

Conductance distribution of disordered quasi one-dimensional wires

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Abstract. We determine analytically the distribution of conductances of quasi one-dimensional disordered electron systems, neglecting electron-electron interaction, for all strengths of disorder. We find that in the crossover region between the metallic and insulating regimes, $P(g)$ is highly asymmetric. The average and the variance of $P(g)$ for $\langle g \rangle \gg 1$ and $\langle g \rangle \ll 1$ are shown to agree with exact results.

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1 Introduction

The conductance g (in units of e^2/h) of a mesoscopic disordered electron system is known to fluctuate strongly from sample to sample, or as a function of an external parameter such as a magnetic field or a gate voltage controlling the electron density [1]. In the metallic regime these fluctuations are universal and of Gaussian nature, i.e. the variance of g is given by a pure number independent of the specifics of the system, depending only on the presence (or absence) of time reversal symmetry with respect to orbital or spin motion (orthogonal, unitary and symplectic cases) [2]. For increasing disorder the fluctuations grow and are no longer universal, Gaussian and symmetric about the average value. When the variance becomes as large as the average conductance it is necessary to consider the full distribution of conductances, $P(g)$. The situation is simple again in the localized regime, where $P(g)$ is known to be a log-normal distribution, with variance $\sim \langle \ln(1/g) \rangle$ [3]. Except for numerical studies of finite size systems [4,5,6] little is known about the conductance distribution in the crossover regime. These studies suggest that the distribution is highly asymmetric [5], with $-\ln P(g)$ increasing like a power of g for $g \rightarrow \infty$ and like $(\ln g)^2$ for $g \rightarrow 0$ [6]. The shape of $P(g)$ in the crossover regime depends on the spatial dimension, but appears to be compatible with one-parameter scaling, and hence universality at a true metal-insulator transition in $d \geq 3$.

On the other hand, analytical results for finite systems (length L) in $d = 2 + \epsilon$ dimensions ($\epsilon \ll 1$), where a weak disorder approximation can be applied, showed

that the higher moments of $P(g)$ are non-universal and diverge in the limit $L \rightarrow \infty$ [7]. It has been proposed, however, that these results are not incompatible with a universal distribution at the critical point, which was determined to be a Gaussian with power law tails [8]. This may seem surprising in view of the numerical results [4,6]. One should keep in mind, however, that in $d = 2 + \epsilon$ dimensions the critical conductance at the transition is large $\langle g \rangle_c = 1/\epsilon \gg 1$, which is deep in the metallic regime and hence is quite different from the critical value $\langle g \rangle_c \sim 1$ expected, e.g. in $d = 3$ dimensions.

Here we consider the conductance distribution for the simpler case of a quasi one-dimensional wire of width $W \ll \ell$, where ℓ is the mean free path due to elastic scattering, and length $L \gg \ell$. Although in this case (for orthogonal and unitary symmetry) all states are localized in the thermodynamic limit $L \rightarrow \infty$, for finite length L the system exhibits well defined metallic and insulating regimes, and a smooth crossover between them. To be more precise, this is the case for a quantum wire with ideal leads of the same cross section, for which the perpendicular momenta at given energy E_F are quantized into N discrete levels, providing N channels of transport. The localization length $\xi = N\ell$ in this case, which for $N \gg 1$ allows for a metallic regime to be realized in short wires ($\xi \ll L$), whereas for long wires ($L \gg \xi$) the system is of insulating character. For strictly one-dimensional weakly disordered systems the conductance distribution may be obtained analytically [9], but in this case a metallic regime is absent. The dimensionless conductance g of a quantum wire can be expressed in terms of the N transmission eigenvalues T_i of the corresponding scattering problem as $g = \sum_{i=1}^N T_i$ [10]. The joint probability distribution $P_T(\{T_i\})$ of the T_i may be obtained [11] from a Fokker-Planck equation known as the Dorokhov-Mello-Pereyra-Kumar (DMPK) equation [12], in the limit of large N . The distribution $P_T(\{T_i\})$ depends only on the parameter L/ξ . The DMPK approach has been shown to be in agreement with the exact formulation of the problem in terms of a supersymmetric nonlinear sigma model [13]. Within the latter formulation the average and the variance of the conductance have been calculated for all values of L/ξ [14].

To calculate the conductance distribution $P(g)$ from the joint distribution $P_T(\{T_i\})$ an N -fold integration is required, subject to the constraints $0 \leq T_i \leq 1$ and $\sum_i T_i = g$, which has only been done in the limiting cases of $L/\xi \ll 1$ (metal) and $L/\xi \gg 1$ (insulator). Here we describe a systematic and simple method, valid for all values of L/ξ , to obtain $P(g)$ from $P_T(\{T_i\})$ essentially analytically. We employ a generalized saddlepoint approximation, which recovers all the known results in the limiting cases, and provides results in the crossover regime in semiquantitative agreement with numerical data and with analytical results for the average and the variance of g . In particular, we find that $P(g)$ for $L/\xi \sim 1$ is given by a “one-sided” log-normal distribution for $g < 1$, with a Gaussian tail at $g > 1$ [15].

