

# Brownian motion model of a $q$ -deformed random matrix ensemble

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**Abstract.** The effect of an external perturbation on the energy spectrum of a mesoscopic quantum conductor can be described by a Brownian motion model developed by Dyson who wrote a Fokker–Planck equation for the evolution of the joint probability distribution of the energy levels. For weakly disordered conductors, which can be described by a Gaussian random matrix ensemble, the solution of the Fokker–Planck equation has recently been obtained to give the correlation of level densities at different energies and different parameter values. In this paper we generalize this calculation to the case of a  $q$ -random matrix ensemble which should be relevant for conductors at stronger disorder.

## 1. Introduction

Weakly disordered mesoscopic quantum conductors are well described by Gaussian random matrix ensembles [1]. The joint probability distribution of the eigenvalues of such an ensemble of  $N \times N$  Hermitian matrices is given by [2]

$$P(E_i) = \prod_i^N |E_i - E_j|^\beta \exp[-V(E_i)]. \quad (1)$$

Here the factor  $\prod_i^N |E_i - E_j|^\beta$  comes from the Jacobian associated with the transformation from the space of Hermitian matrices to the space of eigenvalues  $E_i$ , and  $V(E) = cE^2/2$  for Gaussian ensembles where  $c$  is an arbitrary positive constant which determines the mean level spacing. The parameter  $\beta$  depends on the symmetry of the ensemble; it is 1, 2 or 4 if the ensemble is orthogonal, unitary or symplectic respectively. Dyson [3] showed that the above joint probability distribution can be obtained within a one-dimensional Brownian motion model with some fictitious ‘time’  $\tau$ . In this model  $N$  classical particles at positions  $E_i(\tau)$  evolve in a fictitious viscous fluid with friction coefficient  $\mu$  and temperature  $1/\beta$  according to the Fokker–Planck equation

$$\mu \frac{\partial}{\partial \tau} P(E_i, \tau) = \sum_{i=1}^N \frac{\partial}{\partial E_i} \left( P \frac{\partial W}{\partial E_i} + \frac{1}{\beta} \frac{\partial P}{\partial E_i} \right). \quad (2)$$

Here

$$W(E_i) = - \sum_{i < j} \ln |E_i - E_j| + \sum_i V(E_i). \quad (3)$$

Distribution (1) is then obtained as a ( $\tau \rightarrow \infty$ ) limiting equilibrium distribution of  $P(E_i, \tau)$ . Beenakker and Rejaei [4] have recently shown that one can actually identify the fictitious

time  $\tau$  with an external perturbation parameter  $X$  (which can be, for example, an electric or magnetic field acting on the system) if one defines  $\tau = X^2$ . The eigenvalues  $E_i(X)$  will now depend parametrically on  $X$ . This then allows one to obtain various energy level correlation functions at different values of the external perturbation parameter, usually called parametric correlations.

The Brownian motion model of Gaussian random matrix ensembles have been used [4] to obtain the parametric density correlation function defined as

$$S(E, X, E', X') = \sum_{i,j} \langle \delta(E - E_i(X)) \delta(E' - E_j(X')) \rangle - \left( \sum_i \langle \delta(E - E_i(X)) \rangle \right) \left( \sum_j \langle \delta(E' - E_j(X')) \rangle \right) \quad (4)$$

where  $\langle \rangle$  denotes an average over an ensemble of particles with different impurity configurations. Their calculation reproduces the earlier results of Szafer and Altshuler [5] obtained from microscopic theory for disordered conductors and valid for weak disorder. In particular, the Brownian motion model reproduces a remarkable universality first obtained in [5] in the level ‘velocity autocorrelation function’ defined as

$$C(X) = \frac{1}{\Delta^2} \langle \partial_{X'} E_i(X') \partial_{X'} E_i(X' + X) \rangle \quad (5)$$

where  $\Delta$  is the mean level spacing.

In this paper we will argue that a generalization of the Fokker–Planck equation (2) to include the strong disorder regime naturally leads to the Brownian motion model of a  $q$ -ensemble [6]. We will first motivate the generalization and define the model, and then solve it exactly to obtain the correlation functions defined in (4) and (5) as functions of the deformation parameter  $q$  which plays the role of disorder.

