Emergence of Anomalous Distributions in Disordered Condensed Matter Systems

K. A. Muttalib[1] and P. Wölfle[2]

[1]*Department of Physics, University of Florida, P.O. Box 118440, Gainesville, FL 32611-8440*

[2]*Institut für Theorie der Kondensierten Materie, Universität Karlsruhe, D-76128 Karlsruhe, Germany.*

Abstract

The present day non-Gaussian distribution of mass density of the universe evolved from an initial Gaussian distribution in the presence of non-linear interactions. We discuss an analog in disordered condensed matter system where increasing the disorder changes the distribution of conductances from a Gaussian at weak disorder to a log-normal at strong disorder. The highly asymmetric 'one-sided' log-normal distribution in the intermediate crossover regime can be understood as a simple hybrid of these two limiting distributions.

I. Introduction:

It is believed that the distribution of mass density in the early universe started as Gaussian, but it evolved towards the present day non-Gaussian distribution due to the nonlinearities associated with cluster formation in the presence of gravity. One can equivalently study the present day distribution as a function of scale or size, which is Gaussian at large scales but has long tails at small scales. In disordered condensed matter systems, a similar situation occurs when one starts with a Gaussian distribution for an observable and there exists an *experimental knob* that can be turned (e.g. changing the number of scattering centers in a given sample) to change the distribution. In such a case one can in principle study in a systematic and controlled way how a Gaussian distribution suddenly starts *'growing a tail'* as one keeps on turning the experimental knob. An even more interesting situation occurs if the system undergoes a phase transition while the knob is being turned, and leads to a non-trivial limiting distribution on the other side of the transition. Then the Gaussian distribution not only develops a tail, but has to change dramatically through the phase transition. In this review we will consider such an example.

In particular, the distribution on the other side of the transition we consider is log-normal, which is a Gaussian distribution for the logarithm of the same variable. In other words, the distribution changes from Gaussian to log-normal as a an experimental parameter is changed. At the intermediate crossover or transition regime, the distribution is of a novel type which is highly asymmetric with diverging moments and can not be studied within a perturbative scheme. Nevertheless, it turns out that the novel distribution can be understood as a simple hybrid of the two limiting distributions.

II. The conductance:

We will discuss a system where non interacting electrons move in a periodic lattice with random scattering centers and experiments measure electrical transport properties of the system. The density of scattering centers can be increased systematically by e.g. bombarding the pure sample with specific dose of neutrons or alpha particles to create defects. Thus there exists an experimental knob to change the amount of disorder in the system. This apparently simple system shows a surprisingly rich behavior. Without the randomness, the system behaves as a perfect conductor with infinite conductivity. This is because the solution of the Schrödinger equation for a perfect periodic lattice is a Bloch wave, which does not scatter from the lattice points. In the presence of 'weak' disorder, the system is a metal; because of the repeated scattering from the random scattering centers, the electrons 'diffuse' through the system as in a random walk problem, giving rise to a finite conductivity. For sufficiently large disorder, the electrons get trapped in one of the deep minima of the random potential, and coherent backscattering from nearby scattering centers lead to an exponentially localized wavefunction which does not carry any current; the system becomes an insulator [1]. In three dimensions, there exists a critical disorder such that the system undergoes a second order phase transition from the metallic to the insulating behavior as the disorder is increased through the critical disorder [2].

We can imagine a disordered conductor to be described by a scattering matrix S, which relates the incoming flux of electrons to the outgoing flux, $S\binom{I}{I'} = \binom{O}{O'}$, where I, O, I', O' describe the wave amplitudes on the left and right, respectively. In terms of the transmission and reflection matrices t, t' and r, r' the S matrix can be written in the form

$$S = \begin{pmatrix} t & r \\ r' & t' \end{pmatrix}.$$

Then the dimensionless conductance $g = Tr(tt^\dagger)$ is a measure of the capacity of the system to carry current from one side of the system to the other when a voltage is applied [3]. In this dimensionless form, a metallic sample of finite size will correspond to having $g \gg 1$ (weak disorder), and an insulator will correspond to $g \ll 1$ (strong disorder). The metal-insulator transition occurs at the critical conductance $g \sim 1$ (critical disorder).

