

The generalized DMPK equation revisited: towards a systematic derivation

This content has been downloaded from IOPscience. Please scroll down to see the full text.

2014 J. Phys. A: Math. Theor. 47 125103

(<http://iopscience.iop.org/1751-8121/47/12/125103>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 128.227.227.148

This content was downloaded on 19/03/2014 at 21:36

Please note that [terms and conditions apply](#).

The generalized DMPK equation revisited: towards a systematic derivation

Andrew Douglas¹, Peter Markoš² and K A Muttalib³

¹ Department of Physics, University of North Florida, 1 UNF Dr., Jacksonville, FL 32224-7699, USA

² Department of Physics, INPE, Faculty of Electrical Engineering and Information Technology, Slovak University of Technology, 81219 Bratislava, Slovakia

³ Department of Physics, University of Florida, PO Box 118440, Gainesville, FL 32611-8440, USA

E-mail: a.douglas@unf.edu, peter.markos@stuba.sk and muttalib@phys.ufl.edu

Received 7 December 2013, revised 3 February 2014

Accepted for publication 17 February 2014

Published 10 March 2014

Abstract

The generalized Dorokov–Mello–Pereyra–Kumar (DMPK) equation has recently been used to obtain a family of very broad and highly asymmetric conductance distributions for three-dimensional disordered conductors. However, there are two major criticisms of the derivation of the generalized DMPK equation: (1) certain eigenvector correlations were neglected based on qualitative arguments that cannot be valid for all strengths of disorder, and (2) the repulsion between two closely spaced eigenvalues were not rigorously governed by symmetry considerations. In this work we show that it is possible to address both criticisms by including the eigenvalue and eigenvector correlations in a systematic and controlled way. It turns out that the added correlations determine the evolution of the Jacobian, without affecting the evaluation of the conductance distributions. They also guarantee the symmetry requirements. In addition, we obtain an exact relationship between the eigenvectors and the Lyapunov exponents leading to a sum rule for the latter at all disorder strengths.

Keywords: Anderson transition, generalized DMPK, Lyapunov exponents

PACS numbers: 73.23.-b, 71.30., 72.10.-d

(Some figures may appear in colour only in the online journal)

1. Introduction

A highly non-trivial, non-Gaussian distribution $P(g)$ of conductances g for disordered three-dimensional (3d) conductors in the large disorder limit has recently been predicted [1, 2]. The predictions include, among others, a strong deviation from the expected log-normal

distribution in the deeply insulating limit in agreement with numerical results obtained from tight-binding Anderson model [3, 4, 5], variance that grows with disorder consistent with numerical simulations [6], and an asymmetry that changes sign near the metal–insulator transition as the disorder is decreased from the deeply insulating limit. The distribution was obtained using the so-called generalized DMPK (GDMPK) equation [7]. The original Dorokov–Mello–Pereyra–Kumar (DMPK) equation [8] has been shown to be valid for quasi one-dimensional (1d) systems [9], where there is no Anderson metal–insulator transition. The GDMPK equation, however, has been claimed to be valid beyond quasi 1d in the large disorder regime. Since the full distribution $P(g)$ in 3d at strong disorder is beyond the scope of the conventional field theory framework [10, 11], and since a broad and highly asymmetric distribution can have interesting consequences for the Anderson transition, it is important to critically evaluate the validity of the GDMPK equation.

For a given N -channel conductor of length L_z and cross section L^2 , the $2N \times 2N$ transfer matrix M relates the wavefunction on the left of the sample to that on the right via: $\Psi_R = M\Psi_L$. For the case of time reversal and spin rotation symmetry (orthogonal ensemble), M can be parameterized by the $N \times N$ unitary matrices u and v , and $N \times N$ diagonal matrix λ :

$$M = \begin{pmatrix} u & 0 \\ 0 & u^* \end{pmatrix} \begin{pmatrix} \sqrt{1+\lambda} & \sqrt{\lambda} \\ \sqrt{\lambda} & \sqrt{1+\lambda} \end{pmatrix} \begin{pmatrix} v & 0 \\ 0 & v^* \end{pmatrix}. \quad (1)$$