2 Generalized saddlepoint approximation

It is useful to introduce variables λ_i and x_i defined by $T_i = (1 + \lambda_i)^{-1}$, $\lambda_i = \sinh^2 x_i$, in terms of which the conductance distribution may be represented as

$$P(g) = \frac{1}{Z} \int_{-\infty}^{\infty} \frac{d\tau}{2\pi} e^{i\tau g} \int_0^{\infty} (\prod_{i=1}^N d\lambda_i) \exp \left[- F(\{\lambda_i\}; \tau) \right] \tag{1}$$

The “free energy” F for unitary symmetry (for orthogonal and symplectic symmetry the calculation is analogous) is obtained from the DMPK equation [11]

$$F = 2 \sum_i V(\lambda_i) + \sum_{i,j} u(\lambda_i, \lambda_j) + \sum_i \frac{i\tau}{1 + \lambda_i} \tag{2}$$

where $u(\lambda_i, \lambda_j)$ is generated by the Jacobian of the integration over the transfer matrix elements and leads to “level repulsion”. Here one may interpret $V(\lambda_i) = (\xi/2L)x_i^2$ as a “one-body potential” and $u(\lambda_i, \lambda_j) = -\frac{1}{2}(u_1 + u_2)$, with $u_1(\lambda_i, \lambda_j) = \ell n | \lambda_i - \lambda_j |$ and $u_2(\lambda_i, \lambda_j) = \ell n | x_i^2 - x_j^2 |$ as an “interaction potential” of charges at positions λ_i . In the metallic regime $V(\lambda)$ gives rise to a confinement of the charges in the regime $\lambda_i < 1$ (note $V(\lambda) \propto \lambda^2, \lambda < 1$), such that a description in terms of a charge density $\rho(\lambda)$ is appropriate. In the insulating regime ($V(\lambda) \sim \ell n^2 \lambda, \lambda \gg 1$) the logarithmic repulsion between the charges dominates the potential $V(\lambda)$, leading to an exponentially large separation between the charges, of which only the one closest to the origin is of importance.

To capture both aspects we keep the first eigenvalue λ_1 separate and represent all the other eigenvalues by a continuum density $\rho(\lambda)$, beginning at a lower limit $\lambda_2 > \lambda_1$. The free energy then takes the form

$$F(\rho(\lambda); \lambda_1, \lambda_2; \tau) = 2 \int_{\lambda_2}^{\infty} d\lambda \rho(\lambda) V_{tot}(\lambda) + 2V(\lambda_1) + \int_{\lambda_2}^{\infty} d\lambda d\lambda' \rho(\lambda) u(\lambda, \lambda') \rho(\lambda') + \frac{i\tau}{1 + \lambda_1} \tag{3}$$

where $V_{tot}(\lambda) = V(\lambda) + u(\lambda, \lambda_1) + \frac{i\tau}{1 + \lambda}$. The integration over variables $\lambda_3, \dots, \lambda_N$ in (1) is replaced by a functional integration $D[\rho(\lambda)]$. The latter is done in saddlepoint approximation, leading to the integral equation for $\rho(\lambda)$

$$\int_0^{\infty} d\zeta' [u_1(\zeta - \zeta') + u_2(\zeta + \lambda_2, \zeta' + \lambda_2)] \rho(\zeta' + \lambda_2) = 2V_{tot}(\zeta + \lambda_2) \tag{4}$$

where $\zeta = \lambda - \lambda_2$ has been introduced. This integral equation can be solved approximately by putting $u_2(\zeta + \lambda_2, \zeta' + \lambda_2) = u_2(\zeta, \zeta') + \Delta u$ and neglecting Δu in lowest order, which is exact in the limits $\zeta, \zeta' \gg \lambda_2$ and $\zeta, \zeta' \ll \lambda_2$. The leading correction term $\Delta u \propto \lambda_2$ in the metallic regime can be treated perturbatively by replacing ρ in the integral involving Δu by the saddlepoint solution for $\Delta u = 0$, (the results of this approximation will be presented below).

