LETTER TO THE EDITOR

Random matrix models with additional interactions

K A Muttalib

Department of Physics, University of Florida, Gainesville, FL 32611, USA

Abstract. It has been argued that, despite remarkable success, existing random matrix theories are not adequate at describing disordered conductors in the metallic regime, due to the presence of certain two-body interactions in the effective Hamiltonian for the eigenvalues, in addition to the standard logarithmic interaction that arises entirely from symmetry considerations. We present a new method that allows exact solution of random matrix models with such additional two-body interactions. This should broaden the scope of random matrix models in general.

From a phenomenological point of view, random matrix models have proved very useful in our understanding of a wide variety of physical systems including complex nuclei [1], disordered metals [2] and chaotic systems [3]. Although the physical systems are very diverse, the local statistical properties of the characteristic levels of these systems in the bulk of the spectrum turn out to be universal, similar to the well known universal properties of the distribution of eigenvalues of random matrices as proposed originally by Wigner [4]. Recently the models have been generalized to include transitions in spectral statistics [5] that are characteristic of metal–insulator or chaotic–regular transitions in finite systems. This has opened up the possibility of describing such transitions in this powerful mathematical framework, allowing exact evaluation of correlation functions. However, on one hand the statistical properties of numerically solved microscopic models with random disorder describing mesoscopic conductors show remarkable agreement with predictions of the generalized random matrix theory over a wide range of disorder [5, 6]; on the other hand there are indications that the appropriate random matrix model for disordered conductor is, while highly accurate, not exact in the metallic regime [7, 8]. An exact solution [9] for the Fokker–Planck equation describing the probability distribution of the transmission coefficients [10] shows that the resulting matrix model should include a small correction term which apparently destroys the solvability of the model. This correction is responsible for a very small correction to the magnitude of the universal conductance fluctuation, but at the same time this also resolves a small discrepancy between the random matrix result and the perturbative result from microscopic theory [11]. While it is not clear how important the correction term is, e.g. to the question of transition from metal to insulator, the fact that the correction exactly reproduces the result of the microscopic theory makes it qualitatively non-negligible. It is therefore believed that, despite remarkable success, the usefulness of the phenomenological random matrix approach for the problem of disordered conductors will be severely restricted if such corrections cannot be accommodated within a solvable framework.

In this work we present a new method to accommodate certain types of corrections to the standard random matrix models. These corrections are similar to those arising in the problem of disordered conductors. The method generalizes the approach based on the

theory of orthogonal polynomials and allows exact solutions for physically relevant models in terms of known functions.

The basic ansatz of the random matrix theory is that for a physical system described by an $N \times N$ matrix X with eigenvalues x_n, $n = 1, \ldots, N$, the joint probabilty distribution (JPD) for the ensemble of all random X matrices consistent with given symmetries (hermiticity, time reversal etc) and subject to some physical constraint (e.g. given average density of eigenvalues) can be written quite generally in the form [4]

$$P(x_1, \ldots, x_N) = \prod_{m<n} |x_m - x_n|^\alpha \prod_n e^{-V(x_n)}. \tag{1}$$

Here α is a symmetry parameter and is equal to 1, 2 or 4 for orthogonal, unitary and symplectic symmetries respectively. For example, for disordered conductors, a good ansatz [12] is to use the $2N \times 2N$ matrix $X = \frac{1}{4}[TT^\dagger + (TT^\dagger)^{-1} - 2I]$, where T is the transfer matrix characterizing the conductor and I is the unit matrix. The doubly degenerate real eigenvalues x are restricted between 0 and ∞ by current conservation and directly gives the conductance $g = \sum_n 1/(1 + x_n)$. It is useful to describe the probability distribution in terms of an effective 'Hamiltonian', H, of the eigenvalues defined by $P = \exp(-\alpha H)$, where

$$H(x_n) = -\sum_{m<n} \ln |x_m - x_n| + \frac{1}{\alpha} V(x_n). \tag{2}$$

The repulsive logarithmic 'interaction' term arises from symmetry considerations alone, while the confining 'single-particle potential' $V(x)$ is the Lagrange multiplier function which takes care of the physical constraint [13] mentioned above and, in general, depends on various physical parameters. For example $V(x) = tx$, where t depends on disorder, describes the disordered metal quite well [2, 14].