III. Distribution of conductances; Gaussian vs log-normal:

If we take several different samples with the *same disorder*, which means the same density of scattering centers but *different realizations* of the randomness, then the measured values of the conductances will not be identical, giving rise to a distribution of conductances $P(g)$. It is known from perturbation theory and experiments that in the metallic regime, the distribution $P(g)$ is Gaussian. Typically a macroscopic system is *self averaging*; a single measurement corresponds to an already ensemble averaged value. This is because one can think of the current as flowing along many parallel paths, each characterized by a partial conductance. To the extent that these conductances are statistically independent, the total conductance of the sample, which is the sum of the partial conductances, has a Gaussian distribution. For such systems, it turns out, somewhat surprisingly, that the variance of the distribution is a *universal* quantity of order unity (2/15 for a quasi one dimensional system considered here), independent of the size of the system or the mean value of the conductance, as long as the system remains metallic. This is known as the universal conductance fluctuation [4]. This means that as one turns the disorder knob, the mean conductance decreases with increasing disorder, but the distribution $P(g)$ around that mean remains Gaussian and the width of the distribution does not get broader as one would have naively expected. Clearly changing disorder in this regime does not lead to growing tails to the Gaussian distribution.

On the other hand, the distribution of conductances is known to be log-normal in the insulating regime [5]. This is because the electron wavefunctions in this regime are exponentially localized within a region of the size of the so called localization length. In this case electron transport is taking place predominantly via a single wavefunction spanning the system, all the other states giving an exponentially smaller contribution. The conductance is thus determined by the statistically fluctuating (from sample to sample) localization length of the dominating wave function. Assuming a Gaussian distribution of the inverse localization length, it follows that the logarithm of the conductance will be Gaussian distributed, which is a log-normal distribution for $P(g)$. This is a highly asymmetric distribution with a very long tail. The implication is that although $P(g)$ remained Gaussian and even the width did not change with increasing disorder for small disorder, once the disorder becomes sufficiently strong, $P(g)$ changes from a Gaussian on the metallic side to a log-normal on the insulating side. It must be true then that somewhere near the critical disorder, the Gaussian suddenly starts to grow tails.

IV. Transition from Gaussian to log-normal distribution:

If the critical distribution is not symmetric and has long tails, then the usual description of the phase transition in terms of the mean (or the most probable) conductance is no longer adequate because the tails may dominate the physical properties. In this case the phase transition should be described in terms of a transition between two distributions. Then we need to ask: what is the nature of the distribution near the critical regime? How does it emerge from either the Gaussian or the log-normal distributions as one turns the disorder knob one way or the other?

Clearly, perturbation theory in disorder (or $1/g$) is inadequate to understand the anomalous distribution near the transition, because significant changes in the Gaussian occur only at strong disorder near the critical value of $g \sim 1$. Moreover, the n-th moment of the log-normal distribution diverge faster than e^{n^2}, which means that the distribution can not be uniquely reconstructed from the moments. If the moments of the critical distribution also diverge similarly, then one needs to consider the full distribution $P(g)$ directly rather than trying to extract it from the moments. Indeed, a combination of perturbation theory in $d = 2 + \epsilon$ dimensions [6] and an attempt to reconstruct $P(g)$ from the resulting diverging moments lead to a prediction for the critical distribution having a Gaussian head (centered at $1/\epsilon$) and symmetric power law tails [7]. This picture is not even remotely close to the experimental [8] or numerical evidences [9], which show a highly asymmetric critical distribution with log-normal tail on the insulating side and a very sharp cutoff on the metallic side at $g \sim 1$. A possible reason for this failure is that the critical conductance in the $2 + \epsilon$ dimension happens to be at $g_c \sim 1/\epsilon$ which, for the perturbation theory to be valid, must be at $\epsilon \ll 1$, i.e. $g_c \gg 1$. This is deep in the metallic regime, and fails to capture the actual critical regime at $g_c \sim 1$ which is beyond the perturbative limit.

We have developed a novel non-perturbative method to study the distribution $P(g)$ directly without having to compute the moments. The method is valid only in quasi-one dimensions ('thick quantum wire'), where localization of the wavefunction along the direction transverse to the current is not fully taken into account. This limits us to the situation where there is only a crossover between the metallic and insulating regimes and no true phase transition as in three dimensions. However, since the localization length diverges near the critical point, even in quasi-one dimension the crossover regime gives a qualitatively good description of the critical distribution. In fact, our predictions at the middle of the crossover agree quite well with those obtained numerically for the true phase transition in three dimensions.

