Random Matrix Theory and the Scaling Theory of Localization

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We consider the most probable value of conductance of a disordered quantum conductor in the framework of the random matrix theory developed earlier. Analytic calculations are possible in the metallic as well as strongly localized regimes. We make a simple assumption on the eigenvalue density, as suggested by numerical work, and explore the consequences of a one-parameter distribution. A key result of the one-parameter scaling theory, that there is true metallic behavior only in dimensions greater than two, is recovered naturally.

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The one-parameter scaling theory of localization has been enormously successful in describing a variety of weak-localization phenomena in disordered quantum conductors.[1] However, the presence of universal conductance fluctuations in metals[2] raises the issue of how to reconcile the non-self-averaging behavior characteristic of the conductance g with the smooth scaling behavior of g assumed in the localization theory. We proposed earlier,[3] based on a conjecture by Imry,[4] that the theory of random multiplicative transfer matrices, together with a global maximum-entropy hypothesis, provides a natural framework to study the distribution of g in terms of the distribution of the eigenvalues of the transfer matrices.[5] In particular, we showed that the eigenvalue distribution factorizes naturally into a universal term which comes from symmetry considerations alone and describes the conductance fluctuations, and a parameter-dependent term which characterizes the global eigenvalue density. The one-parameter scaling theory can then be reformulated in the presence of these fluctuations by demanding that this eigenvalue density be a function of a single parameter describing the degree of disorder, size, dimension, Fermi energy, etc., of the system. Independent numerical work strongly suggests[6] that the logarithm of the eigenvalues is approximately uniformly distributed in the metallic regime, which can be obtained simply from a one-parameter eigenvalue distribution. For stronger disorder, it has been argued that additional parameters are needed.[7]

In the present work we assume that the eigenvalue density in the metallic regime is well approximated by the numerical work on tight-binding Anderson Hamiltonians. The distribution of the eigenvalues of the transfer matrices can then be described by a single parameter, although not necessarily equal to the classical conductance. We then consider the most probable distribution, where analytic calculations are possible in both the metallic and strongly localized regimes. We calculate the conductance as a function of the parameter and determine the length and dimension dependence of the parameter. We show that such a dependence implies

that there can be metallic regimes in the thermodynamic limit in dimensions greater than two only, in accordance with the one-parameter scaling theory. In addition, we find that in the strongly localized regime, the eigenvalue distribution is again very regular, described by a single localization length. We find that the most probable distribution is entirely characterized by the smallest eigenvalue, which describes not only the localization length in the insulating regime but also the conductance in the metallic regime. It is encouraging that such detailed qualitative conclusions follow naturally within the formulation of the random matrix theory and the maximum-entropy hypothesis, which may provide an alternative framework to study localization where fluctuation effects arising from symmetry considerations are already included.

A disordered conductor of length L and cross-sectional area L^{d-1} attached to perfectly ordered leads with N propagating momentum channels can be characterized by a $2N \times 2N$ multiplicative transfer matrix T giving the flux amplitudes to the right of the conductor in terms of the incoming and outgoing fluxes on the left. The two-probe conductance g is then given by[8]

$$g = \sum_{i=1}^{N} \frac{1}{1+X_i}, \qquad (1)$$

where $X_i \geq 0$ are the N nondegenerate eigenvalues of the matrix $X = \frac{1}{4}[T^\dagger T + (T^\dagger T)^{-1} - 2]$. The distribution of g is then given in terms of the distribution of X_i, which was obtained within the random matrix theory and the maximum-entropy hypothesis:[3]

$$P(\{X_i\}) = \prod_{i<j} |X_i - X_j|^\beta \prod_i \exp[-f(X_i)], \qquad (2)$$

where β can be 1 or 2 depending on whether the system has time-reversal symmetry or not ($\beta = 4$ in the presence of spin-orbit coupling[9]). The factor $\prod |X_i - X_j|^\beta$ is independent of any system parameter and describes the universal conductance fluctuations as conjectured by Imry. The system parameters enter the distribution via the function $f(X)$, which also determines the global den-

sity of the eigenvalues. The distribution will be consistent with the one-parameter scaling theory provided $f(X)$ depends on a single parameter.