## 2. The $q$ -Hermite ensemble

Although the Gaussian ensemble has been defined by choosing  $V(E) = cE^2/2$  in (1), in general one can think of  $V(E)$  as a Lagrange multiplier function which can be chosen to yield, for example, a particular density of levels appropriate for the system being described. It is known that the correlation functions obtained from the probability distribution (1) become independent of  $V(E)$  for a wide variety of choices once they are expressed in terms of a variable in which the mean level spacing is unity. This is the source of universality in the random matrix ensembles. However, as pointed out in [6], this universality breaks down for sufficiently ‘weak’  $V(E)$  such that  $V(E)$  behaves as  $\ln^2 E$  for large  $E$  instead of any power of  $E$ . In particular if  $V$  is qualitatively of the form  $V(E) = b \ln^2[1 + E]$  (where  $b$  is a constant) such that it crosses over from a power law behaviour at small  $E$  to a  $\ln^2 E$  behaviour for large  $E$ , then this behaviour can be well represented by the weight function of a  $q$ -Hermite polynomial, given by [7]

$$V(E; q) = \ln \theta_2(i \sinh^{-1} E; \sqrt{q}) + \text{constant} \quad (6)$$

where  $\theta_2$  is the second Jacobi theta function. The resulting random matrix ensemble is a  $q$ -Hermite ensemble with properties that show a crossover from the Gaussian ensemble (for  $q = 1$ ) towards an uncorrelated Poisson ensemble, as a function of the deformation parameter  $q$ . At a phenomenological level, the  $q$ -ensembles could be thought of as the strong disorder generalization of the Gaussian random matrix ensemble. The nearest-neighbour spacing distribution and the number variance as a function of disorder obtained

numerically from the microscopic tight-binding Anderson model for a strongly disordered three-dimensional conductor agrees very well [8] with those calculated for the  $q$ -ensembles. Moreover, the numerically obtained shape of the spacing distribution at the critical regime agrees quantitatively well with the predictions of the  $q$ -model for a particular value of the deformation parameter, and the numerically obtained number variance agrees quantitatively well with the prediction of the  $q$ -model for that *same value* of the parameter<sup>†</sup>. It has also been shown [10] that the two-level correlation function obtained for the  $q$ -ensemble is identical to that obtained for a matrix model at the critical regime where the effect of strong disorder is incorporated by including a preferential basis [11]<sup>‡</sup>. These agreements indicate that the  $q$ -ensemble captures the essentials of the statistical properties of the system at stronger disorder near the critical regime. It is therefore natural to try to generalize the Brownian motion model along the same lines and consider a ‘ $q$ -deformed’ Fokker–Planck equation as a phenomenological model that includes stronger disorder. For simplicity, we will consider only the unitary ensemble. From the mathematical point of view, properties of such  $q$ -deformed Brownian motion have not been obtained before and should be of intrinsic interest. Our calculations will generalize the method of [4] for the parametric density correlation function  $S(X)$ . In particular we will use the density correlation function to explicitly calculate the level velocity autocorrelation function  $C(X)$  and predict how the universality at weak disorder breaks down with increasing disorder.

The above discussion suggests that the natural choice for the generalization of the Fokker–Planck equation (2) to include strong disorder is to replace  $V(E)$  in (3) with  $V(E; q)$  given in (6). We will call this the Brownian motion model of the  $q$ -Hermite ensemble, because the limiting distribution  $P(E_i; q, \tau \rightarrow \infty)$  will correspond to the equilibrium distribution for the  $q$ -Hermite ensemble. This generalization is clearly not unique. For example, one other possible route is to replace the derivatives in (2) with  $q$ -difference operators. Our generalization is motivated by the role of  $V(E)$  in the corresponding phenomenological equilibrium model which seems to be relevant for strongly disordered conductors.

### 3. Brownian motion model of $q$ -Hermite ensemble

In order to calculate the parametric density correlation function (4), we need to know the eigenfunctions and eigenvalues of the Fokker–Planck equation (2). Sutherland [12] obtained them for the particular case of  $V(E) = cE^2/2$  by mapping (2) onto a Schrödinger equation. In the special case of  $\beta = 2$ , the mapping yields a set of non-interacting fermions for which the eigenfunctions and eigenvalues are known. We will follow his method and show that although for the  $q$ -ensemble the mapping yields a very complicated interacting system even at  $\beta = 2$ , it is still possible to identify the eigenfunctions and eigenvalues with known functions and therefore obtain  $S(X)$  as well as  $C(X)$ .