The saddlepoint density $\rho_{sp}(\lambda)$ is found to develop negative parts for small λ_2 , although $\rho(\lambda)$ is positive by definition. We take this as a signal that configurations of charges with $\lambda_2 < \lambda_c$ (for which $\rho_{sp}(\lambda)$ starts to turn negative at small λ) are unphysical and should be deleted. This is done by limiting the integration on λ_2 to $\lambda_2 > \lambda_c + \lambda_1$. The free energy after the saddlepoint integration on $\rho(\lambda)$ is found as

$$F(\lambda_1, \lambda_2; \tau) = \int_{\lambda_2}^{\infty} d\lambda V_{eff}(\lambda) \rho_{sp}(\lambda) + 2V(\lambda_1) + \frac{i\tau}{1 + \lambda_1} + F_{f\ell} \tag{5}$$

where $F_{f\ell}$ is the fluctuation part of the functional integral on $\rho(\lambda)$ and may be shown to depend on λ_1, λ_2 as $F_{f\ell} = \frac{1}{2}\ell n(\lambda_2 - \lambda_1) + \text{const.}$

Since V_{eff} and ρ_{sp} are linear functions in τ , $F(\lambda_1, \lambda_2; \tau) = F^0 + i\tau F' + \frac{1}{2}(i\tau)^2 F''$ is a quadratic form in τ leading to a Gaussian integral over τ in (1), with the result

$$P(g) = \frac{1}{Z} \int_0^\infty d\lambda_1 \int_{\lambda_1 + \lambda_c}^\infty d\lambda_2 e^{-S} \quad (6)$$

where $S = -(g - F')^2/2F'' + F^0$. The remaining integrals on λ_1, λ_2 can be done numerically, or again in saddlepoint approximation.

3 Results

In the metallic regime ($L/\xi \ll 1$), the relevant values of λ_1 and λ_2 are small of order L/ξ . In the limit $\lambda_1, \lambda_2 \rightarrow 0$ we find $|F''| = \frac{1}{15}$ and $F' = \xi/L$, whereas F^0 tends to a constant at the saddlepoint $\lambda_2 = \lambda_c + \lambda_1$. Thus $P(g)$ is given by a Gaussian centered at $g = \xi/L$, of variance $1/15$, in agreement with known results [1].

We have calculated the expressions for F^0 and F' analytically up to and including all terms of order L/ξ (the correction to F'' is $0[(L/\xi)^2]$). The correction to the average conductance in the metallic regime to this order is found as $\langle g \rangle = \xi/L - \eta L/\xi$, with $\eta = 0.027$, which compares well with the exact result [15] $\eta = 1/45 \simeq 0.022$. There is no correction to the variance in order L/ξ , in agreement with [15].

In the insulating regime, $L/\xi \gg 1$, the typical values of x_1 and x_2 are both $\gg 1$. In fact, the requirement of positivity of the density for $L/\xi > \pi^2/2$ can only be satisfied if $x_2 \rightarrow \infty$, independent of x_1 . Using the saddlepoint values of $x_1 = -\frac{1}{2}\ell n(g/4)$ and $F^0 = (\xi/L)x_1^2 - x_1$, $F' = 4e^{-2x_1}$ and $F'' = -\frac{4}{3}e^{-4x_2} \rightarrow 0$ one finds

$$P(g) = \frac{1}{Z} \frac{1}{g} \exp \left[-\frac{\xi}{4L} (\ell n(g/4) + L/\xi)^2 \right] \quad (7)$$

a log-normal distribution in agreement with [3].

In the crossover regime on the insulating side, where $\xi/L < 1$, we make use of the fact that the typical values of x_1, x_2 are $x_2 \gg 1$, but $x_1 < x_2$, otherwise arbitrary. We then find $F^0 \simeq (1/3)(\xi/L)^2 x_2^3 - (\xi/L)(x_2^2 - x_1^2) + x_2 - (1/2)\ell n(x_1 \sinh(2x_1))$; $F' \simeq \cosh^{-2} x_1$ and $F'' = -\sinh^{-2}(2x_2) [\frac{1}{3} - \frac{1}{4x_2^2} + \sinh^{-2}(2x_2)]$.

The saddlepoint equation for x_1 is given by $\cosh x_1 = g^{-1/2}$, which has a solution only for $g \leq 1$. For $g > 1$, instead the boundary values $x_1 = 0$, $x_2 = (2L/\pi\xi)$ give the minimum of F . The corresponding results for $P(g)$ are

$$P(g) = \frac{1}{Z} \exp -a(g-1)^2 \quad , \quad g > 1 \quad (8)$$

$$P(g) = \frac{1}{Z} \frac{1}{g} \left[\frac{\text{arsech} \sqrt{g}}{g\sqrt{1-g}} \right]^{1/2} \exp \left[-\frac{\xi}{L} \left(\text{arsech} \sqrt{g} \right)^2 \right], \quad g < 1 \quad (9)$$

Here $a = F''(x_2 = 2L/\pi\xi)$ controls the Gaussian cut-off of $P(g)$ for $g > 1$. For $L/\xi \gg 1$ and $g^{-1} \gg 1$ Eq. (9) reduces to the log-normal distribution (7).