The functional dependence of $f(X)$ cannot be calculated solely from symmetry considerations. However, independent numerical calculations based on the tight-binding Anderson Hamiltonian strongly suggest that the variable v defined by $\cosh v = 2X - 1$ is uniformly distributed for a wide range of parameters in the metallic regime.[6,10] It is therefore more useful to write the distribution in terms of the variable v. It is easy to show that a function of the form $f(v) \sim v^2$ gives a uniform charge density. So, at least in the metallic regime an approximate form for the function f can be taken to be $f(z,v) = (z/2)v^2$, where z is an unknown parameter. (It has been pointed out by Pichard et al.[10] that if this same form for f holds for all v then it leads to a log-normal distribution for g in the localized regime. We shall discuss this regime later.) Given the function $f(v)$, the distribution $P(v)$ can be written in terms of an effective Hamiltonain[11] H defined by $P(v) = \exp(-\beta H)$, where (we use $\beta = 1$ for notational simplicity)

$$H(\{v\}) = -\sum_{i<j}^{N} \ln|\cosh v_i - \cosh v_j|$$
$$-\sum_{j}^{N} \ln|2\sinh v_j| + \tfrac{1}{2} z \sum_{j}^{N} v_j^2 . \quad (3)$$

It is clear that the parameter z, which we leave unspecified at this point, can be varied to obtain both the metallic and the localized regimes. For sufficiently large z the confining term $\sim v^2$ dominates over the logarithmically repulsive term in Eq. (3) and many of the eigenvalues are close to or less than 1. This, according to Eq. (1), gives rise to large g and hence a metal. On the other hand, for sufficiently small z all the eigenvalues will be very large, contributing insignificantly to the conductance, and we have an insulator. In order to make this statement more quantitative we need to solve for the Hamiltonian as a function of the parameter z. In the metallic regime, $v_i \ll 1$, we can expand the hyperbolic functions and the Hamiltonian becomes

$$H(\{\mu\}) = -\sum_{i<j}^{N} \ln|\mu_i - \mu_j| - \tfrac{1}{2} \sum_{j}^{N} \ln\mu_j + \tfrac{1}{2} z \sum \mu_j , \quad (4)$$

where we have defined $\mu = v^2$, and additional constants have been ignored. In principle, one should use this Hamiltonian to calculate the average conductance and its higher moments, but even in the restricted small-eigenvalue regime an analytic calculation is not possible. However, progress can be made if we restrict ourselves to the most probable configuration of the eigenvalues, defined by putting $\partial H/\partial\mu_j = 0$ and calculate the conductance for that configuration. The most probable distribution of the eigenvalues is then given by the solution of

the equation

$$\sum_{i\neq j} \frac{1}{\mu_i - \mu_j} + \frac{1}{2\mu_j} - \frac{z}{2} = 0 , \quad (5)$$

which can be solved by Stieltjes' method.[11] We define a polynomial $p(\mu)$ of order N such that $p(\mu_j) = 0$. It is easy to show that

$$\sum_{i\neq j} \frac{1}{\mu_i - \mu_j} = \frac{1}{2} \frac{p''(\mu_j)}{p'(\mu_j)} , \quad (6)$$

where a prime denotes a derivative with respect to μ. Equation (5) can then be rewritten in terms of the $p(\mu_j)$ as

$$\mu_j p''(\mu_j) + (1 - z\mu_j)p'(\mu_j) = 0 . \quad (7)$$

The polynomial $\mu p''(\mu) + (1 - z\mu)p'(\mu)$ has exactly N zeros and therefore must be proportional to $p(\mu)$. We can obtain the proportionality constant by equating the coefficients of μ^N. The result is that the polynomial $p(\mu)$ satisfies the differential equation

$$\mu p''(\mu) + (1 - z\mu)p'(\mu) + zNp(\mu) = 0 , \quad (8)$$

whose solution is the Nth-order Laguerre polynomial $L_N(z\mu)$. Thus in the most probable distribution the eigenvalues v_j sit at the N zeros of the Laguerre polynomial $L_N(zv^2)$. In particular, the first zero of the Laguerre polynomial is not at the zero of its argument, so that the smallest eigenvalue is not zero, but is given by $v_{\min} = (zN)^{-1/2}$. This is very important because it is the smallest eigenvalue that determines whether the system is insulating or metallic, and therefore must become large with large disorder.

Keeping in mind the fact that the smallest eigenvalue is nonzero, we can define an average density of zeros of the Laguerre polynomial, which will give the eigenvalue density. In terms of the variable v this is given by

$$\sigma(v) \approx \tfrac{1}{2} [z(4N - zv^2)]^{1/2} , \quad (9)$$

which can be taken to be a constant equal to $(zN)^{1/2}$, the largest value being $v_{\max} = 2(N/z)^{1/2}$. We can now calculate the conductance explicitly, as long as there are many eigenvalues much less than 1:

$$g = \sum_i \frac{1}{1+X_i} = \frac{1}{2} \int_0^{v_{\max}} \frac{\sigma(v)dv}{1+\cosh v}$$
$$\approx 2\sqrt{zN} \tanh \tfrac{1}{2} \sqrt{N/z} . \quad (10)$$