In order to map the Fokker–Planck equation (2) onto a Schrödinger equation, we define [4]

$$P(E_i; \tau) = \Psi_0(E_i)\phi(E_i; \tau) \quad (7)$$

<sup>†</sup> The numerical work of [8] was done for a model with confining potential  $V(E) = A \ln^2(|E|)$ ,  $|E| \rightarrow \infty$ . As emphasized in [6] and confirmed in [9], this is equivalent to the  $q$ -model.

<sup>‡</sup> The two-level correlation function of [11] in the *critical regime* coincide with the  $N \rightarrow \infty$  limit of the  $q$ -model of [6]. The weak disorder regime in both models correspond to a scaling of the parameter such that  $\gamma N \rightarrow 0$  as  $N \rightarrow \infty$ .

where

$$\Psi_0(E_i) = e^{-\beta W(E_i)/2} \quad (8)$$

and  $W$  is given by (3). It is easy to check that  $\phi(E_i; \tau) = e^{-\beta W(E_i)/2}$  is a solution of the time-independent Fokker–Planck equation. Substituting (7) into (2) yields the following equation for  $\phi$ ,

$$\beta\mu \frac{\partial \phi}{\partial \tau} = \sum_j \left[ \frac{\partial^2}{\partial E_j^2} - \frac{1}{\Psi_0} \frac{\partial^2 \Psi}{\partial E_j^2} \right] \phi. \quad (9)$$

For any choice of  $V$ , from the definition of  $\Psi_0$  we obtain

$$\begin{aligned} \frac{1}{\Psi_0} \sum_k \frac{\partial^2 \Psi}{\partial E_k^2} &= \sum_{k < l} \frac{\beta(\beta - 2)}{2(E_k - E_l)^2} - \sum_{k < l} \frac{\beta}{(E_k - E_l)} \left[ \frac{\partial V(E_k)}{\partial E_k} - \frac{\partial V(E_l)}{\partial E_l} \right] \\ &\quad - \sum_k \left[ \frac{\partial^2 V(E_k)}{\partial E_k^2} - \left( \frac{\partial V(E_k)}{\partial E_k} \right)^2 \right]. \end{aligned} \quad (10)$$

We can identify the right-hand side of (10) with some  $(\epsilon - U(E))$ , and write,

$$H\Psi_0 = \epsilon\Psi_0 \quad (11)$$

where

$$H = \sum_j \frac{\partial^2}{\partial E_j^2} + U. \quad (12)$$

Thus  $\Psi_0$  satisfies a time-independent Schrödinger equation with Hamiltonian  $H$ , and eigenvalue  $\epsilon$ . Substituting (10) into (9), we obtain

$$-\frac{\partial \phi}{\partial \tau} = \frac{1}{\beta\mu} \left[ - \sum_j \frac{\partial^2}{\partial E_j^2} - U + \epsilon \right] \phi = -(H - \epsilon)\phi. \quad (13)$$

Thus for  $\tau = it$ ,  $\phi$  satisfies a time-dependent Schrödinger equation, whose time evolution is determined by the Hamiltonian  $H$ , which in turn satisfies the time-independent equation (11).

For  $V(E) = cE^2/2$ ,

$$U = \sum_{k < l} \frac{\beta(\beta - 2)}{2(E_k - E_l)^2} + c^2 \sum_k E_k^2. \quad (14)$$

In this case (12) becomes the well known Sutherland Hamiltonian with inverse square interaction and a harmonic oscillator potential, and eigenvalue  $\epsilon = Nc + \beta N(N - 1)/2$ . For the special case of  $\beta = 2$ , the interaction term in  $U$  becomes zero and the Hamiltonian is simply that of a set of non-interacting particles in a harmonic oscillator potential. The time evolution of the single particle wavefunctions are then given by the eigenvalues and eigenfunctions of a harmonic oscillator and can be used to write the parametric density correlation function [4]:

$$S(E, E'; X) = c \sum_{n=N}^{\infty} \sum_{m=0}^{N-1} \phi_n(E\sqrt{c})\phi_n(E'\sqrt{c})\phi_m(E\sqrt{c})\phi_m(E'\sqrt{c})e^{(\epsilon_m - \epsilon_n)X^2} \quad (15)$$

where  $\phi_n(E\sqrt{c})$  and  $\epsilon_n$  are the normalized eigenfunctions and energy eigenvalues of the harmonic oscillator with frequency  $\omega = 2c/\mu$ , and we have used  $\tau = X^2$ .