The solvability of the model has so far relied crucially on the fact that the only interacting term in (2) is the logarithmic repulsion which arises entirely from symmetry considerations; in other words any relevant physical constraint must give rise to only a single-particle potential. Given this restriction, the universal distributions for nearest-neighbour spacing or the so-called Δ_3 statistics in the bulk of the spectrum, which we will generically call the Wigner distributions [4], follow from the above JPD when $V(x)$ is taken to be linear or quadratic in x. In these cases the potential is strong enough to overcome the logarithmic repulsion and the density of levels scale with the number of levels. When $V(x)$ is not strong enough, the universality breaks down; in particular for $V(x) \to [\ln(x)]^2$ for large x, there is a transition from the Wigner distribution to an uncorrelated Poisson distribution as a function of a parameter [5]. Nevertheless, the model remains exactly solvable.

The first hint that a disordered conductor in the metallic regime in higher than one dimension may not be exactly described by a simple logarithmic interaction of the above form came from attempts to check detailed predictions of random matrix results numerically [7]; but the nature of the correction needed came from exact solutions [9] of the Fokker–Planck equation satisfied by the transmission eigenvalues in the metallic regime [10]. The solution showed, when mapped to a random matrix Hamiltonian, that the two-particle interaction part has the form

$$-\frac{1}{2} \sum_{m<n} \ln |x_m - x_n| - \frac{1}{2} \sum_{m<n} \ln | \arcsinh^2(x_m^{1/2}) - \arcsinh^2(x_n^{1/2})|. \tag{3}$$

For $x \ll 1$, this reduces to the standard logarithmic repulsion, but for $x \approx 1$ the additional term makes it non-logarithmic. The difference is enough to change the variance of conductance from the random matrix result $\frac{1}{8}$ to the microscopic perturbative result $\frac{2}{15}$.

It is important to establish how significant this small difference is as far as the qualitative statistical properties are concerned. Although the existence of such additional two-body terms can be understood as arising from some physical constraint that need not be of a single-particle form, the question of if or how it affects the known random matrix results could not be addressed within the current random matrix framework because any such additional two-body interaction destroys the existing criterion for solvability and, therefore, the usefulness of the model.

We will show below that, with an additional two-body interaction given by a simplification of equation (3), it is still possible to solve the model exactly but now using a new method. While the models constructed are appropriate for disordered conductors, the solvability of such models broadens the scope of random matrix theory in general.

As a first step towards constructing a model that can be solved exactly, and is close to a physical model, we approximate the arcsinh function in equation (3) by a polynomial $s_k(x)$, of degree k; in the metallic regime where model (3) is valid this should give a good approximation. For simplicity and purpose of illustration we will discuss the case $s_k(x) = x^k$ in detail. We will indicate at the end how the method can, in principle, be generalized to arbitrary polynomials.

We will, therefore, consider in detail the model described by an additional two-body interaction of the form $\ln(x_m^k - x_n^k)$, which is equivalent to a JPD of the form

$$P(x_1, \ldots, x_N) = \prod_{m<n}(x_m - x_n)(x_m^k - x_n^k)\prod_n e^{-V(x_n)} \tag{4}$$

where k is a positive integer. Note that for $k = 1$ the model reduces to the standard unitary random matrix ensemble. An exact solution of this model should allow us to understand at least the qualitative effects of the additional two-body corrections.

