,\text{J}$$

How much work was done by gravity? (Only the component $\|$ to displacement counts (here anti-parallel $\rightarrow W < 0$)

$$W_g = -F_g \sin 30^0 d = -4.9\,\text{J}$$

Where did this energy go?

Into potential energy (Chapter 8):

$$\Delta U = -W = 4.9\,\text{J}$$

increased the potential energy (went up).

How much work was done by the frictional force?

$$W_f = F_f d = -\mu_K mg \cos 30^o d = -1.7\,\text{J}$$

extracts energy from the object which would have otherwise be stored in the object as kinetic energy.

Where did this energy go? Thermal energy, heated up the mass and the ramp.

$$10\,\text{J} - 4.9\,\text{J} - 1.7\,\text{J} = 3.4\,\text{J}$$

**Spring Force:**

$$\vec{F}_S = -k\vec{d} \qquad \text{we do only 1-D:} \quad F = -kx$$

where $\vec{d}$ or $x$ is the displacement of the spring's free end from its relaxed position.

Work done by a spring force:

$$W = \int_{x_i}^{x_f} F\,dx = \frac{1}{2}kx_i^2 - \frac{1}{2}kx_f^2 \qquad \text{for } x_i = 0 \qquad W = -\frac{1}{2}kx^2$$

Example: Same mass pushed up a hill but now we add a spring with $k = 1\text{N/m}$ to it that is attached to the mass and pulls the mass up when $x < 0.6\,\text{m}$ and pushes back when $x > 0.6\,\text{m}$ (Spring is relaxed when mass moved by $0.6\,\text{m}$).

Net force is now position dependent.

$$F_{net}(x) = F_a - mg\cos 30^0 - \mu_K mg\sin 30^0 - k(x - 0.6\,\mathrm{m})$$

Spring Force $> 0$ when $x < 0.6\,\mathrm{m}$, changes sign at $x = 0.6\,\mathrm{m}$.

The net work is now:

$$W_{net} = \int_{x_i=0}^{x_f=1\mathrm{m}} F_{net}(x)dx =$$

$$(F_a - mg\cos 30^o - \mu_K mg\sin 30^o)x|_0^{1\mathrm{m}} - k(\frac{1}{2}x^2 - 0.6\,\mathrm{m}x)|_0^{1\mathrm{m}}$$

$$= 3.4\,\mathrm{J} + 0.1\,\mathrm{J} = 3.5\,\mathrm{J}$$

Other way to get this:

$$W_{net} = F_{avg}d$$

averaged force over the distance. Only spring force is changing during motion. Goes from $+0.6\,\mathrm{N}$ to $-0.4\,\mathrm{N}$. The average force between $x = 0.2\,\mathrm{m}$ and $x = 1\,\mathrm{m}$ is 0 (goes from $+0.4\,\mathrm{N}$ to $-0.4\,\mathrm{N}$). The average force on the first $20\,\mathrm{cm}$ is $0.5\,\mathrm{N}$.

$$W_{spring} = 0.5\,\mathrm{N} \cdot 0.2\,\mathrm{m} = 0.1\,\mathrm{J}$$

**Chapter 8**

**Conservative Forces**

Gravitational Force and Spring Force (Elastic Force)

Non-conservative Forces: Friction

Closed loop criteria: Net work moving a particle around a closed loop is zero.

**Potential Energy:**

Work done by conservative force changes potential energy:

$$\Delta U = -W$$

Gravity:

$$\Delta U = mg\Delta y = mgh$$

Spring:

$$\Delta U = \frac{1}{2}k(x_f^2 - x_i^2)$$

Energy Conservation:

Isolated system (no external forces) and only conservative forces:

$$\Rightarrow \qquad E_{mec} = K_i + U_i = K_f + U_f$$

Potential Energy Curves:

$$F(x) = -\frac{dU(x)}{dx}$$

Conservative systems:

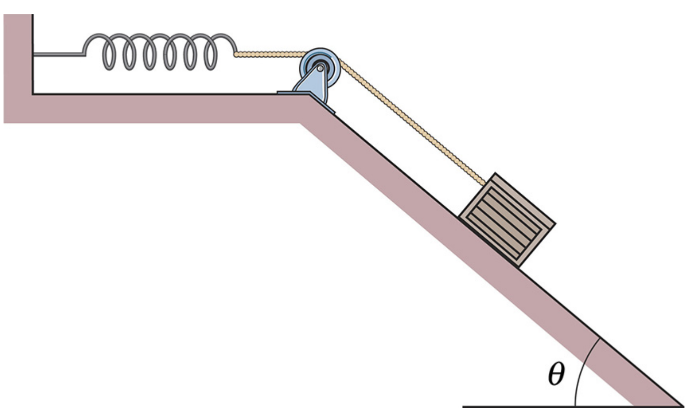
$$K(x) = E_{mec} - U(x)$$

If non-conservative rest goes into thermal energy: $E_{th} = F_f d$ (see last Example)

Example (28)

A $2.0\,$kg breadbox on a frictionless incline of angle $\theta = 40^0$ is connected, by a cord that runs over a pulley, to a light spring ($k = 120\,$N$/$m). The box is released from rest when the spring is unstretched. Assume pulley is massless (no angular moment of inertia!) and frictionless.

What is the speed of the box when it has moved $d = 10\,$cm down the incline?



**Energy conservation:**

Potential Energy of box: $U_i^B = 0 \qquad U_f^B = -mgh = -mgd\sin 40^o$

Potential Energy of spring: $U_i^S = 0 \qquad U_f^S = \frac{1}{2}kd^2$

Kinetic Energy of box: $K_i = 0 \qquad K_f = \frac{1}{2}mv_f^2$

$$K_f + U_f^S + U_f^B = 0$$

Solve for $v$:

$$v_f = \sqrt{\frac{2}{m}(mgd\sin 40^o - \frac{1}{2}kd^2)} = 0.81\,\frac{\text{m}}{\text{s}}$$

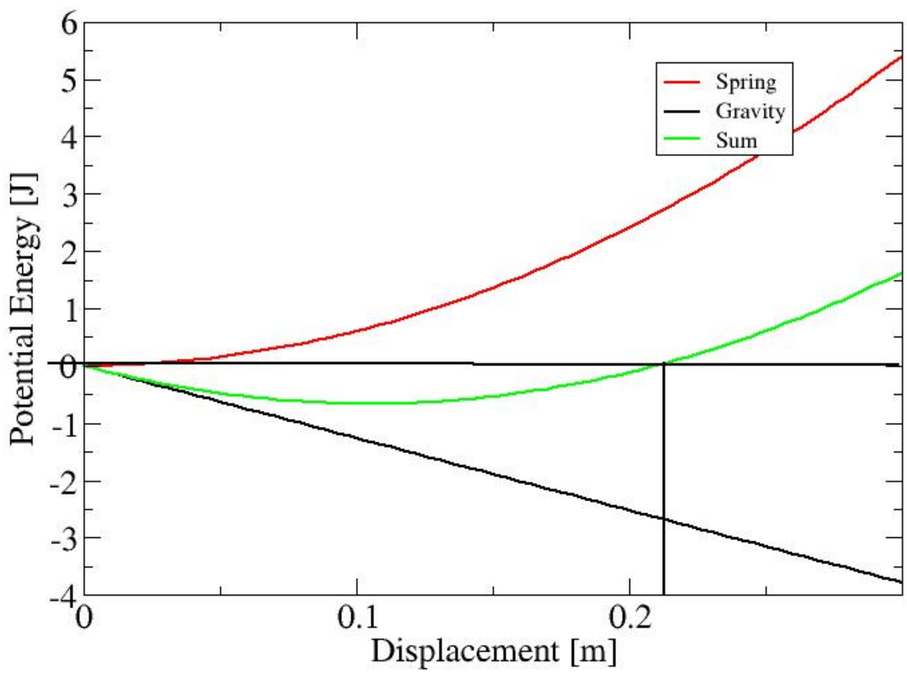
Where does the box stop?

