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Rotationally invariant family of Lévy-like random matrix ensembles

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Online at stacks.iop.org/JPhysA/42/152001**Abstract**

We introduce a family of rotationally invariant random matrix ensembles characterized by a parameter λ . While $\lambda = 1$ corresponds to well-known critical ensembles, we show that $\lambda \neq 1$ describes ‘Lévy-like’ ensembles, characterized by power-law eigenvalue densities. For $\lambda > 1$ the density is bounded, as in Gaussian ensembles, but $\lambda < 1$ describes ensembles characterized by densities with long tails. In particular, the model allows us to evaluate, in terms of a novel family of orthogonal polynomials, the eigenvalue correlations for Lévy-like ensembles. These correlations differ qualitatively from those in either the Gaussian or the critical ensembles.

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1. Introduction

Gaussian random matrix ensembles (RMEs) were proposed by Wigner about half a century ago to describe the statistical properties of the eigenvalues and eigenfunctions of complex many-body quantum systems in which the Hamiltonians are considered only in a probabilistic way [1, 2]. Over the past several decades, they proved to be a very useful tool in the studies of equilibrium and transport properties of disordered quantum systems, classically chaotic systems with a few degrees of freedom, two-dimensional gravity, conformal field theory and chiral phase transition in quantum chromodynamics [3]. This wide applicability results from certain universal properties of the correlation of the eigenvalues known as the Wigner distributions, as opposed to the Poisson distributions that result from random sets of eigenvalues.

More recently, attempts have been made to construct generalized ensembles that show a crossover from a Wigner to a Poisson distribution as a function of a parameter, as seen in many physical systems [4]. One such generalization is the family of ‘q-random matrix ensembles’ (q-RMEs) [5, 6]. These were later shown to be models of ‘critical ensembles’,

with statistical properties different from the Gaussian RMEs and relevant for systems near a metal–insulator transition [7]. While there are other models that also describe critical statistics [8], one advantage of the q-RMEs is that they are rotationally invariant, and therefore can be analytically studied in great detail by the powerful method of orthogonal polynomials [2]. In particular, the differences between the Gaussian and the critical RMEs can be traced to the differences in the asymptotic properties of the classical versus q-orthogonal polynomials [9, 10], illustrating how the universality of the Gaussian RMEs breaks down and gives rise to a different kind of universality for the critical ensembles.

In this work, we take the generalization one step further to include ‘Lévy-like’ ensembles. Lévy ensembles were introduced by Cizeau and Bouchaud (CB) [11] where the matrix elements are drawn from a power-law distribution according to $P(H_{ij}) \sim 1/|H_{ij}|^{1+\mu}$, $H_{ij} \gg 1$, $0 < \mu < 1$. The eigenvalue density for such ensembles falls off as $1/x^{1+\mu}$. Such matrices with a broad distribution of matrix elements have been studied in the context of a wide variety of systems including financial markets, earthquakes, scale free networks, communication systems, etc as well as quantum chaotic systems such as the Coulomb billiard and kicked rotor with singularities [12]. The statistical properties of the eigenvalues and eigenvectors of such matrices are, in general, quite different from the universal properties of the Gaussian or the critical ensembles. Indeed, it has been argued [11] that matrices of the Lévy type are relevant for describing localization transition of interacting electrons in infinite dimensions. In particular, while the eigenvectors of the Gaussian RMEs are all extended, the Lévy-type matrices contain signatures of mobility edges, separating localized states from extended states within the spectrum.