For our  $q$ -ensemble,  $V$  is given by (6). Although the coefficient of the inverse square term still vanishes for  $\beta = 2$ , clearly the long-range  $(E_k - E_l)^{-1}$  term does not vanish in this

case for any  $\beta$ , and the corresponding Hamiltonian  $H$  remains that of a very complicated interacting set of particles [13]. Nevertheless, the important point is that the many-body wavefunction for this Hamiltonian is known from (11) to be  $\Psi_0$ , which has the form of a Vandermonde determinant as seen from (8) and (3). We know that such a determinant can be written as an antisymmetrized product of orthogonal polynomials [2] with  $V(E; q)$  as its weight function, so that the many-body wavefunction has the form of products of single-particle wavefunctions given by the orthogonal polynomials. The time evolution of  $\phi$  is then determined by the single-particle Hamiltonian that corresponds to these single-particle wavefunctions. For  $V = cE^2/2$  and  $\beta = 2$ , these are the Hermite polynomials which are wavefunctions of the harmonic oscillator and therefore we recover the results of [4]. For  $V$  given by (6) and  $\beta = 2$ , these are  $q$ -Hermite polynomials which are the wavefunctions of a  $q$ -harmonic oscillator. In other words, the many-body Hamiltonian in this case actually corresponds to a set of independent  $q$ -oscillators.

#### 4. Parametric density correlation function

We can now write the parametric density correlation function for the  $q$ -ensemble. Since the time evolution is determined by non-interacting  $q$ -oscillators, the density correlation function has exactly the same form as (15), except that the  $\phi_n(E)$  and  $\epsilon_n$  are now the normalized eigenfunctions and energy eigenvalues of the  $q$ -harmonic oscillator. These are given in Ismail and Masson [14] and Ismail [15],

$$\phi_n(E; q) = \frac{1}{k_n} h_n(E; q) e^{-V(E; q)/2} \quad (16)$$

and

$$\epsilon_n(q) = \left( n_q + \frac{1}{2} \right) \omega \quad n_q = \frac{1 - q^n}{1 - q} \quad \omega = \frac{2}{\mu \ln \frac{1}{q}} \quad (17)$$

where

$$k_n = \frac{q^{n(n+1)/4}}{\sqrt{\ln \frac{1}{q} (q; q)_n (q; q)_\infty}} \quad (18)$$

are the normalization constants,

$$h_n(E; q) = \sum_{k=0}^n \frac{(q)_n}{(q)_n (q)_{n-k}} (-1)^k q^{k(k-n)} \left( E + \sqrt{E^2 + 1} \right)^{n-2k} \quad (19)$$

are the  $q$ -Hermite polynomials and we have used the notation  $(q)_k = \prod_{j=1}^k (1 - q^j)$ .

Note first that the eigenvalues and eigenfunctions reduce to those of an ordinary harmonic oscillator in the limit  $q = 1$ , and we retrieve the results of [4], valid for weak disorder. The parameter  $\mu$  plays the role of disorder in this limit. In order to understand how the  $q$ -deformation affects the weak disorder results, we first consider a simple case  $E = E' = 0$  in the large  $N$  limit. Using (15)–(19), the density correlator in this limit can be written as

$$S(X) = \frac{2c}{\pi\gamma} \sum_{m=0}^N q^m g_m e^{m_q \omega X^2} \sum_{n=N}^{\infty} q^n g_n e^{-n_q \omega X^2} \quad (20)$$

where we defined  $q = e^{-\gamma \dagger}$ , and  $g_n = \prod_{i=1}^n (1 - q^{2i-1}) / (1 - q^{2i})$ .

† We change the notation from [6] in order to avoid confusion with the symmetry parameter  $\beta$ .

Since we need to take the limit  $N \rightarrow \infty$  at the end, the weak disorder limit  $q = 1$  has to be defined more carefully. It is clear from (19) that the appropriate limit should be  $q^N \rightarrow 1$  as  $N \rightarrow \infty$ . In terms of  $\gamma$ , the weak disorder limit becomes  $\gamma N \rightarrow 0$  as  $N \rightarrow \infty$ , which requires that we consider  $\gamma$  as an  $N$ -dependent parameter. The strong disorder limit will then be obtained when  $\gamma N = \text{constant}$  in the limit  $N \rightarrow \infty$ . Since both of these limits can be obtained for  $\gamma \ll 1$ , we will restrict ourselves to only this limit in the following without any loss of generality.