Mathematically: where the term under the $\sqrt{}$ becomes 0.

$$mgd\sin 40^0 = \frac{1}{2}kd^2 \Rightarrow d = \frac{2mg\sin 40^0}{k} = 0.21\,\text{m}$$

How does the potential energy curve of this system look like?

$$U(x) = \frac{1}{2}kx^2 - mgx\sin 40^o$$




Gives us the turning point and also the stable equilibrium point where it would come to rest if friction or drag slows the motion down.

If we would do the same experiment on the moon, how would this change?

The moons gravitational pull is only 1/6 of the pull on earth. What would change in the above equations?

$g_{Moon} = \frac{g_{Earth}}{6} \approx 1.64\,\mathrm{m/s^2}$, the slope of the gravitational potential would be shallower and the equilibrium point would shift further to the left.

## Chapter 9

## Center of Mass and linear momentum

The basic motivation for ***center of mass*** is that we will later split up the motion of a solid body (or collection of point particles) into a center of mass motion and a rotation around an axis going through the center of mass.

$$\vec{r}_{Com} = \frac{1}{M}\int \vec{r}dm \overbrace{=}^{\text{for particles}} \frac{1}{M_{tot}}\sum m_i\vec{r}_i$$

Newton's 2nd law:

$$\vec{F}_{Net} = m\vec{a}_{com} \qquad \text{all internal forces cancel}$$

Linear momentum:

$$\vec{p} = m\vec{v} \qquad \text{for single particle}$$

$$\vec{P} = M\vec{v}_{com} \qquad \text{for system of particles or solid body}$$

$$\vec{F}_{net} = \frac{d\vec{P}}{dt} \qquad \vec{P} \text{ is constant if } \vec{F}_{net} = 0$$

**Impuls** caused by a Force acting during a certain time interval:

$$\vec{J} = \int_{t_i}^{t_f} \vec{F}(t)dt = \vec{F}_{avg}\Delta t = \Delta\vec{p}$$

equal to change in momentum.

In daily life, impulse is often seen as something that happens instantaneously, unresolved in time:

- Someone makes an impulsive decision: Not derived by weighting arguments, not transparent how she/he came to this decision.

It is used in physics in a very similar way. The body will undergo an abrupt (short time scale) change in momentum and we are usually either not able to resolve (or just not interested in) the details of the process. This is especially useful when the details don't change the outcome.

Note also: The involved masses don't change their positions (significantly) during the time the force acts. The involved masses only change the direction of the propagation (change in momenta). All this is only true in first order; looking into the details of an egg shattering on the floor is possible and could be very interesting, but wouldn't change the outcome. Keep that in mind when looking at problems.

Examples:

- Egg shattering by the kitchen floor. The kitchen floor applies a large but short in duration force on the egg.

- Two billiard balls hitting each other.

- Two cars colliding.

In all these cases:

- Gravity, the normal force, friction are forces which act over long time scales while the involved masses change their positions significantly.

Checking your understanding:

A paratrooper whose chute fails to open lands in snow; he is hurt slightly. Had he landed on bare ground his stopping time would be 30 times shorter and the collision lethal. Does the presence of the snow increase, decrease, or leave unchanged the values of

- the paratroopers change in momentum?
- the impulse stopping the paratrooper?
- the force stopping the paratrooper?

What is the impulse on the paratrooper?

Mass $m = 100\,\text{kg}$, Velocity: $v_f = 0 \qquad v_i = -56\,\text{m/s}$ (terminal speed, downward).

$\vec{J} = \Delta\vec{p} = \vec{p}_f - \vec{p}_i = m(v_f - v_i) = 5600\,\text{kg m/s} \qquad$ upward

The other equation for impulse:

$$\vec{J} = \int_{t_i}^{t_F} \vec{F}(t)dt = \vec{F}_{avg}(t_f - t_i) = \vec{F}_{avg}\Delta t$$

Can be used to calculate average forces or impulses if you know the duration of the impulse.