However, the CB model is very difficult to study because it is not rotationally invariant. While eigenvalue densities for certain Lévy ensembles have been obtained analytically [13], little progress has been made in any systematic study of the eigenvalue *correlations* where novel universal features can be expected. In the present work we will consider ‘Lévy-like’ ensembles, characterized by a power-law eigenvalue density $1/x^{1+\mu}$, without specifying whether such a density arises from a distribution of the matrix elements as in the CB model. For this purpose, we introduce a rotationally invariant model characterized by a parameter λ . We show that $\lambda = 1$ corresponds to a critical ensemble, while $\lambda > 1$ and $\lambda < 1$ describe Lévy-like ensembles with bounded and unbounded (long tail) densities, respectively. In particular, the model allows us to calculate the two-level kernel for Lévy-like ensembles, from which all correlations can be evaluated, in terms of a novel family of orthogonal polynomials. These correlations turn out to be qualitatively different from either the Gaussian or the critical ensembles.

Consider the set of all $N \times N$ Hermitian matrices M (we will restrict ourselves to the unitary ensembles only) randomly chosen with the following probability measure $P_N^V(M) dM \propto \exp[-\text{Tr}(V(M))] dM$, where $V(x)$ is a suitably increasing function of x , Tr is the matrix trace and dM is the Haar measure. By going over to the eigenvalues–eigenvectors representation, it can be shown that the joint probability distribution of the eigenvalues $X = (x_i, i = 1, 2, \dots, N)$ of the matrices can then be written in the form [2]

$$P_N^V(X) \propto \prod_{1 \leq i < j \leq N} (x_i - x_j)^2 \prod_{i=1}^N e^{-2V(x_i)}. \quad (1)$$

Here the factor $\prod (x_i - x_j)^2$ arises from the Jacobian of a change of variables.

The level correlations can be determined exactly by recognizing that the distribution can be written as a product of Vandermonde determinants of a set of (monic) polynomials $\phi_n(x)$ that are orthogonal with respect to the weight function $w(x) = e^{-V(x)}$ [2], i.e.

$$\int_{-\infty}^{\infty} e^{-V(x)} \phi_n(x) \phi_m(x) dx = \delta_{mn}. \quad (2)$$

The main quantity of interest is the large N limit of the two-level kernel

$$K_N^V(x, y) \equiv e^{-(V(x)+V(y))/2} \sum_{n=0}^{N-1} \phi_n(x)\phi_n(y), \tag{3}$$

from which all correlation functions can be obtained. The Gaussian RMEs follow when one chooses the ‘confinement potential’ $V(x) = x^2$ that defines the Hermite polynomials. Since all ‘Freud-type’ weight functions $e^{-V(x)}$, with monotonically increasing *polynomial* $V(x)$, lead to orthogonal polynomials with qualitatively similar asymptotic behavior in the large- N limit, they all share the same correlations as the Gaussian REMs. This is at the root of the wide applicability of the Gaussian RMEs. On the other hand, q-RMEs follow when one chooses $V(x) \sim \ln^2 x$ for large x [5], which leads to the q-polynomials. All q-polynomials have similar large- N asymptotic behavior, which differ qualitatively from those of the Freud-like classical polynomials. Thus all q-RMEs characterized by different q-polynomials share the same correlation properties as those of the critical ensembles, which are different from those of the Gaussian ensembles.

2. Model for Lévy-like ensembles

It turns out that all q-RMEs have density of eigenvalues falling off as $1/x$ [14]. Comparing this with the fact that Lévy-like ensembles have density falling off as $1/x^{1+\mu}$, this suggests that we consider a generalization of the q-RME model where the asymptotic $\ln^2 x$ behavior of the potential is extended to include other powers of the logarithm. We therefore introduce a one-parameter generalization of the q-RMEs characterized by

$$V(x) = \frac{1}{\ln(1/q)} [\sinh^{-1} x]^{1+\lambda}, \quad \lambda > 0, \quad q < 1. \tag{4}$$