For large  $N$ , approximating the sums in equation (20) by integrals and using the approximation  $g_n \approx 1/\sqrt{nq}$ , we obtain

$$S(X) = \frac{1}{2} \int_0^1 ds \int_1^{\frac{1}{\sqrt{\gamma N q}}} dt e^{(s^2 - t^2)X^2/\mu}. \quad (21)$$

The strong disorder parameter  $\gamma$  appears only in the combination  $\gamma N q$  (note that for  $\gamma \ll 1$ ,  $\gamma N q = 1 - e^{-\gamma N}$ ) and only in the upper limit of the integral over  $t$ . For weak disorder ( $\gamma N \rightarrow 0$ ) this limit becomes  $\infty$  and we retrieve the result of [16, 4]. For finite  $\gamma N$ , we can write a simple expression for large  $X$ :

$$S(X) \approx \frac{\mu^2}{2\pi^4 X^4} \left( 1 - \sqrt{1 - e^{-\gamma N}} e^{-\frac{X^2}{\mu} e^{-\gamma N}} \right) \quad X^2 \gg \mu. \quad (22)$$

Thus even for  $\gamma N \neq 0$ , we recover the weak disorder result [16, 4]  $S = \mu^2/2\pi^4 X^4$  in the limit  $X^2 \gg \mu e^{\gamma N}$ . However, for  $\mu \ll X^2 \ll \mu e^{\gamma N}$  we obtain a qualitatively different behaviour given by  $S = \mu e^{-\gamma N}/X^2$ . Thus, we see that  $q$ -deformation sets a new scale  $\gamma N$  which determines the breakdown of the weak disorder limit.

## 5. Level velocity autocorrelator

We are now in a position to study the level velocity autocorrelator  $C(X)$  defined in (5). It was shown in [5] from microscopic theory that  $C(X)$  for large  $X$  has a universal behaviour independent of the microscopic parameters. This result is valid for weak disorder only and was reproduced within the Brownian motion model of the Gaussian random matrix ensemble in [4]. Simon and Altshuler [16] later showed that a reparametrization  $\epsilon_i(X) = E_i(X)/\Delta$  and  $x = \sqrt{C(0)}X$ , makes the quantity  $c(x) = C(x)/C(0)$  universal in the weak disorder limit and obtained explicit expressions for large and small  $x$ . The quantity  $C(0)$  defines a generalized conductance for arbitrary perturbation  $X$  in the weak disorder limit. This result has not been reproduced within the Brownian motion model of the Gaussian random matrix ensemble because the  $X \rightarrow 0$  limit has a singularity at weak disorder so that  $C(0)$  and therefore the scaling is no longer well defined, and requires an artificial broadening of the levels [16]. Numerical studies of the tight-binding Anderson model with various disorder, where  $X$  is either a magnetic flux or nonuniform background potential, agrees with the above predictions for weak disorder. We will now obtain  $C(X)$  for the  $q$ -ensemble which we expect will give us at least qualitatively correct features at strong disorder.

First we rewrite  $C$  in terms of  $S$  [4],

$$C(E, E', X - X') = \frac{\partial^2}{\partial X \partial X'} \int dE \int dE' S(E, E'; X - X'). \quad (23)$$

For  $E = E' = 0$ , using equations (15)–(19), we obtain the following general form:

$$C(X - X') = c \sum_{m=0}^{N-1} \sum_{n=N}^{\infty} \left( \int_{-\infty}^0 h_n(\sqrt{c}E; q) h_m(\sqrt{c}E; q) e^{-V(\sqrt{c}E; q)} dE \right)^2$$

$$\times \frac{\partial^2}{\partial X \partial X'} e^{(m_q - n_q)} \omega(X - X')^2. \tag{24}$$

Using repeatedly the  $q$ -analogue of integration by parts given in [15], we obtain, e.g.