Lets assume the paratrooper slowed down in $\Delta t_1 = 0.1\,\text{s}$. This can be used to get the average:

$$\text{acceleration} \qquad a_{avg,1} = \frac{\Delta v}{\Delta t_1} = 560\frac{\text{m}}{\text{s}^2}$$

$$\text{force} \qquad F_{avg,1} = m a_{avg,1} = 56000\,\text{N}$$

Can you calculate from this the depths of the snow through which the paratrooper fell? No!

The same average accelerations and average forces over the same time interval generate different distances.

Case 1: We have a constant acceleration during $\Delta t$:

$$a(t) = a_{avg}$$

Then

$$x(\Delta t) = \frac{1}{2} a_{avg} \Delta t^2 + v_i \Delta t = -2.8\,\text{m}$$

Case 2: We have a constant acceleration of:

$$a(t) = 2a_{avg}$$

during the first half of $\Delta t$ and zero acceleration during the second half of $\Delta t$. Gives the same final velocity $(v_f = 0)$ and average acceleration but:

$$x(\Delta t) = \frac{1}{2}(2a_{avg}) \left(\frac{\Delta t}{2}\right)^2 + v_i \frac{\Delta t}{2} = -1.4\,\text{m}$$

So be careful with this.

**Conservation of momentum: No external forces.**

Internal forces don't contribute to the net force when we look at center of mass motions. (Compensate each other: Actio=Reactio)

$$\vec{F}_{net} = \frac{d\vec{P}}{dt} \qquad \text{if} \qquad F_{x,net} = 0 \qquad \Rightarrow \qquad P_x \text{ is conserved}$$

$P_x$ is total linear momentum in x-direction. Applicable to Collisions, Explosions, Rockets

Collisions:

- Elastic Collisions: Energy is conserved

- Inelastic Collisions: Energy is not conserved

– colliding bodies stick together: Completely inelastic collision. $v_f$ is equal for all bodies and $v_f = v_{Com}$. Helps to solve equations. Type of collision where as much kinetic energy as possible is transfered into other energy forms (thermal or elastic deformation)

– colliding bodies don't stick together: Outcome can not be calculated from initial conditions.

Example:

Two trains on a track, train A has mass $m_1 = 10^6$kg moving west with $v_1 = 40\,\text{m/s}$ velocity, train B has mass $m_2 = 5 \times 10^5$kg moving west with $v_2 = 60\,\text{m/s}$. They will undergo a completely inelastic collision (stuck together, trains do this in most collisions). What is the velocity of the trains after collision?

$$P_{final} = (m_1 + m_2)v_f = m_1 v_1 + m_2 v_2 = P_i$$

$$v_f = \frac{10^6 \cdot 40 + 5 \times 10^5 \cdot 60}{1.5 \times 10^6} \frac{\text{m}}{\text{s}} = 46.67 \frac{\text{m}}{\text{s}} = v_{Com}$$

How much kinetic energy is tranformed into thermal energy damaging the trains?

$$E_{th} = E_{kin.1} + E_{kin,2} - E_{kin,f}$$

$$E_{th} = \frac{1}{2} m_1 v_1^2 + \frac{1}{2} m_2 v_2^2 - \frac{1}{2}(m_1 + m_2) v_{Com}^2 = 6.67 \times 10^7 \text{J}$$

It is sometimes useful to think about this in a coordinate system where the center of mass is at rest:

$$w_{Com} = 0 \qquad \text{Changed from} \qquad v_{Com} = 46.67\,\frac{\text{m}}{\text{s}}$$

So in that system ($w = v - v_{Com}$):

$$w_1 = -6.67\,\frac{\text{m}}{\text{s}} \qquad w_2 = 13.33\frac{\text{m}}{\text{s}}$$

The two trains stand still after collision (no kinetic energy anymore):

$$w_f = \frac{10^6 \cdot (-6.67) + 5 \times 10^5 \cdot 13.33}{1.5 \times 10^6}\frac{\text{m}}{\text{s}} = 0 = w_{com}$$

The thermal energy is then:

$$E_{th} = E_{kin,1} + E_{kin,2} = \frac{1}{2}m_1 w_1^2 + \frac{1}{2}m_2 w_2^2 = 6.67 \times 10^7\text{J}$$

has to be the same.