For $\lambda = 1$, this is a model for critical ensembles [6], defined by the Ismail–Masson q-polynomials [10]. Since the asymptotic behavior of the potential is $V(x) \sim \ln^{1+\lambda}(x)$, any $\lambda \neq 1$ is qualitatively different from a q-ensemble. However, there is no direct way to obtain orthogonal polynomials for any arbitrary non-trivial weight function. In particular, the orthogonal polynomials corresponding to $\lambda \neq 1$ are not known, and we cannot write down the two-level kernel (equation (3)) directly. We therefore follow an indirect but systematic method [15] that allows us to obtain the polynomials recursively for any given potential. It is well known that every orthogonal system of real-valued polynomials satisfy a three term recursion relation [9, 10]

$$x\phi_n(x) = \phi_{n+1}(x) + S_n\phi_n(x) + R_n\phi_{n-1}(x). \tag{5}$$

Following [15], we define a set of integrals

$$Q_{n,m} \equiv \int_{-\infty}^{\infty} x^m e^{-2V(x)} \phi_n(x) dx, \tag{6}$$

which in turn satisfy the recursion relation

$$Q_{n,m} = Q_{n-1,m-1} - S_{n-1}Q_{n-1,m} - R_{n-1}Q_{n-2,m}. \tag{7}$$

Thus, the determination of the coefficients R_n and S_n necessary to calculate the polynomials of degree $n \leq N - 1$ requires only the knowledge of the $2N + 1$ integrals $Q_{0,m}$ for $m = 0, 1, \dots, 2N$.

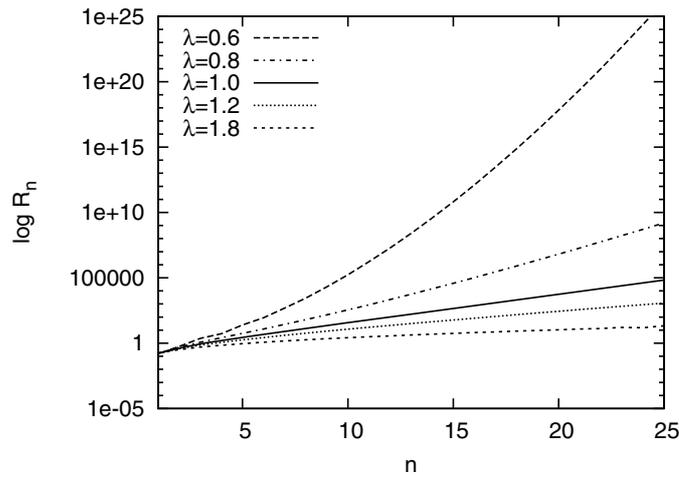


Figure 1. Log R_n as a function of n for different values of λ . The solid line corresponds to the critical ensemble $\lambda = 1$, which separates two qualitatively distinct classes $\lambda > 1$ and $\lambda < 1$.

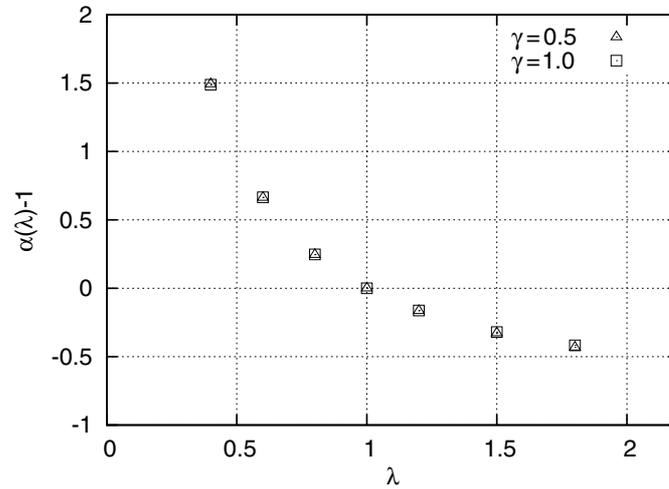


Figure 2. The exponent $\alpha(\lambda)$ as a function of λ for two different values of $\gamma \equiv \ln(1/q)$. $\lambda = 1$ corresponds to the q -polynomials describing the critical ensembles.

