

# Parametric number variance of disordered systems in the multifractal regime

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**Abstract.** We study the effect of an external field  $X$  on the energy levels of a disordered system by evaluating the parametric number variance (PNV). The weak disorder regime is studied within the Gaussian random matrix theory, while the multifractal regime is studied by considering the  $q$ -deformed random matrices. The PNV at both small and large values of  $X$  has distinct features in the weak disorder and multifractal regimes that should be observable in numerical studies.

## 1. Introduction

The study of various statistical properties of energy levels [1] and transmission eigenvalues [2] has played an important role in our understanding of a wide variety of complex quantum systems. The energy level statistics in a classically chaotic system is characterized by level repulsion and universal spectral rigidity, and is well described by the Gaussian random matrix ensembles introduced by Wigner and Dyson [3]. The level statistics of classically integrable systems, on the other hand, follow the universal Poisson statistics, with no level repulsion. In the particularly interesting case of a mesoscopic disordered quantum conductor, the energy levels follow the Wigner–Dyson statistics in the weakly disordered metallic regime where the wavefunctions are extended [4], while the statistics become Poissonian in the strongly disordered insulating regime where the wavefunctions are localized at different points and therefore do not overlap with one another. Moreover, in three dimensions, the system exhibits the Anderson metal–insulator transition at a critical intermediate disorder, where the wavefunctions are multifractal [5]. The level statistics in this critical region also turns out to be universal, but different from both Wigner–Dyson and Poisson statistics [6]. Thus, for example, the level number variance  $\Sigma_2(N) = \langle (\delta N)^2 \rangle$  is logarithmic in  $N$  for large  $N$  in the Wigner–Dyson regime [7], linear in  $N$  with a slope  $\chi = 1$  in the Poisson regime, and linear in  $N$  with  $\chi < 1$  in the critical regime [8] where  $\chi$  is related to the multifractal exponent describing overlapping of local densities [9]. Clearly, the level statistics contains important signatures of the various phases, including the critical region.

In the commonly used statistical measures such as the spacing distribution or the number variance, the ensemble average is taken either as a spectral average on an ensemble member over different intervals, or over essentially independent members, such as different nuclei or different realizations of randomness in a quantum conductor. Goldberg *et al* [10] introduced a different kind of statistical measure, the *parametric* number variance (PNV) which measures the correlation of fluctuations in levels as a parameter *external* to the system is varied. The

parameter can represent the strength of an external perturbation such as a magnetic field, and is therefore experimentally tunable. The levels as a function of the parameter will show multiple crossings if the system is Poissonian, while a chaotic system will show avoided crossings due to level repulsion. Simons and Altshuler [11] have shown that the PNV becomes universal in the weakly disordered regime of a quantum conductor if the level and the parameter are both appropriately scaled. This universality breaks down at stronger disorder as observed in numerical studies. No attempt has yet been made to study the PNV in the critical region, which is clearly of interest. In this paper we obtain the PNVs in the critical regime for various strengths of multifractality and compare them with the universal weakly disordered regime.

Our study is based on a recent observation that there exists a new class of random matrix ensembles which describes critical states with multifractal eigenfunction statistics [12]. In particular, the  $q$ -Hermite random matrix ensembles introduced in [13] describe, in the limit  $N \rightarrow \infty$  where  $N$  is the number of levels, critical states with different strengths of multifractality given by the parameter  $\gamma = \ln(1/q)$ . The advantage of studying the  $q$ -ensembles is that the problem is exactly solvable; the correlation functions can be written down for all strengths of multifractality in terms of properties of  $q$ -Hermite polynomials. In the present paper we shall use the  $q$ -Hermite ensemble to obtain the PNV in an external perturbation  $X$  in the critical regime. We shall restrict ourselves to the case of unitary ensembles only, which corresponds to a broken time-reversal symmetry.

The paper is organized as follows. In section 2 we define PNV and introduce Dyson's Brownian motion model [14] to study parametric correlations [15]. In section 3 we introduce the  $q$ -Hermite model and obtain PNVs for large  $X$  within a hydrodynamic approximation. In section 4 we use a recently obtained exact solution to evaluate PNVs at zero energy but for all  $X$ . We then present the summary and conclusion in section 5.