Reverse complete inelastic collisions and we have an explosion:

- Center of mass motion is 'conserved' (described by external forces like gravity)

- Momentum is conserved during the explosion

Typical problem:

Some object flies under the influence of gravity, explodes during free fall:

– Center of mass follows standard free-fall parabola

– Transfer into a coordinate system that co-moves with the center of mass

– Calculate the velocities of all pieces following conservation law ($\vec{P}_{Net} = 0$ in that coordinate system)

– Add the center of mass velocity to each velocity to get the velocities with respect to ground.

Other example:

A hammer hits a nail. The weight of the hammer head is $m_1 = 0.5\,$kg, the speed of the hammer head when it strikes the nail is $v_1 = 200\,$m/s. The head then bounces back with a speed of $v_2 = 100\,$m/s (opposite direction). The nails mass is $m_2 = 5g$. If the frictional force between the nail and the wood is $F_f = 10^4$N, how far will the nail go into the wood?

Step 1: Need to know the nails initial velocity from conservation of linear momentum during the stride.

Step 2: Then we have a constant force decelerating the nail with a constant acceleration. This drives the velocity back to zero. Can calculate the time this takes.

Step 3: Once we have the time, we can calculate the distance.

Step 1:

$$p_{nail} = p_{hammer,1} - p_{hammer,2} = m_1(v_1 - v_2)$$

$$p_{nail} = 0.5\text{kg} \cdot (200\frac{\text{m}}{\text{s}} - (-100\frac{\text{m}}{\text{s}}) = 150\,\text{kg}\frac{\text{m}}{\text{s}}$$

$$\Rightarrow v_{nail} = \frac{p_{nail}}{m_{nail}} = \frac{150}{5 \times 10^{-3}}\frac{\text{m}}{\text{s}} = 30000\frac{\text{m}}{\text{s}}$$

Step 2:

$$F_f = m_{nail}a_f \Rightarrow a_f = \frac{10^8}{5 \times 10^{-3}}\frac{\text{m}}{\text{s}^2} = 2 \times 10^{10}\frac{\text{m}}{\text{s}^2}$$

$$\Delta t = \frac{v_{nail}}{a_f} = 1.5\,\mu\text{s}$$

Step 3: Acceleration is in opposite direction than velocity (decceleration):

$$d = \frac{1}{2}a_f\Delta t^2 - v_{nail}\Delta t = 2.25\,\text{cm}$$

Rotation and Rolling:

Equation of motions are identical to the equations used for linear motion if we use the following replacements:

$$\vec{r} - \vec{r}_0 \Rightarrow \theta - \theta_0 \qquad \vec{v} \Rightarrow \vec{\omega} \qquad \vec{a} \Rightarrow \vec{\alpha}$$

$$m \Rightarrow I \qquad \vec{F} \Rightarrow \vec{\tau} \qquad \vec{p} \Rightarrow \vec{l}$$

Virtually identical conservation laws:

$$\frac{d\vec{P}}{dt} = \vec{F}_{net} \Rightarrow \frac{d\vec{L}}{dt} = \vec{\tau}_{net}$$

We distinguish between three different types of motion:

1. Linear

2. Rotation

3. Rolling = Linear+Rotation

**Rotation:**

Circular motion of a solid body around a fixed axis.

Rotational inertia:

$$I = \int r^2 dm \qquad r: \text{ perpendicular distance to rot. axis}$$

Further away from axis increases rotational inertia ('the mass of rotation').