$$\int_0^\infty h_{2n+1}(\sqrt{c}E; q) h_{2m}(\sqrt{c}E; q) e^{-V(\sqrt{c}E; q)} dE = \frac{(1-q)^{(2n+2m+1)/2} ((2n+1)_q!!)((2m-1)_q!!)}{q^{2m+m(2m+1)+(2n+1)(n+1)} (2n-2m-1)_q}. \tag{25}$$

Note that similar integrals for even-order polynomials  $h_{2m}h_{2n}e^{-V}$  vanish. After some algebra, the expression for  $C(X - X')$  takes the form

$$C(X - X') = \frac{c}{2\gamma} \left[ \sum_{m=0}^{[N/2]} \sum_{n=[N/2]+1}^\infty \frac{c_{n,m} f_{2n+1,2m}(X - X')}{((2n-1)_q - (2m)_q)^2} + \sum_{m=1}^{[N/2]-1} \sum_{n=[N/2]+1}^\infty \frac{c_{m,n} f_{2n,2m+1}(X - X')}{((2n-1)_q - (2m)_q)^2} \right] \tag{26}$$

where

$$c_{n,m} = \frac{q^{2n+2m} [(2n+1)_q!!]^2 [(2m-1)_q!!]^2}{(2n+1)_q! (2m)_q!} \tag{27}$$

and

$$f_{n,m}(X - X') = \frac{\partial^2}{\partial X \partial X'} e^{[(2m)_q - (2n-3)_q] \omega(X - X')^2}. \tag{28}$$

In the large  $N$  limit, we can obtain a rather simple integral form by noting that  $[(2n+1)_q!!]^2 / (2n)_q!$  can be well approximated by  $\sqrt{n_q/\pi}$ . Without any loss of generality we will also choose the constant  $c$  such that the density of levels  $\sigma(E) = S(E, X = X' = 0)$  is unity, so that the mean level spacing is also unity. This gives  $c = \pi^2 \gamma / 2N_q$ ,  $\omega = \pi^2 / 2\mu N_q$ . Replacing the sums by integrals over  $n$  and  $m$  and neglecting terms of order  $1/N$  we obtain

$$C(X) = \frac{1}{2\pi^2} \int_0^{N/2} dm \int_{N/2}^\infty dn \frac{q^{2n+2m}}{[(2n)_q - (2m)_q]^2} f_{n,m}(X) \left[ \sqrt{\frac{n_q}{m_q}} + \sqrt{\frac{m_q}{n_q}} \right]. \tag{29}$$

Substituting  $n_q = 2Nt$  and  $m_q = 2Ns$ , we obtain the final expression for  $C$ ,

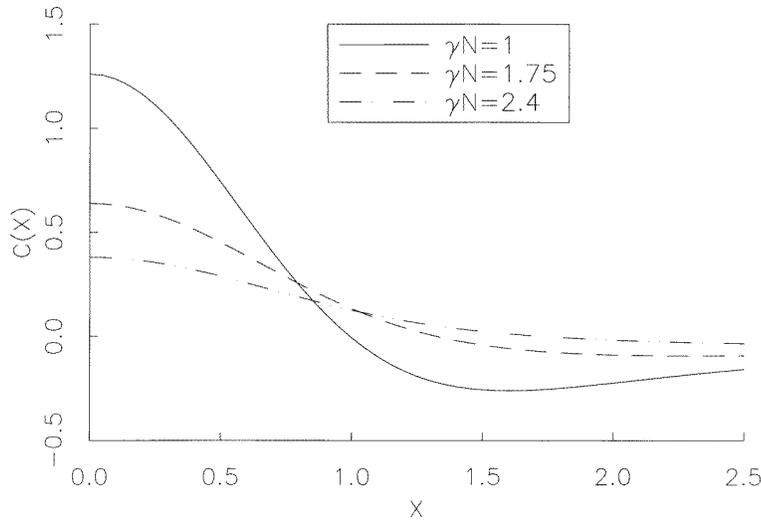
$$C(X) = \frac{1}{\mu} \int_0^1 ds \int_1^{\frac{1}{\sqrt{\gamma N_q}}} dt \frac{s^2 + t^2}{s^2 - t^2} e^{(s^2 - t^2) X^2 \pi^2 / 2\mu} - \frac{\pi^2 X^2}{2\mu^2} \int_0^1 ds \int_1^{\frac{1}{\sqrt{\gamma N_q}}} dt (s^2 + t^2) e^{(s^2 - t^2) X^2 \pi^2 / 2\mu}. \tag{30}$$

This is a remarkably simple expression for  $C(X)$  for the  $q$ -ensemble. As in the density correlator in (21), the deformation parameter  $\gamma$  appears only in the upper limit of the integral over  $t$ , in the combination  $\gamma N_q$ , and (30) reduces to the weak disorder limit of [4, 16] when  $\gamma N \rightarrow 0$ .