The method is based on a Fokker-Planck equation for the joint probability distribution (jpd) of the transmission eigenvalues for an N channel wire of length L, which describes the 'evolution' of the distribution as the length is increased [10]. The parameter $\Gamma = Nl/L$ where $l \ll L$ is the mean free path plays the role of the strength of disorder. (The thermodynamic limit is taken as $N \to \infty$, $L \to \infty$, keeping the ratio Γ to be finite.) Conductance $g = Tr(tt^{\dagger})$ is directly related to the real, non-negative transmission eigenvalues λ_i by the relation

$$g = \sum_{i=1}^{N} \frac{1}{1+\lambda_i},$$

where a metal consists of a fraction l/L of λ_i, each much smaller than unity (these are the 'open' channels), so that the conductance is $g = Nl/L = \Gamma$; an insulator will consist of all $\lambda_i \gg 1$, giving rise to a conductance $g \ll 1$. $P(g)$ can then be written down as an N-dimensional integral of the jpd of the transmission eigenvalues λ_i which are real and non-negative [11]. By writing the jpd as the exponential of an effective Hamiltonian, one can then map the problem on to an electrostatic problem of N charges on a line (from 0 to ∞) with two-particle repulsive interactions and a single particle confinement potential. The electrostatic problem is then solved within a variational scheme [12]. Note that the distribution for all disorder in this framework turns out to be a function of only one parameter, Γ, with $\Gamma \gg 1$ corresponding to a metal and $\Gamma \ll 1$ corresponding to an insulator. The crossover occurs at $\Gamma \sim 1$.

It turns out that understanding how tails *grow* in the Gaussian distribution is far more complicated than understanding how the tails in the log-normal distribution get *cut off* as one approaches the critical regime from the insulating side. In our framework this is because the metallic distribution ($g \gg 1$) involves the jpd of small $\lambda_i \ll 1$ which are strongly correlated due to their mutual repulsive interaction, while the insulating distribution ($g \ll 1$) involves only the smallest one or two eigenvalues which are exponentially separated from the rest. In terms of the physical picture presented above conduction in the metallic regime ($g > 1$) involves different current paths coupled in parallel. For samples of this type the distribution will be Gaussian, even though only the tail of the distribution may fall into the regime $g > 1$. As soon as one current path (transmission channel) is dominant ($g < 1$), the distribution changes to log-normal. As a consequence, the transition between the two regimes $g < 1$ and $g > 1$ in $P(g)$ turns out to be remarkably sharp. Our results predict [12] that in the insulating regime (in the absence of time reversal symmetry) the distribution has the form

$$P[\ln(g)] \approx \begin{cases} \sqrt{\frac{x_1 \sinh 2x_1}{1-g}} e^{-\Gamma x_1^2}, & g < 1, \\ \sqrt{2} g e^{-a(g-1)^2}, & g > 1, \end{cases}$$

where $x_1 = \cosh^{-1}(1/\sqrt{g})$ and a depends on the disorder parameter Γ.

Figure 1 shows plots of $P[\ln(g)]$ as a function of $\ln(g)$ given by the above equation for four values of Γ. In the deeply insulating regime $\Gamma \ll 1$, the distribution is log-normal with peak at $< \ln(g) >= 1/\Gamma$. As the disorder is decreased, Γ increases and the peak of the log-normal distribution shifts towards larger g with increasing g. For these cases, the distribution gets the $g > 1$ part of its tail cut off by a Gaussian. This gives rise to a highly asymmetric distribution which eventually leads to a novel 'one-sided log-normal' distribution when the peak of the log-normal distribution shifts to $g = 1$. The Gaussian cut off at $g = 1$ is indicative of the fact that $g = 1$ requires one 'open' channel for the current, corresponding to the first transmission eigenvalue $\lambda_1 \ll 1$. However, the repulsive interaction between the eigenvalues forces the next one exponentially far for $\Gamma \ll 1$, leading to a sharp cutoff for the distribution beyond $g > 1$.

'One-sided log-normal' distributions predicted above have already been observed (at least with the same qualitative features) in numerical investigations of the crossover/transition regimes of integer Quantum Hall systems in quasi-one as well as two dimensions and Anderson metal-insulator transitions in three dimensions [9]. The details of how the log-normal tail gets cut off at $g \sim 1$ have however not yet been studied numerically. It would be interesting to see if such highly asymmetric hybrid distributions appear generically in crossover situations between two different limiting distributions.