We have kept the upper limit equal to $v_{\max}$ in order to allow for the case where $v_{\max} < 1$. For the case $v_{\max} > 1$, the upper limit should be replaced by 1, since our expansion of the hyperbolic functions is not valid beyond it, but the effects are negligible. We can now separate three distinct regimes as functions of z from the above expression of the conductance. For $z \gg N$, $v_{\max} \ll 1$, i.e., all eigenvalues are much less than 1. In this regime $g = N$ which is the ballistic regime. For $z \ll N$, $v_{\max} \gg 1$

and the smallest eigenvalue is at $v_{min} = (zN)^{-1/2}$. Thus as long as $zN \gg 1$, there will be many eigenvalues less than 1 and the system would be metallic. The conductance in this regime is given by $g = (zN)^{1/2}$. This is the Ohmic regime. For $z \ll 1/N$, all the eigenvalues are much larger than 1, and our expansion of the hyperbolic functions in the Hamiltonian is no longer valid. This is the non-Ohmic regime. The calculations become simple again for the strongly localized regime which we shall discuss later.

We can obtain an explicit length dependence of the parameter z by demanding that Ohm's law be valid in the Ohmic regime. Thus for $1/N \ll z \ll N$, we have $g = (zN)^{1/2} = Nl/L$. This gives $z = N(l/L)^2$. If we use $N \sim L^{d-1}$, where d is the dimension, we get $z \sim L^{d-3}$, which implies that $v_{min} = (zN)^{-1/2} = L/Nl \sim L^{2-d}$. This has the immediate consequence that $d = 2$ is the marginal dimension, in accordance with the one-parameter scaling theory.[1] For a given size L, v_{min} will be determined by many system parameters via its dependence on z, but as the size L is increased, v_{min} increases with L in less than two dimensions, always leading to an insulator at sufficiently large L. In dimensions greater than two, v_{min} decreases with increasing L, leading to a metal for small enough disorder. It is encouraging that such qualitative results follow naturally within the framework of the random matrix theory.

We now consider the strongly localized regime. Following Pichard et al. we assume that $f(v)$ remains of the same form even in this regime. We emphasize here that there is no numerical evidence that this simple one-parameter form for $f(v)$ holds for strong disorder as well. In fact, Cohen, Roth, and Shapiro[7] have argued, based on a Migdal-Kadanoff–type scaling approach, that the distribution is generally expected to have two independent parameters and only in the metallic limit does a one-parameter distribution emerge. Nevertheless, it is useful to determine the consequences of such a simple assumption over the entire range of disorder. The strongly localized regime is particularly simple to consider. In this regime z is very small and all the eigenvalues are much larger than 1 so that it is more convenient to go back to the variable X. The most probable set of eigenvalues in the limit $X \gg 1$ is given by $\partial H(\{X_i\})/\partial X_i = 0$ which gives

$$\sum_{i \neq j}^{N} \frac{1}{X_i - X_j} + \frac{1 - z \ln X_i}{X_i} = 0. \tag{11}$$

The conductance is determined entirely by the smallest eigenvalue. Let us arrange the eigenvalues in order of increasing magnitude so that X_1 is the smallest one. Then all X_i for $i > 1$ are much smaller than X_1 and we can neglect the first term in Eq. (11). Then X_1 is given by $X_1 = \exp(1/z)$ and the conductance is $g = 1/X_1 = \exp(-1/z)$. The nth eigenvalue X_n is obtained from the above equation by neglecting all X_m for $m < n$ in the first term and neglecting the first term for all $m > n$.

This leads to $X_n = \exp(n/z)$, so that $X_n = X_1^n$. Thus the logarithm of the eigenvalues is again uniformly distributed. Note, however, that in the localized regime g has the length dependence $g = \exp(-v_1) \sim \exp(-L/\zeta)$, where ζ is the localization length, which implies that $z \sim \zeta/L$ and $v_{min} \sim L/\zeta$, independent of dimensions. This is different from the length dependence in the metallic regime. We plan to study the conductance for all values of z numerically in order to better understand the crossover region, which will also allow us to calculate the weak-localization correction.

In summary, we show explicitly that a key result in the one-parameter scaling theory of localization, namely, that there can be true metallic systems only in dimensions greater than two, can be obtained naturally within the random matrix theory formulation proposed earlier. The scaling theory does not take into account the universal conductance fluctuations at length scales small compared to the inelastic length in the system, but describes the weak-localization phenomena very well. If the random matrix formulation can recover key results of the scaling theory, one of which we show in the present work, then it would be an alternative framework to study the metal-insulator transition where such fluctuation effects are already taken into account.

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