In particular, for large  $X$ ,

$$C(X) \approx -\frac{1}{\pi^2 X^2} + \frac{\left( \sqrt{\gamma N_q} + \frac{1}{\sqrt{\gamma N_q}} \right)}{2\pi^2 X^2} e^{-\frac{\pi^2 X^2}{2\mu} \left( \frac{1}{\gamma N_q} - 1 \right)} \frac{\pi^2 X^2}{2\mu} \gg 1. \tag{31}$$

Similar to our results for the density correlator  $S(X)$ , we find that even for  $\gamma N \neq 0$  we retrieve the universal result of weak disorder  $C(X) = -1/\pi^2 X^2$  at sufficiently large  $X$ .



**Figure 1.** Level velocity correlator  $C(X)$  as a function of the external perturbation parameter  $X$  for different values of  $\gamma N$  obtained from numerical evaluation of equation (30). The parameter  $\gamma N$  increases with increasing disorder.

The existence of a new scale set by  $\gamma N_q$  makes the nonuniversal corrections important for large  $\gamma N$ . For small  $X$ ,

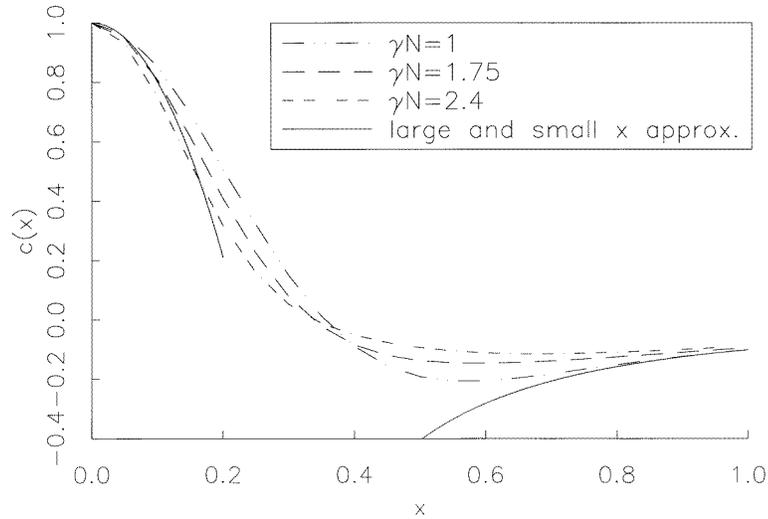
$$C(X) \approx C(0) - \frac{2X^2}{\mu^2(\gamma N_q)^{3/2}} [1 + \gamma N_q - 2(\gamma N_q)^{3/2}] \quad \frac{X^2}{\mu \gamma N_q} \ll 1 \quad (32)$$

where

$$C(0) = \frac{1 - \gamma N_q}{\mu \gamma N_q} \ln \frac{1 + \sqrt{\gamma N_q}}{1 - \sqrt{\gamma N_q}}. \quad (33)$$

Figure 1 shows  $C(X)$  as a function of  $X$  for various values of  $\gamma N$  obtained from numerical evaluation of equation (30) where the weak disorder parameter  $\mu$  has been set equal to unity. The approximate limits (31) and (32) show the qualitative features correctly. Note that  $C(0)$  can be related to the conductance in the weak disorder limit [16]. However, the same is not necessarily true in the strong disorder limit, and therefore at this point it is not possible to identify our parameter  $\gamma N$  quantitatively with the conductance. This prevents us from making any quantitative predictions, or comparing figure 1 with possible real experimental numbers. Nevertheless,  $\gamma N$  increases monotonically with disorder, and it should be possible to compare the above qualitative behaviour with, for example, numerical results for the tight-binding Anderson model at strong disorder.