3. Results

We first note that for the present model, because we choose the potential to be symmetric, all $S_n = 0$. Therefore, R_n completely determines the polynomials. For simplicity, we will use a fixed value of the parameter $\ln(1/q) \equiv \gamma = 0.5$ except in figure 2 where we also show results for $\gamma = 1$. Figure 1 shows the n -dependence of R_n for different values of λ obtained by evaluating the $Q_{0,m}$ numerically. For all λ , the large- n behavior of R_n has the form

$$R_n \propto q^{-n\alpha(\lambda)}. \tag{8}$$

In contrast, for all Freud-like classical orthogonal polynomials, $R_n \propto n$. Figure 2 shows the λ dependence of the exponent α .

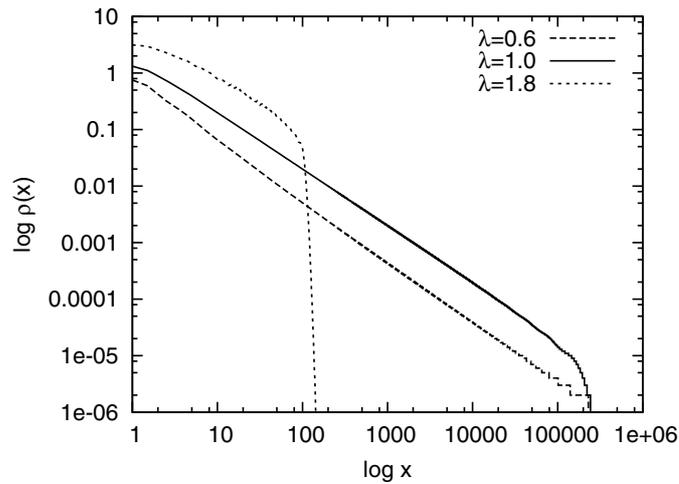


Figure 3. Density of eigenvalues for different values of λ . $\lambda = 1$ is the critical ensemble. For $\lambda > 1$, the density has a sharp edge, while for $\lambda < 1$ there is a true ‘Lévy-like’ long tail.

For $\lambda = 1$, $\alpha(\lambda) = 1$, which recovers the known recursion relation¹ for the q-orthogonal polynomials. On the other hand, for $\lambda > 1$, $\alpha(\lambda) < 1$, while for $\lambda < 1$, $\alpha(\lambda) > 1$. Note that the distribution of the zeros of the polynomials are determined by the exponent α , such that when $\alpha = 1$, the logarithm of the zeros of the corresponding Ismail–Masson q-polynomials are uniformly distributed. For $\alpha < 1$ the zeros are ‘bunched together’ while for $\alpha > 1$ they are ‘unbunched’, highlighting their differences from the q-polynomials. Thus, $\lambda \neq 1$ defines a novel family of orthogonal polynomials that generalizes the q-polynomials. However, the asymptotic properties of these ‘generalized q-polynomials’ are not known.

The density of eigenvalues $\rho(x) = K_N^V(x, x)$ can now be obtained for different values of λ from equation (3) by summing the products numerically. The results are shown in figure 3. As already known, the density for $\lambda = 1$ falls off as $1/x$. For $\lambda \neq 1$, the density falls off as $1/x^{1+\mu(\lambda)}$, as expected for Lévy-like ensembles. However, for $\lambda > 1$, $\mu(\lambda) < 0$ and the density falls off slower than $1/x$; the normalization condition (that there are N eigenvalues in total for a matrix of size N) then forces the density to have a sharp edge, similar to the edge of the semicircular density of the Gaussian RMEs. Thus, even though all $\lambda \neq 1$ have power-law densities, the long tail characteristic of a Lévy ensemble is cut off for $\lambda > 1$ by a sharp edge. For $\lambda < 1$, $\mu(\lambda) > 0$ and the density has true power-law tails. It should be noted that nonextensive ensembles [16] characterized by one parameter q also show a similar power-law behavior; for $q > 1$, the distributions of eigenvalue density show true long tails and for $q < 1$, the distributions have compact support. However, the parameter q in this case depends on the dimensionality N of the ensemble such that in the large- N limit where universal behavior is expected, the maximum q allowed for the nonextensive ensembles approaches unity.