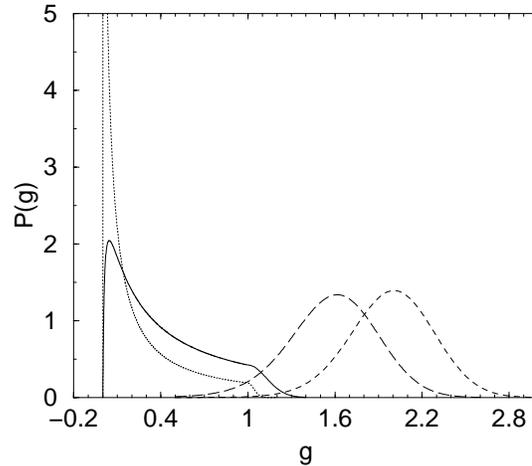


Fig. 1 Conductance distribution $P(g)$ versus g for $\xi/L = 0.4, 0.7, 1.6, 2.0$ (dotted, solid, long-dashed, short-dashed lines)

Fig. 1 shows $P(g)$ versus g for several values of $\xi/L = 0.4, 0.7, 1.6, 2.0$. In Fig. 2 the results for $\xi/L = 0.4$ and 0.7 are again shown plotted versus $\ln(1/g)$ together with results for $\xi/L = 0.25, 0.1$. In the logarithmic plot (Fig. 2) one clearly recognizes a log-normal distribution centered at $\ln(1/g) \equiv L/\xi - \ln 4$ and of variance $\text{var}[\ln(1/g)] \cong 2L/\xi$, cut-off at $\ln g = 0 (g = 1)$. The abrupt qualitative change of the shape of the distribution $P(g)$ in the crossover regime ($L/\xi \sim 1$) as one goes from values $g < 1$ to $g > 1$ is a consequence of the small value of F'' . Note that $|F''| \ll 1$ even in the metallic regime, decreasing exponentially in the insulating regime. The term $\propto (g - F')^2$ is thus multiplied by the large number $1/|F''|$, which forces the saddlepoint equation $F' = g$, which, however, has a solution only for $g < 1$. This leads to a power law dependence of S on $\ln g$. At $g > 1$ the minimum of S is attained at the boundary of the integration regime $x_1 = 0$, where the term $\propto (g - F')^2$ dominates, resulting in a Gaussian cut-off of $P(g)$.

This result might suggest that quite generally the statistics of the conductance in the crossover regime is Gaussian centered at $g = 1$ for $g > 1$ and log-normal centered at $\langle \ln g \rangle = L/\xi$ for $g < 1$. This may be made plausible in the following way. If the center of the distribution is located at $g \sim 1$, the lowest eigenvalue λ_1 still must be dominant, meaning that $\lambda_2 \gg 1$. The statistics of λ_1 is then essentially determined by the single particle potential $V(\lambda_1)$, giving rise to Gaussian statistics for $\lambda_1 < 1$, where $V(\lambda_1) \propto \lambda_1^2$ and to log-normal statistics for $\lambda_1 > 1$, where $V(\lambda_1) \propto \ln^2 \lambda_1$. These dependences carry over to the statistics of $g \simeq 1/(1 + \lambda_1)$.

The shape of $P(g)$ in higher dimensions, as determined numerically [5,6] shares the feature of an abrupt change at $g = 1$ from approximately log-normal to exponential behavior with our results. Thus, although the DMPK approach followed here is applicable only for quasi one-dimensional systems, the qualitative behavior found here

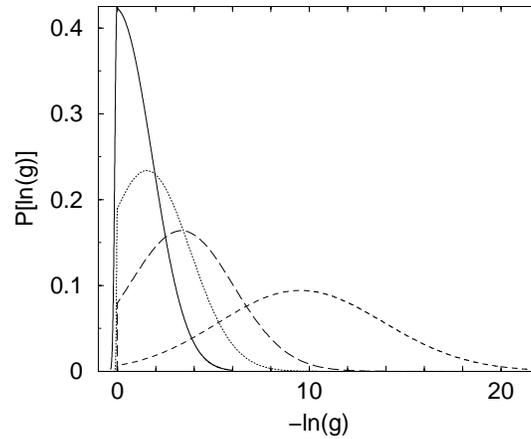


Fig. 2 Conductance distribution versus $\ln(1/g)$ for $\xi/L = 0.7, 0.4, 0.25, 0.1$ (solid, dotted, long-dashed, short-dashed lines)

may be more generally valid.

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