## 2. PNV and Dyson's Brownian motion model

We start with Dyson's Brownian motion model [14] of random matrix ensembles given by the Fokker-Planck equation

$$\mu \partial_\tau P = \sum_{i=1}^N \partial_{E_i} \left( P \partial_{E_i} W + \frac{1}{\beta} \partial_{E_i} P \right) \quad W(E_i) = - \sum_{i < j}^N \ln |E_i - E_j| + \sum_i^N V(E_i). \quad (1)$$

Here, the fictitious time  $\tau$  describes the dynamics of the  $N$  eigenvalues at position  $E_i(\tau)$  and the parameter  $\mu$  represents the friction coefficient. The limiting distribution  $P(\{E_i\}, \tau \rightarrow \infty)$  is then given by

$$P(E_i) = \prod_i^N (E_i - E_j)^2 \exp[-V(E_i)]. \quad (2)$$

For  $V(E) = cE^2/2$ , where  $c$  is an arbitrary positive constant (which determines the mean level spacing), this is the well-known joint probability distribution of the Gaussian unitary ensemble [7]. Following Beenakker [15], we identify  $\tau = X^2$  where  $X$  is an external perturbation, which then gives us the dispersion of the energy levels in response to the external field, and will allow us to obtain various energy level correlation functions at different values of the external perturbation parameter. This is by no means a unique way of introducing parametric dependence in random matrix ensembles; neither is it clear in terms of microscopic theory what particular kind of perturbation it corresponds to. As a phenomenological model, it reproduces the density and velocity correlation functions of a non-interacting spinless particle in a disordered ring subject to both a background potential and an Aharonov-Bohm flux [11, 16].

In this work we shall be primarily interested in the PNV defined as [10]

$$U(X) = \langle [N(E, X + X') - N(E, X - X')]^2 \rangle \quad (3)$$

where  $N(E, X) = \sum_i \Theta(E - E_i(X))$  is the spectral staircase function which counts the number of levels up to an energy  $E$ , and  $\langle \dots \rangle$  represents an average over intervals of length  $X$  at fixed  $E$ . Note that  $U(X \rightarrow 0) = 0$ . The large- $X$  behaviour can also be determined easily,  $U(X \rightarrow \infty) = 2\Sigma_2(E)$  where  $\Sigma_2$  is the number variance. Using  $N = \int^E \rho(E', X) dE'$  and the parametric density–density correlation function

$$S(E, E', X) = \langle \rho(E, X^2) \rho(E', 0) \rangle \quad (4)$$

where

$$\rho(E, \tau) = \int_{-\infty}^{\infty} dE_1 \dots dE_n P(\{E_n\}, \tau) \sum_i^N \delta(E - E_i) \quad (5)$$

is the density of levels, one can rewrite  $U(X)$  as

$$\langle [N(E, X' + X) - N(E, X - X')]^2 \rangle = \int^E \int^E dE_1 dE_2 (S(E_1, E_2, 0) - S(E_1, E_2, X)). \quad (6)$$

We shall first use Dyson's solution for the density in the  $N \rightarrow \infty$  limit and use the hydrodynamic approximation to obtain the density–density correlation function. However, the hydrodynamic approximation fails in the important limit of both  $E$  and  $X \rightarrow 0$ . We shall then use a recently obtained exact finite- $N$  result for  $E = 0$  to evaluate  $U(X)$  for all  $X$ . The two results combined give us a fairly detailed description of  $U(X)$  for the  $q$ -Hermite ensembles.

### 3. The hydrodynamic approximation

Dyson [14] solved the Fokker–Planck equation (1) to obtain an evolution equation for the density at  $N \rightarrow \infty$

$$\mu \frac{\partial}{\partial \tau} \rho(E, \tau) = \frac{\partial}{\partial E} \left[ \rho(E, \tau) \frac{\partial}{\partial E} \left( V(E) - \int_{-\infty}^{\infty} dE' \rho(E', \tau) \ln |E - E'| \right) \right]. \quad (7)$$