Our results now allow us to obtain exact correlations for Lévy-like ensembles. Using equation (3) for the two-level kernel we calculate the ‘unfolded’ cluster function² $Y(r) \equiv |K_N^V(r)|^2$, where the variable r is such that the density is uniform and unity: $\bar{\rho}(r) = 1$. It is

¹ Exact expression for all n is $R_n = (q^{-n} - 1)/4$, see [10].

² We restrict ourselves to a range of parameters where the kernel can be considered translationally invariant.

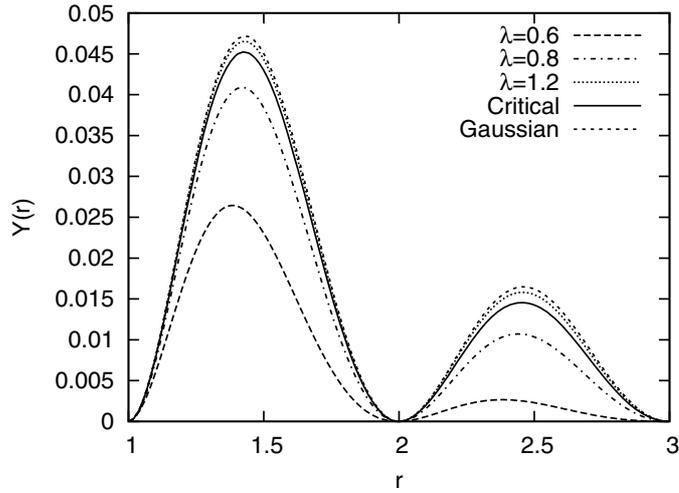


Figure 4. The cluster function $Y(r)$ for Lévy-like ensembles for different values of λ . The dotted and the solid lines agree with the known results for the Gaussian and the critical ensemble (with $\gamma = 0.5$), respectively.

only in this rescaled variable r that the universality of a given ensemble is revealed if it exists. The results are shown in figure 4. Note that cluster functions for $\lambda > 1$ are intermediate between the Gaussian (Y^G) and the critical (Y^C) ensembles

$$Y^G = \left[\frac{\sin(\pi r)}{\pi r} \right]^2, \quad Y^C = \left[\frac{\gamma}{2\pi} \frac{\sin(\pi r)}{\sinh[(\gamma/2)r]} \right]^2, \quad (9)$$

but for $\lambda < 1$ it is qualitatively different, with peak positions of the lobes shifting toward smaller r values with decreasing λ . This suggests that Y consists of a more general λ -dependent argument as well as power of the \sinh function that reduces to Y^C in the limit $\lambda = 1$. Note that our results are limited to finite system size N only, which makes any attempt to obtain the true asymptotic behavior of Y unreliable. Nevertheless, with sufficiently large N , we can clearly demonstrate that the correlation functions obtained from Y for $\lambda \neq 1$ are qualitatively different from both the Gaussian and the critical ensembles.

As an example of the correlation functions obtained from the cluster function, we evaluated the number variance $\Sigma(L)$ within a range L shown in figure 5. For $\lambda \gg 1$, the number variance tends to that of the Gaussian RMEs, as expected, due to the absence of long tails. On the other hand, for $\lambda < 1$, the number variance shifts toward the uncorrelated Poisson distribution. For $\lambda < 1$, $\Sigma(L)$ for large L seems linear in L , as it is for the critical as well as Poisson cases, although the slope depends on λ in a nontrivial manner. We have also evaluated the gap function which, for critical ensembles, falls off slower compared to the Gaussian RMEs. We find that for $\lambda > 1$ the gap function is in between the critical and the Gaussian RMEs, while for $\lambda < 1$ it falls off slower than the critical ensemble. Again, finite system size prevents us from obtaining a reliable λ dependence of the asymptotic behavior. Determination of the true asymptotic behavior of these correlation functions will require knowledge of the asymptotic properties of the novel orthogonal polynomials introduced here.