For the standard logarithmic interaction part we follow the method of orthogonal polynomials [4] and write the product of the differences $\prod_{m<n}(x_m - x_n)$ as a Vandermonde determinant for which the jth column has elements $x_1^{j-1}, x_2^{j-1}, \ldots, x_N^{j-1}$, j varying from 1 to N. The determinant remains invariant if we add some linear combination of the other columns with lower powers of the x's; the new jth column has elements $Y_{j-1}(x_1), Y_{j-1}(x_2), \ldots, Y_{j-1}(x_N)$, where $Y_j(x) = \sum_{l=0}^{j} b_{jl}x^l$ is a polynomial in x, of degree j; the coefficients b will depend on the choice of the single-particle potential V as we will show later. In a similar way, we write the correction term $\prod_{m<n}(x_m^k - x_n^k)$ as a second Vandermonde determinant, for which the jth column has elements $Z_{j-1}(x_1), Z_{j-1}(x_2), \ldots, Z_{j-1}(x_N)$, where $Z_j(x) = \sum_{l=0}^{j} c_{jl}x^{kl}$ is now a polynomial in x^k, of degree j; the coefficients c will be determined from the choice of V. We now multiply the ith column of each determinant by $\exp[-V(x_i)/2]$, and interchange rows and columns of the second determinant. Equation (4) can then be written as the product of the two determinants, in the form

$$P(x_1, \ldots, x_N) = C_N \det K(x_i, x_j) \tag{5}$$

where C_N is the normalization constant, and

$$K(x_i, x_j) = \exp\left[-\tfrac{1}{2}(V(x_i) + V(x_j))\right]\sum_{l=0}^{N-1} Y_l(x_i)Z_l(x_j). \tag{6}$$

The reason for writing the JPD as a determinant is the following: our ability to evaluate the n-point correlation function defined by

$$R_n(x_1, \ldots, x_n) = \frac{N!}{(N-n)!}\int \cdots \int dx_{n+1} \ldots dx_N P_N(x_1, \ldots, x_N) \tag{7}$$

depends on our ability to integrate the JPD over arbitrary number of variables. These integrals can be done in a very simple way [4] if the JPD can be expressed as a determinant as in equation (5), providing the following two conditions are satisfied:

$$\int K(x,x)\,d\mu(x) = \text{constant} \qquad \text{and} \qquad \int K(x,y)K(y,z)\,d\mu(y) = K(x,z) \qquad (8)$$

where $d\mu$ is a suitable measure. This is where the restriction of the standard logarithmic interaction, equivalent to the case $k = 1$, comes in. For $k = 1$, the polynomial $Z(x)$ is the same as $Y(x)$, and they can be chosen such that they form an orthonormal set $p(x)$ with respect to the measure $\exp[-V(x)]\,dx$, i.e.

$$\int p_m(x)p_n(x)e^{-V(x)}\,dx = \delta_{mn}. \qquad (9)$$

Then the above two conditions in (8) follow from the orthogonality and normalizability of the polynomials. The additional two-body interaction forces the polynomials to be distinct, destroying the orthogonality. We will now show that even for distinct polynomials $Y(x)$ and $Z(x)$, the two conditions in (8) can be satisfied under certain conditions making it possible to obtain exact solutions for the correlation functions for these generalized models.

Let us choose the coefficients b and c in such a way that the polynomials Y and Z satisfy the following:

$$\int Y_n(x)x^{kj}e^{-V(x)}\,dx = 0 \qquad j = 0, 1, \ldots, n-1$$
$$\qquad\qquad\qquad\qquad \neq 0 \qquad j = n$$
$$\int Z_n(x)x^j e^{-V(x)}\,dx = 0 \qquad j = 0, 1, \ldots, n-1$$
$$\qquad\qquad\qquad\qquad \neq 0 \qquad j = n. \qquad\qquad (10)$$