The hydrodynamic approximation involves linearizing the density  $\rho = \rho_0 + \delta\rho$  around a constant equilibrium density  $\rho_0$ . This is a good approximation in the large- $N$  limit, and for energy  $E \gg \Delta$  where  $\Delta$  is the mean level spacing. In this approximation, the correlation functions become translationally invariant (function of  $E - E'$  only), and the Fourier transform of the parametric density–density correlation function can be written as

$$S(k, X) = \rho_0^2 \delta(k) + (1 - b(k)) \exp(-|k| \eta X^2) \quad (8)$$

where  $b(k)$  is the two-level form factor which is the Fourier transform of the two-level correlation function [7]

$$Y_2(E - E') = -S(E - E', 0) + \rho_0^2. \quad (9)$$

The PNV then becomes

$$U(X) = 4 \int_{-\infty}^{\infty} dk \frac{1 - \cos(kE)}{k^2} (1 - b(k)) \left( 1 - e^{-\eta |k| X^2} \right) \quad (10)$$

where  $\eta = \pi \rho_0 / \mu$ . Note that because of the hydrodynamic approximation, the equation for  $S(k, X)$  is not valid for  $k > 1/\Delta$ . However, for  $X > \Delta \sqrt{\mu}$ ,  $S(k, X)$  is exponentially small, so the integrated measure  $U(X)$  is valid in this range of  $X$ . We can rewrite the PNV in terms of its large- $X$  asymptotics as

$$U(X \gg \Delta \sqrt{\mu}) = 2\Sigma_2(E) - \int_{-\infty}^{\infty} dk f(|k|) e^{-|k| X^2} \quad (11)$$

where

$$f(|k|) = \frac{4(1 - \cos(kE))}{k^2} (1 - b(|k|)). \quad (12)$$

Note that we have explicitly used the fact that  $b(k)$  is an even function of  $k$ .

The Gaussian random matrix ensemble is defined by the choice  $V(E) = cE^2$  in equation (2). The two-level correlation function  $Y_2(s)$ , which is well known in this case, is given by [7]

$$Y_2(E - E') = \left[ \frac{\sin(E - E')}{\pi(E - E')} \right]^2 \quad (13)$$

which leads to the form factor

$$b(k) = \begin{cases} 1 - |k|/2\pi & |k| < \pi \\ 0 & |k| \geq \pi \end{cases}. \quad (14)$$

To extract the leading behaviour for large  $X$ , we can expand  $f(k)$  in powers of  $k$ . For the Gaussian random matrix ensemble this is simply

$$f(|k|) = \frac{1 - \cos(kE)}{2\pi|k|} = \frac{E^2 k}{4\pi} + O(k^3). \quad (15)$$

This gives for the PNV the following large- $X$  behaviour in the weak disorder regime:

$$U(X \gg \Delta\sqrt{\mu}) = 2\Sigma_2(E) - \frac{E^2}{2\pi X^4} \quad (16)$$

where the number variance  $\Sigma_2(E)$  increases logarithmically with  $E$  for large  $E$  [7].

It was argued in [12] that the  $q$ -Hermite ensemble, given by the choice [13]

$$V(E; q) = \ln \theta_2(i \sinh^{-1} E; \sqrt{q}) + \text{constant} \quad (17)$$

where  $\theta_2$  is the second Jacobi theta function, represents an ensemble of critical states. While this particular choice makes the distribution solvable because  $V(E; q)$  is the weight factor of the  $q$ -Hermite orthogonal polynomials, and one can therefore write down the correlation functions in terms of these polynomials, the important property that makes this choice relevant for the critical distribution is the fact that  $V(E; q)$  grows only as  $\ln^2 E$  for large  $E$  as opposed to a power law. The two-level correlation function for the  $q$ -Hermite ensemble in the weakly multifractal translationally invariant regime was obtained in [13]

$$Y_2(E - E'; q) = \frac{\gamma^2}{4\pi^2} \frac{\sin^2[\pi(E - E')]}{\sinh^2[\gamma(E - E')/2]} \quad \gamma \ll 2\pi^2 \quad (18)$$

where the parameter  $\gamma = \ln(1/q)$  is related to the multifractality exponent  $D_2$  [12]. The static form factor  $b(k)$  in this regime was obtained in [17], and is given by

$$b(k) = \frac{1}{4\pi} \left( (|k| + 2\pi) \coth \left[ \frac{(|k| + 2\pi)\pi}{\gamma} \right] + (|k| - 2\pi) \coth \left[ \frac{(|k| - 2\pi)\pi}{\gamma} \right] - 2|k| \coth \left( \frac{|k|\pi}{\gamma} \right) \right). \quad (19)$$