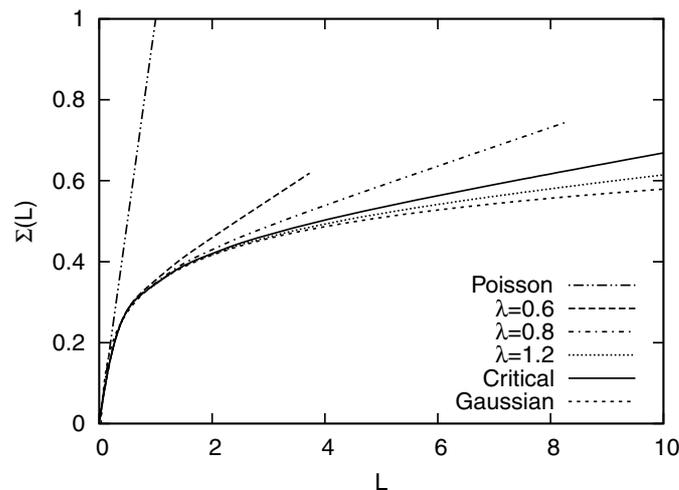


Figure 5. Number variance $\Sigma(L)$ for Lévy-like ensembles for different values of λ .

4. Summary and conclusion

In summary, we have introduced a rotationally invariant model of random matrix ensembles by defining a confinement potential $V(x)$ that includes a parameter λ . We obtained the orthogonal polynomials associated with the weight function $w(x) = e^{-V(x)}$ for different values of λ by using a recursive method. For $\lambda = 1$, we recover the well-known q-polynomials that describe the critical ensemble, with eigenvalue density $\rho(x) \propto 1/x$. For $\lambda \neq 1$, we obtain a novel family of ‘generalized q-polynomials’. While $\rho(x) \propto 1/x^{1+\mu(\lambda)}$ for all $\lambda \neq 1$, only $\lambda < 1$ correspond to Lévy-like ensembles with true long tails. We use the generalized q-polynomials (for finite N) to evaluate the two-level kernel from which all eigenvalue correlations can be obtained. We find that for $\lambda \neq 1$, the correlations are of novel types, differing from both the Gaussian and the critical ensembles. Clearly, studies of the asymptotic properties of the generalized q-polynomials introduced above would be very useful in better understanding the properties of Lévy-like ensembles.

Although we have considered only the variation with respect to the parameter λ , the model has another parameter, q , which we kept fixed. We have confirmed that the two parameters are independent, so that a single effective parameter cannot describe the separate dependence of the two parameters. It would be important to find out if the q-parameter can describe e.g. any ‘transition’ within the Lévy-like ensembles, as argued in [11]. While the mobility edge or the localization transition has clear meaning in the context of electron transport in disordered systems, it would be very interesting if it exists in Lévy-like ensembles, in the context of complex systems like earthquakes and scale-free networks.

References

- [1] Wigner E P 1951 *Proc. Camb. Phil. Soc.* **47** 790
Dyson F J 1962 *J. Math. Phys.* **3** 140
- [2] Mehta M L 1991 *Random Matrices* 2nd edn (Boston, MA: Academic)
- [3] For recent reviews and applications see:
Guhr T, Mueller-Groeling A and Weidenmueller H 1998 *Phys. Rep.* **299** 189