The present day non-Gaussian distribution of mass density in the universe look remarkably close to a log-normal distribution [13]. Is it possible that the distribution is actually a hybrid of two simple limiting distributions in some intermediate regime of its time evolution?

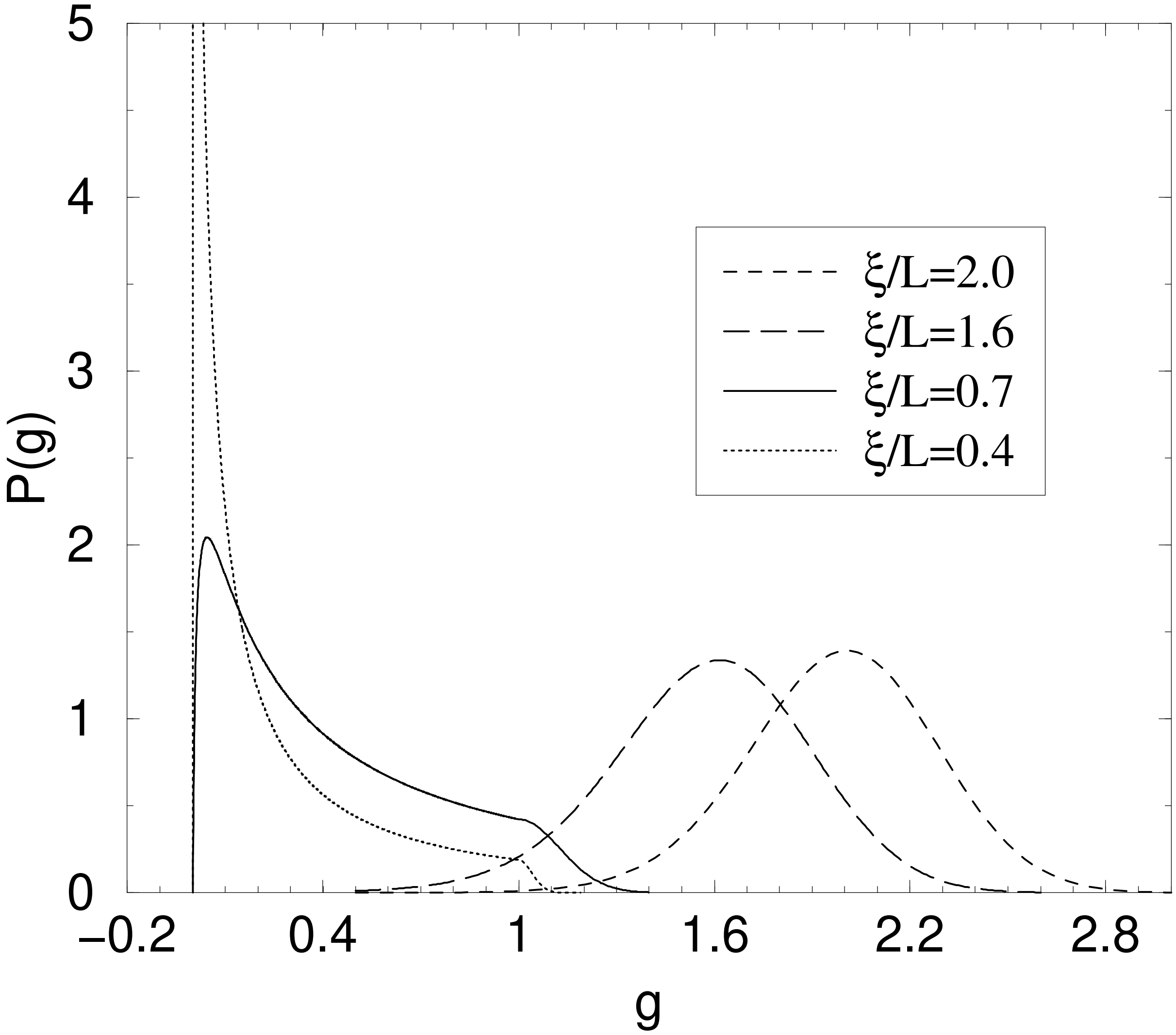

FIG. 1. $P[\ln(g)]$ as a function of $\ln(g)$ for four different values of Γ in the insulating regime $\Gamma < 1$. Dashed line corresponds to $\Gamma = 0.1$, long dashed line $\Gamma = 0.25$, dotted line $\Gamma = 0.4$ and solid line $\Gamma = 0.7$, which is close to the crossover regime.

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