It can then be shown [15] that the two polynomials form a 'biorthogonal' pair, defined by

$$\int e^{-V(x)}Y_n(x)Z_m(x)\,dx = g_n\delta_{mn}. \qquad (11)$$

We will always choose an overall multiplicative factor such that $g_n = 1$, i.e. the polynomials are normalized. Clearly the two conditions in (8) are satisfied again:

$$\int K(x,x)\,dx = \int e^{-V(x)}\sum_{l=0}^{N-1} Y_l(x)Z_l(x)\,dx = \sum_{l=0}^{N-1}\int e^{-V(x)}Y_l(x)Z_l(x)\,dx = N \qquad (12)$$

where we have used the normalization, and

$$\int K(x,y)K(y,z)\,dy = \int e^{-\frac{1}{2}[V(x)+V(y)]}e^{-\frac{1}{2}[V(y)+V(z)]}\sum_{j,l=0}^{N-1} Y_j(x)Z_j(y)Y_l(y)Z_l(z)\,dy$$

$$= e^{-\frac{1}{2}[V(x)+V(z)]}\sum_{j,l=0}^{N-1} Y_j(x)Z_l(z)\int e^{-V(y)}Y_l(y)Z_j(y)\,dy = K(x,z) \qquad (13)$$

where we have used the biorthogonality of the polynomials. Given these properties, the integration over $N - n$ variables $x_{n+1}, \ldots, x_N$ in the JPD can be explicitly carried out [4], and we obtain

$$R_n = \det[K(x_i, x_j)]_{i,j=1,\ldots,n} \qquad (14)$$

where the kernel $K(x_i, x_j)$ is given by equation (6). In particular, the density is given by $K(x, x)$ and the two-particle kernel, from which the nearest-neighbour spacing distribution or the Δ_3 statistics can be calculated, is given by $K(x, y)K(y, x)$.

The model is, then, exactly solvable if for a given choice of the single-particle potential $V(x)$ the corresponding biorthogonal polynomials can be obtained. The physically interesting model that already gives a very good description of the metallic regime of a disordered conductor in the $k = 1$ limit is given by $V(x) = x$, $0 \leqslant x \leqslant \infty$. The model is exactly solvable in terms of Laguerre polynomials. For arbitrary k, the corresponding biorthogonal polynomials have been explicitly calculated by Konhauser [16]. (For the simplest non-trivial case $k = 2$, these are the polynomials introduced by Spencer and Fano [17] to study penetration of matter by gamma rays, and studied later by Preiser [18].) Therefore using the new method the exact solution can be immediately written down in terms of these Konhauser biorthogonal polynomials. It has been argued [14] that an appropriate generalization for all disorder, in the $k = 1$ limit, is given by the choice

$$V(x; q) = \sum_{n=0}^{\infty} \ln[1 + (1 - q)q^n x] \qquad 0 \leqslant q < 1. \tag{15}$$

As $q \to 1^-$, $V(x) \to x$, and one recovers the metallic regime, while increasing disorder corresponds to decreasing q. This model is exactly solvable in terms of the q–Laguerre polynomials. For arbitrary k, again the corresponding biorthogonal polynomials are explicitly known [19] and the exact solution can be written down in terms of these 'q–Konhauser' biorthogonal polynomials. The detailed properties of these solutions are under investigation.

It is also possible to consider a more general form of the JPD given by

$$P(x_1, \ldots, x_N) = \prod_{m < n} [r(x_m) - r(x_n)][s(x_m) - s(x_n)] \prod_n e^{-V(x_n)} \tag{16}$$

where $r(x)$ is a polynomial of degree h and $s(x)$ is a polynomial of degree k. Defining Y and Z as polynomials in $r(x)$ and $s(x)$ respectively, the above method should be applicable if conditions (10) are replaced by [15]