The corresponding leading term in  $f(k)$  now has a  $k$ -independent term proportional to  $1 - b(0) \neq 0$  for  $\gamma \neq 0$ :

$$f(k) = \frac{E^2}{2} (1 - b(0)) \quad (20)$$

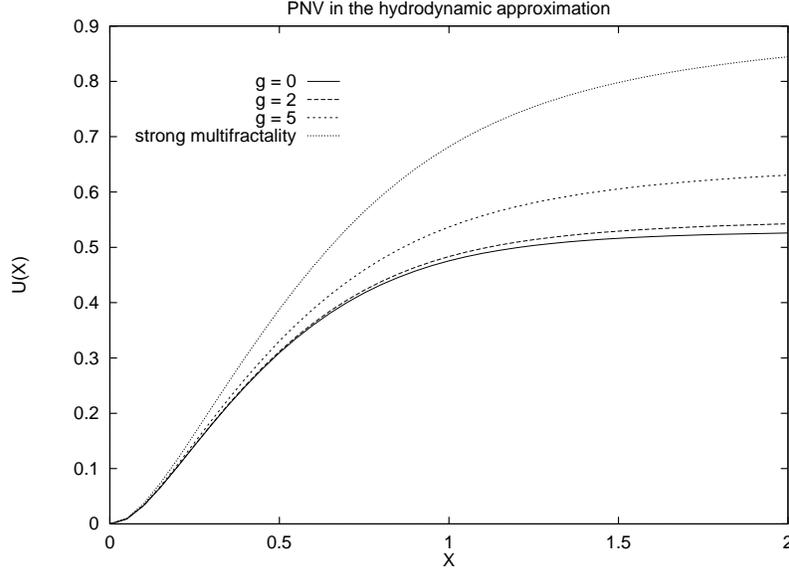
where

$$b(0) = \coth\left(\frac{2\pi^2}{\gamma}\right) - \frac{\gamma}{2\pi^2}. \quad (21)$$

This results in a  $1/X^2$  dependence for large  $X$

$$U(X \gg \Delta\sqrt{\mu}) = 2\Sigma_2(E) - \frac{E^2}{\pi X^2}(1 - b(0)) \quad (22)$$

different from the  $1/X^4$  dependence of equation (16).



**Figure 1.** PNV in the hydrodynamic approximation for different values of  $g = \gamma$  in the weak multifractality regime. The limiting strong multifractality result is also included.

The result of numerical integration of equation (10) is plotted in figure 1 for various values of  $\gamma \ll 2\pi^2$ . Note that the limit  $\gamma \rightarrow 0$  agrees with the weak disorder limit obtained in equation (16). However for any finite  $\gamma$ , the large- $X$  behaviour becomes qualitatively different from the weak disorder limit. This is expected, because the asymptotic behaviour in the  $X \rightarrow \infty$  limit is given by twice the ordinary number variance,  $\Sigma_2(E)$ . The number variance was obtained in [17], with the leading term proportional to  $(E - E')$  with a coefficient  $1 - b(0)$  which is proportional to  $\gamma$ . Thus, the deviation from the weak disorder limit is larger for stronger multifractality. Note that the magnitude of PNV for a given  $X$  increases with increasing  $\gamma$ .

In the opposite limit of strong multifractality, the two-level correlation function for the  $q$ -Hermite model was proposed in [12] to be of the triangular form:

$$Y_2(E - E') = \begin{cases} 1 - 2|E - E'| & |E - E'| < 1/2 \\ 0 & \text{otherwise.} \end{cases} \quad (23)$$

The corresponding form factor is

$$b(k) = \frac{4}{k^2}(1 - \cos(k/2)). \quad (24)$$

The PNV in this case becomes

$$U(X) = 2\Sigma_2(E) - \frac{3E^2}{2\pi X^2}, \quad (25)$$

The number variance in this regime is simply one half of the Poisson value [12], so  $U(X)$  is independent of any disorder parameter.

#### 4. PNV for $E = E' = 0$

The above results are valid in the large- $X$  regime only, owing to the hydrodynamic approximation used. In order to obtain the small- $X$  behaviour of the PNV, we will use the Brownian motion model of the  $q$ -Hermite ensemble recently introduced in [18]. Again the model is not a unique way of incorporating the parametric dependence, and it is not clear whether the results are model independent. Nevertheless, the velocity correlation function within the model reproduces at least the qualitative features in the deviations from universality at strong disorder seen in numerical studies [11].