Parallel axis theorem: $I = I_{com} + Mh^2$

Circular motion variables are related to angular motion variables via geometry:

$$ds = r d\alpha \qquad v_t = r\omega \qquad a_t = r\alpha$$

Newton's second law for rotation:

$$\vec{\tau} = I\vec{\alpha} \qquad \text{Compare with: } \vec{F} = m\vec{a}$$

Kinetic Energy of rotation:

$$K = \frac{1}{2} I \omega^2 \qquad \left( = \frac{1}{2} \int v^2 dm = \frac{1}{2} \int r^2 \omega^2 dm \right)$$

Nothing else than the kinetic energy of each mass element moving with a speed which is set by the rotational motion.

Problem 10.79: A thin uniform rod has a length of $2$ m and can pivot about a horizontal, frictionless pin through one end. It is released from rest at angle $\theta = 40^o$ above the horizontal. Use the principle of conservation of energy to determine the angular speed of the rod as it passes through the horizontal position.

Final energy is rotational kinetic energy:

$$K = \frac{1}{2} I \omega^2 \qquad I = \frac{1}{12} m L^2 + m \frac{L^2}{4} = \frac{1}{3} m L^2$$

Work: $W = \tau \Delta\theta$

The torque is generated by gravity which attacks at the center of mass

$$\tau = mg \cdot \frac{L}{2} \qquad \Delta\theta = 40^o \frac{2\pi}{360^o} = 0.7\,\text{rad}$$

$$mg \frac{L}{2} \Delta\theta = \frac{1}{2}\frac{1}{3} m L^2 \omega^2 \qquad \Rightarrow \qquad \omega = \sqrt{3 \frac{g}{L} \Delta\theta} = 3.2 \frac{\text{rad}}{\text{s}}$$

**Rolling=Rotation+Translation:**

Two types of kinetic energy:

$$K_{rot} = \frac{1}{2} I_{Com} \omega^2 \qquad K_{Trans} = \frac{1}{2} m v_{Com}^2$$

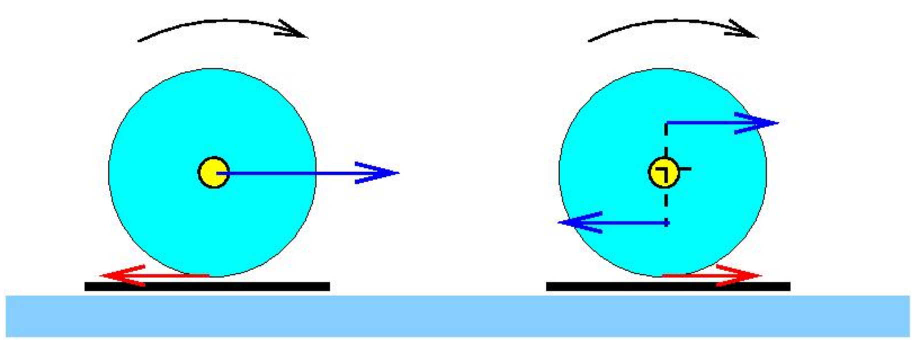
This is the sum over all kinetic energies of all mass elements building up the rolling mass.

Without slipping:

$$r\Delta\theta = \Delta s \qquad r\omega = v \qquad r\alpha = a$$

Couples the center of mass motion and the rotational motion.

Rolling w/o slipping: Requires friction.



**Two different situations:**

1. Apply a force $F$ to a mass that can roll. Friction (or tension of a string in for example: Yo-yo) reduces center of mass acceleration and starts the rotation:

$$ma_{com} = F - f_S \qquad \tau = -rf_S = I\alpha \qquad r\alpha = a_{Com}$$

2. Apply a torque $\tau$ to a mass that can roll. Friction reduces the rotation and starts the center of mass acceleration:

$$\tau - f_S r = I\alpha \qquad ma_{com} = f_S \qquad r\alpha = a_{com}$$

Note:

- The direction of the frictional force changes in the two situation.