An expected difference from the weak disorder result is that we have a well-defined limit at  $X \rightarrow 0$ . The sum rule  $\int_0^\infty C(X) dX = 0$  is explicitly satisfied. The width of the peak at  $X = 0$  is of order  $\gamma N$ , which goes to zero in the weak disorder limit. As we mentioned before, it is this singularity at  $X \rightarrow 0$  which complicates the evaluation of the scaled  $c(x)$  in the undeformed case, where one needs an additional broadening function [16] to evaluate  $C(0)$ . The same problem remains here as long as the width is much smaller than the average dimensionless level spacing, which we have chosen to be unity. We avoid this complication by restricting our subsequent analysis to the case  $\gamma N \geq 1$ . The qualitative aspects of the breakdown of universality will be clearer in this regime since we will not



**Figure 2.** The scaled velocity correlator  $c(x)$  obtained from numerical evaluation of equation (30) for different values of  $\gamma N$ . For comparison, the full curve shows the universal weak disorder limit from [16].

need any *ad hoc* broadening procedure. The scaled  $c(x)$  can be expressed from (30)–(33) for  $\gamma N \geq 1$  as

$$\begin{aligned}
 c(x) &= 1 - \frac{\sqrt{\gamma N_q}(1 + \gamma N_q - 2(\gamma N_q)^{3/2})}{(1 - \gamma N_q) \ln[(1 + \sqrt{\gamma N_q})/(1 - \sqrt{\gamma N_q})]} x^2 & \frac{x^2}{\mu \gamma N_q} \ll C(0) & \quad (34) \\
 &= -\frac{1}{\pi^2 x^2} \left[ 1 - \frac{\sqrt{\gamma N_q} + \frac{1}{\sqrt{\gamma N_q}}}{2} e^{-\frac{x^2}{\ln[(1 + \sqrt{\gamma N_q})/(1 - \sqrt{\gamma N_q})]}} \right] & \frac{x^2}{\mu \gamma N_q} \gg C(0). & \quad (35)
 \end{aligned}$$

For all  $\gamma N$  the universality still exists for large enough  $x$  where the correction term becomes exponentially small. However, for smaller  $x$  the strong disorder parameter cannot be scaled away and the universality in this regime breaks down. This breakdown is associated with the breakdown of the universality of spectral correlations at strong disorder [6]. In figure 2 we show the scaled  $c(x)$  for different values of  $\gamma N$  evaluated numerically from the full expression (30). Since both  $x$  and  $c(x)$  are scaled variables, this is a universal plot that can be compared directly with real or numerical experiments.

We mention that it should also be possible to use equations (15)–(19) to study in the strong disorder regime the statistics of level curvature  $\frac{1}{\Delta} (d^2 E_i(X)/dX^2)_{X \rightarrow 0}$  which contains information about quasilocalized states and multifractality [17], as well as the parametric number variance  $\langle [N(E, X' + X) - N(E, X)]^2 \rangle$  where  $N(E, X)$  is the spectral staircase function measuring the number of levels below some energy  $E$  [18].

## 6. Conclusion

The statistical properties of the energy levels of a weakly disordered mesoscopic system in the presence of an external perturbation can be studied using Dyson's Brownian motion model of Gaussian random matrix ensembles given by (2) and (3) with  $V(E) \propto E^2$ . In this

paper we argue that a generalization of the model with  $V(E)$  replaced by  $V(E; q)$  given in (6) corresponds to the generalization from weakly disordered to strongly disordered systems, the parameter  $q$  playing the role of strong disorder. This defines the Brownian motion model of a  $q$ -ensemble because the corresponding limiting equilibrium distribution for the energy levels are given by the the joint probability distribution of levels of the  $q$ -Hermite ensemble. By mapping (2) on to a time-dependent Schrödinger equation and noting that the many-body wavefunction of the corresponding time-independent solution has a Vandermonde determinant form, we obtain the parametric level density correlation function  $S(X)$  defined in (4). Equation (21) gives a simple integral expression for  $S(X)$  in the large  $N$  limit and show how the weak disorder result breaks down due to the presence of an additional scale set by the deformation parameter. Using the  $q$ -analogue of integration by parts for  $q$ -Hermite polynomials, we also obtain the parametric velocity correlation function  $C(X)$  for the  $q$ -ensemble. The explicit expression (30) shows the breakdown in this regime of the weak disorder universality in the appropriately scaled quantity. It would be interesting to test the qualitative aspects of the predictions by comparing these results with numerical studies of the microscopic tight-binding Anderson model at strong disorder.

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