The calculations are simplest in the limit  $E = E' = 0$ . While this will restrict the generality of our results, it will be sufficient for our main goal of obtaining the small- $X$  (as well as large- $X$ ) behaviour of the PNV. It was proposed in [18] that the appropriate Brownian motion model for the  $q$ -Hermite ensemble is described by the same Fokker–Planck equation as (1) where the limiting distribution is given by (2), with the potential  $V(E)$  chosen to be  $V(E; q)$  given by equation (17). The parametric density–density correlation function  $S(E, E'; X)$  can then be written down exactly for finite  $N$  in terms of  $q$ -Hermite polynomials. In the simpler limit of  $E = E' = 0$ ,  $\gamma \ll 1$  and large  $N$  this is given by

$$S(X) = \frac{1}{2} \int_0^1 ds \int_1^{1/\sqrt{\gamma N_q}} dt \exp[(s^2 - t^2)X^2/\mu] \quad (26)$$

where  $N_q = (1 - q^N)/(1 - q)$ . Using equation (6) one obtains

$$U(X) = \int_0^1 ds \int_1^{1/\sqrt{\gamma N_q}} dt \frac{t^2 + s^2}{(t^2 - s^2)^2} (1 - \exp[(s^2 - t^2)X^2/\mu]). \quad (27)$$

Note that we can take advantage of the finite- $N$  result to define the parameter  $\gamma N$  which can vary from 0 to  $\infty$ . While the  $N \rightarrow \infty$  limit of the  $q$ -Hermite model has been shown to correspond to a critical ensemble, it is expected that for the finite- $N$  case, this should correspond to the limit  $N \rightarrow \infty$ ,  $\gamma \rightarrow 0$ ,  $\gamma N = \text{constant}$ . For small  $X$ , in the regime  $X^2 \ll 1/\gamma N_q$ , one can expand the exponential and obtain

$$U(X \ll 1/\sqrt{\gamma N_q}) = X^2 \int_0^1 ds \int_1^{1/\sqrt{\gamma N_q}} dt \frac{t^2 + s^2}{(t^2 - s^2)^2} + O(X^4). \quad (28)$$

Thus, to leading order we obtain

$$U(X \ll 1/\sqrt{\gamma N_q}) = 2X^2[(1 - \gamma N_q/2) + O(\gamma^2 N_q^2)]. \quad (29)$$

This is in contrast to the linear increase for weak disorder in the universal regime [11, 20]. We recover this linear regime for weak disorder by taking the  $\gamma \rightarrow 0$  limit. In this case, the upper limit of the integral in equation (27) becomes  $\infty$ . Rewriting

$$U(X \ll 1, \gamma = 0) = \int dX^2 \int dX^2 \left[ \int_0^1 ds e^{s^2 X^2} \int_1^\infty dt t^2 e^{-t^2 X^2} + \int_0^1 ds s^2 e^{s^2 X^2} \right. \\ \left. \times \int_1^\infty dt e^{-t^2 X^2} \right] \quad (30)$$

and retaining only the leading term ( $1/X^3$ ) from the integrals over  $t$ , we finally obtain