- Forrester P J, Snaith N C and Verbaarschot J J M 2003 Special issue: random matrix theory *J. Phys. A: Math. Gen.* **36** (guest editors)
- Garcia-Garcia A M and Verbaarschot J J M 2000 *Nucl. Phys. B* **586** 668
- Garcia-Garcia A M and Osborn J C 2007 *Phys. Rev. D* **75** 034503
- Osborn J C and Verbaarschot J J M 1998 *Phys. Rev. Lett.* **81** 268
- [4] Brody T A 1973 *Nuovo Cimento Lett.* **7** 482
- Berry M V and Tabor M 1977 *Proc. R. Soc. Lond. A* **356** 375
- Seligman T H, Verbaarschot J J M and Zirnbauer M R 1984 *Phys. Rev. Lett.* **53** 215
- Berry M V and Robnik M 1984 *J. Phys. A: Math. Gen.* **17** 2413
- Bogomolny E B *et al* 1992 *Phys. Rev. Lett.* **69** 1477
- Crosche C 1992 *J. Phys. A: Math. Gen.* **25** 4573
- [5] Muttalib K A *et al* 1993 *Phys. Rev. Lett.* **71** 471
- Blecken C, Chen Y and Muttalib K A 1994 *J. Phys. A: Math. Gen.* **27** L563
- [6] Muttalib K A, Chen Y and Ismail M E H 2001 *Symbolic Computation, Number Theory, Special Functions, Physics and Combinatorics* ed F Garvan and M Ismail (Dordrecht: Kluwer)
- [7] Kravtsov V E and Muttalib K A 1997 *Phys. Rev. Lett.* **79** 1913
- Bogomolny E, Bohigas O and Pato M P 1997 *Phys. Rev. E* **55** 6707
- Nishigaki S M 1999 *Phys. Rev. E* **59** 2853
- [8] Moshe M, Neuberger H and Shapiro B 1994 *Phys. Rev. Lett.* **73** 1497
- Mirlin A D *et al* 1996 *Phys. Rev. E* **54** 3221
- Mirlin A D and Evers F 2000 *Phys. Rev. B* **62** 7920
- Altshuler B L *et al* 1988 *Sov. Phys. JETP* **67** 625
- Shklovskii B I *et al* 1993 *Phys. Rev. B* **47** 11487
- Pichard J-L and Shapiro B 1994 *J. Physique* **4** 623
- Kravtsov V E *et al* 1994 *Phys. Rev. Lett.* **72** 888
- Garcia-Garcia A M and Verbaarschot J J M 2000 *Nucl. Phys. B* **586** 668
- [9] Szegő G 1975 *Orthogonal Polynomials* 4th edn (Providence, RI: American Mathematical Society)
- [10] Ismail M E H 2005 *Classical and Quantum Orthogonal Polynomials in One Variable* (Cambridge: Cambridge University Press)
- [11] Cizeau P and Bouchaud J P 1994 *Phys. Rev. E* **50** 1810
- [12] Laloux L *et al* 1999 *Phys. Rev. Lett.* **83** 1467
- Moustakas A *et al* 2000 *Science* **287** 287
- Skipetrov S E 2003 *Phys. Rev. E* **67** 036621
- Goh K-I, Kahng B and Kim D 2001 *Phys. Rev. E* **64** 051903
- Dorogovtsev S N *et al* 2003 *Phys. Rev. E* **68** 046109
- Rodgers G J, Austin K, Kahng B and Kim D 2005 *J. Phys. A: Math. Gen.* **38** 9431
- Araujo M, Medina E and Aponte E 1999 *Phys. Rev. E* **60** 3580
- Baiesi M and Paczuski M 2004 *Phys. Rev. E* **69** 066106
- Altshuler B L and Levitov L S 1997 *Phys. Rep.* **288** 487
- Garcia-Garcia A M and Wang J 2005 *Phys. Rev. Lett.* **94** 244102
- Hu B *et al* 1998 *Phys. Rev. Lett.* **82** 4224
- Garcia-Garcia A M 2004 *Phys. Rev. E* **69** 244102
- [13] Burda Z *et al* 2002 *Phys. Rev. E* **65** 066216
- [14] Muttalib K A and Ismail Mourad E H 2007 *Phys. Rev. E* **76** 051105
- [15] Slevin K, Pichard J-L and Muttalib K A 1993 *J. Physique (France) I* **3** 1387
- [16] Toscano F, Vallejos R O and Tsallis C 2004 *Phys. Rev. E* **69** 066131