$$\int e^{-V(x)} Y_n(x)[s(x)]^j \, dx = 0 \qquad j = 0, 1, \ldots, n - 1$$

$$\neq 0 \qquad j = n$$

$$\int e^{-V(x)} Z_n(x)[r(x)]^j \, dx = 0 \qquad j = 0, 1, \ldots, n - 1$$

$$\neq 0 \qquad j = n. \tag{17}$$

The case considered before is a special case where $r(x) = x$ and $s(x) = x^k$. Note that writing $r(x_n) - r(x_m) = (x_n - x_m)r(x_m, x_n)$ and $s(x_n) - s(x_m) = (x_n - x_m)s(x_m, x_n)$, we can write the JPD in the form

$$P(x_1, \ldots, x_N) = \prod_{m < n} (x_m - x_n)^2 \prod_{m < n} r(x_m, x_n)s(x_m, x_n) \prod_n e^{-V(x_n)} \tag{18}$$

which may allow more physically interesting models to be solved exactly, if the corresponding biorthogonal polynomials are known.

In summary, we present a new method to accommodate certain two-body interactions in random matrix models, particularly appropriate for the problem of transport in disordered conductors. We show that correlation functions can be written down explicitly in terms of known biorthogonal polynomials. The approach should broaden the scope of random matrix models in general.

I am grateful to Mourad Ismail for many discussions and, in particular, for bringing [19] to my attention.

References

[1]   Porter C E (ed) 1965 *Statistical Theories of Spectra: Fluctuations'* (New York: Academic)
      Brody T A *et al* 1981 *Rev. Mod. Phys.* **53** 385
[2]   Stone A D, Mello P A, Muttalib K A and Pichard J-L 1991 *Mesoscopic Phenomena in Solids* ed B L
      Altshuler, P A Lee and R A Webb (Amsterdam: North-Holland)
[3]   Giannoni M-J, Voros A and Zinn-Justin J (ed) 1991 *Chaos and Quantum Physics (Les Houches LII)*
      (Amsterdam: Elsevier)
      Reichel L E 1992 *The Transition to Chaos* (Berlin: Springer)
[4]   Mehta M L 1991 *Random Matrices* 2nd edn (New York: Academic)
[5]   Muttalib K A, Chen Y, Ismail M E H and Nicopoulos V N 1993 *Phys. Rev. Lett.* **71** 471
      Blecken C, Chen Y and Muttalib K A 1994 *J. Phys. A: Math. Gen.* **27** L563
[6]   Shlovskii B I, Shapiro B, Sears B R, Lambrianides P and Shore H B 1993 *Phys. Rev.* B **47** 11 487
      Hofstetter E and Schreiber M 1993 *Phys. Rev.* B **48** 16 979
[7]   Slevin K, Pichard J-L and Muttalib K A 1993 *J. Physique* **3** 1387
[8]   Beenakker C W J 1993 *Phys. Rev. Lett.* **70** 1155
[9]   Beenakker C W J and Rejaei B 1993 *Phys. Rev. Lett.* **71** 3689
[10] Mello P A, Pereyra P and Kumar N 1988 *Ann. Phys.* **181** 290
[11] Lee P A and Stone A D 1985 *Phys. Rev. Lett.* **55** 1622
[12] Muttalib K A, Pichard J-L and Stone A D 1987 *Phys. Rev. Lett.* **59** 2475
[13] Balian R 1968 *Nuovo Cimento* **57** 183. This is required whenever the eigenvalues are un-bounded, while for
      bounded eigenvalues the confining potential is not necessary.
[14] Chen Y, Ismail M E H and Muttalib K A 1992 *J. Phys.: Condens. Matter* **4** L417; 1993 *J. Phys.: Condens.
      Matter* **5** 177
[15] Konhauser J D E 1965 *J. Math. Anal. Appl.* **11** 242
[16] Konhauser J D E 1967 *Pacific J. Math.* **21** 303
[17] Spencer L V and Fano U 1951 *J. Res. NBS* **46** 446
[18] Preiser S 1962 *J. Math. Anal. Appl.* **4** 38
[19] Al-Salam W A and Verma A 1983 *Pacific J. Math.* **108** 1