$$U(X \ll 1, \gamma = 0) = \sqrt{\pi}|X|. \quad (31)$$

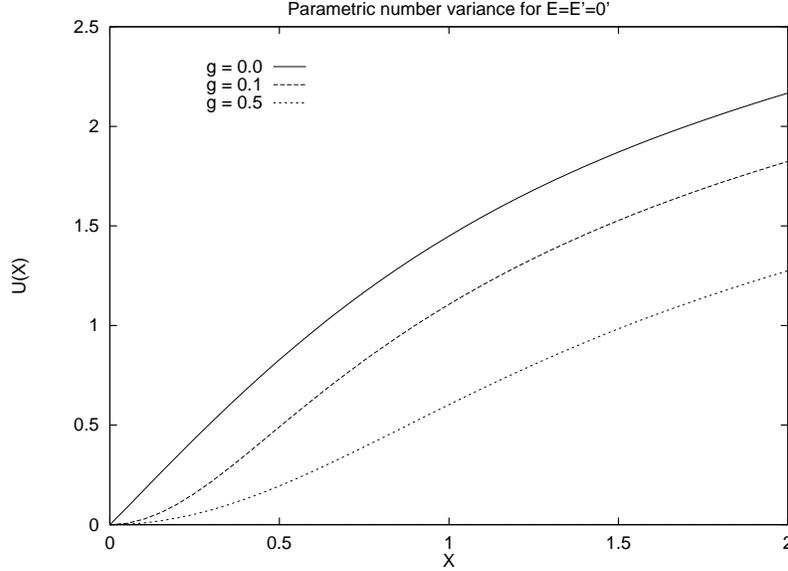
The difference in the  $X$  dependence disappears in the large- $X$  limit. Again, rewriting in the same way as equation (30), where the limits of the  $t$  integrals are now  $1/\sqrt{\gamma N_q}$  instead of  $\infty$ , and keeping only the leading terms in the  $s$  and  $t$  integrals for large  $X$  we obtain

$$U(X \gg 1/\sqrt{\gamma N_q}) = 2 \ln(X) + C + O(1/X^4). \quad (32)$$

Here

$$C = \int_0^1 ds \int_1^{1/\sqrt{\gamma N_q}} dt \frac{t^2 + s^2}{(t^2 - s^2)^2} \quad (33)$$

is weakly dependent on  $\gamma$ . Figure 2 shows  $U(X)$  for various strengths of the parameter  $\gamma N$  obtained directly from numerical integration of equation (27). It shows the clear difference in the  $X$  dependence of  $U(X)$  for small  $X$  for zero and non-zero values of  $\gamma N$ . For any finite  $\gamma N$ ,  $U(X)$  starts with a quadratic dependence in the regime  $X < X_c$  where  $X_c = \sqrt{2\mu\gamma N_q/\pi^2}$ . Beyond  $X_c$ , the linear behaviour is recovered, with the same slope but shifted towards larger  $X$  values compared with the universal curve. Since the maximum value of  $\gamma N_q$  is unity,  $X_c$  eventually saturates for  $\gamma N_q \gg 1$ . Note that, in contrast to the large energy result of figure 1, the magnitude of the PNV for a given  $X$  at small  $E$  decreases with increasing  $\gamma$ . The difference arises from the fact that for large  $E$ , i.e. in the bulk of the spectrum, the two-level correlation function is translationally invariant (equation (18)), and the number variance  $\Sigma_2(E)$  increases with  $\gamma$ . However, in the small- $E$  regime, the correlation function is no longer translationally invariant [13] and, in particular if the energy range contains the origin, there are additional correlations associated with the ‘ghost peaks’ at  $E = -E$  [21]. The resulting PNV is very different. In fact, this difference can be used to determine directly whether the breakdown of translational invariance is of relevance for disordered systems in the multifractal regime, by comparing the PNV from numerical solutions of microscopic models.



**Figure 2.** PNV for  $E = E' = 0$  for different values of  $g = \gamma N_q$ .

## 5. Summary and conclusion

The PNV for disordered systems is a measure of fluctuations of energy levels due to the presence of an external field  $X$ . In the weak disorder regime, the PNV can be obtained from Dyson's Brownian motion model for Gaussian random matrices. We use the  $q$ -deformed random matrices, which are known to describe multifractal states with multifractality  $\gamma$ , to obtain the PNV in the critical regime. It turns out that PNVs at small and large energies have different  $X$  as well as  $\gamma$  dependences, which reflect the translational non-invariance of the two-level correlation function. One striking difference between the weak disorder regime and the multifractal regime appears in the small- $X$  region at small energy, where  $U(X)$  has a linear dependence on  $X$  in the weak disorder regime but a quadratic dependence in the multifractal regime. For a fixed value of  $X$ , the PNV decreases with increasing  $\gamma$  in this regime. The other difference appears at large energies, where  $U(X)$  has a  $1/X^4$  dependence at weak disorder but a  $1/X^2$  dependence in the multifractal regime. For a fixed value of  $X$ , the PNV increases with increasing  $\gamma$  in this regime. These differences should be observable in numerical studies of disordered systems. Since multifractality is expected to be quite generic for critical states, the above features may also be relevant for other physical systems which show a transition from a chaotic to an integrable state.

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