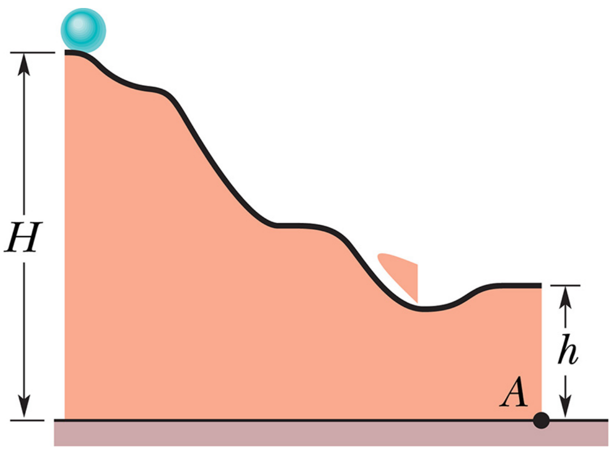
- 3 unknowns ($f_S$, $a_{Com}$, $\alpha$) and 3 equations.

Examples:

1. Rolling down a ramp: $F = mg\sin\theta$.

2. A torque applied to a wheel

Problem 11.11:




Smooth roll from Height $H = 6.0\,\text{m}$ to height $h = 2.0\,\text{m}$. How far from A does the ball land?

What do we need to get the distance? We need the center of mass velocity when the solid ball leaves the ramp.

What can we use to get the velocity? The details of the rolling motion are impossible to get as the details of the ramp are unknown. Leaves energy conservation.

Initial Energy: Potential only $U = mg(H-h)$.

Energy at end of ramp: Kinetic Energy:

$$K = 0.5I\omega^2 + 0.5mv_{com}^2$$

with

$$I = \frac{2}{5}mR^2 \qquad \omega = \frac{v_{Com}}{R} \qquad K = \frac{1}{5}mv_{com}^2 + \frac{1}{2}mv_{com}^2 = \frac{7}{10}mv_{com}^2$$

and

$$v_{com} = \sqrt{\frac{10}{7}g(H-h)} = 7.5\,\frac{\text{m}}{\text{s}}$$

Ramp is horizontal:

$$y(t_0) = 0 = h - \frac{1}{2}gt_0^2 \qquad \Rightarrow \qquad t_0 = \sqrt{\frac{2h}{g}} = 0.64\,\text{s}$$

$$x(t) = v_{com}t_0 = 4.8\,\text{m}$$

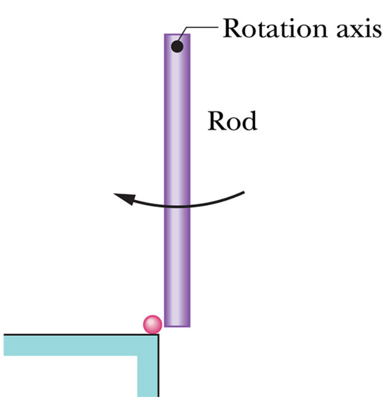
**Angular moment:**

$\vec{l} = \vec{r} \times \vec{p} \qquad \vec{L} = \sum d\vec{l}_i = \sum \vec{r}_i \times \vec{p}_i$ Multiple particles

$\vec{L} = I\vec{\omega}$ Body with rotational inertia $I$

Conserved when net torque is zero (works for each component).

Example: 11-61:




Rod: Length $l = 0.6$ **m,** mass $m_1 = 1.0$ kg, inertia: $I_0 = 0.12$ kg m$^2$. As the rod swings through its lowest point with $\omega_0 = -2.4$ rad/s, a mass of $m_2 = 0.2$ kg sticks to the rod. What is $\omega_1$ just after the collision?

We don't have any external torques $\Rightarrow$ Total angular momentum is conserved:

$$I_0\omega_0 = I_1\omega_1$$

What is $I_1$?

$$I_1 = I_0 + \overbrace{m_2 l^2}^{\text{Contribution from } m_2} = 0.19\,\text{kg m}^2$$

Gives

$$\omega_1 = \frac{I_0\omega_0}{I_1} = 1.5\,\text{rad/s}$$