We will loosely refer to λ and u, v as the eigenvalues and the eigenvectors since they turn out to be the eigenvalues and eigenvectors of a certain combination of MM^\dagger , as will be shown later. The $|u_{ab}|^2$ and $|v_{ab}|^2$ can be thought of as the fraction of current in channel a that enters into channel b from left to right and from right to left, respectively, and $1/(1+\lambda_b)$ is channel b 's contribution to the conductance. M is a random matrix, and its probability distribution can be written as

$$p(M) d\mu(M) = p(\lambda, u, v) d\mu(\lambda) d\mu(u) d\mu(v) \quad (2)$$

where $d\mu(\lambda)$, $d\mu(u)$, and $d\mu(v)$ are the Haar measures of the respective matrices. The measure $d\mu(\lambda)$ can be written in terms of the so-called Jacobian $J(\lambda)$:

$$d\mu(\lambda) = J(\lambda)d^N\lambda; \quad J(\lambda) \equiv \prod_{a \neq b}^N |\lambda_a - \lambda_b|^\beta \quad (3)$$

where the orthogonal symmetry discussed here requires [12] $\beta = 1$. For future reference we will define related functions:

$$\begin{aligned} p(\lambda, v) &\equiv \int d\mu(u) p(\lambda, u, v), \\ p(\lambda) &\equiv \int d\mu(u) d\mu(v) p(\lambda, u, v), \\ P(\lambda) &\equiv p(\lambda)J(\lambda). \end{aligned} \quad (4)$$

The last one is the probability distribution of the eigenvalues λ_i ($i = 1, 2, \dots, N$). One can obtain an evolution equation for $p(\lambda)$ in the following manner. We add a small conductor of length δL_z described by a transfer matrix $M_{\delta L_z} \equiv M'$ to a conductor of length L_z described by $M_{L_z} \equiv M''$. Then taking advantage of the fact that the transfer matrices are multiplicative ($M_{L_z+\delta L_z} \equiv M = M''M'$), an integral equation for the evolution of $p(M)$ with length L_z can be written down:

$$p_{L_z+\delta L_z}(M) = \int d\mu(M') p_{\delta L_z}(M') p_{L_z}(M''). \quad (5)$$

Integrating both sides over $d\mu(u)$ and $d\mu(v)$, we can write the evolution equation for $p(\lambda)$ as

$$p_{L_z+\delta L_z}(\lambda) = \int d\mu(u) d\mu(v) \langle p_{L_z}(\lambda'', u'', v'') \rangle_{M'} \quad (6)$$

where the angular bracket $\langle \dots \rangle_{M'}$ indicates an averaging over M' . When the sample width is much smaller than the localization length ξ ($L \ll \xi$), and the length is much larger than the width ($L_z \gg L$), the matrices u and v are isotropically distributed [8]. In that case, $p(\lambda, u, v)$ only depends on λ . To convert the integral equation to a differential equation, $p(\lambda'')$ is expanded to second order in a Taylor series about $\lambda'' = \lambda$, and the average over M' and integrals over $d\mu(u)$ and $d\mu(v)$ can be done. The resulting evolution equation for $P(\lambda)$ (DMPK equation) is

$$\frac{\partial P(\lambda)}{\partial t} = \frac{2}{N+1} \sum_{a=1}^N \frac{\partial}{\partial \lambda_a} \lambda_a (1 + \lambda_a) \left[\frac{\partial}{\partial \lambda_a} - \sum_{b \neq a} \frac{1}{\lambda_a - \lambda_b} \right] P(\lambda) \quad (7)$$

where $t \equiv L_z/l$, and l is the mean free path. One can obtain the conductance distribution from $P(\lambda)$ via the Landauer formula [13]

$$\mathcal{P}(g) \propto \int \prod_{i=1}^N d^N \lambda P(\lambda) \delta \left(g - \sum_i \frac{1}{1 + \lambda_i} \right). \quad (8)$$

The $\mathcal{P}(g)$ obtained in this way has been shown to be valid for quasi-1d systems at weak disorder, with surprising features near the metal–insulator crossover regime [13–17].

The first attempt towards relaxing the isotropy approximation was made in [18]. $p(\lambda'')$ was expanded out to second order about $\lambda'' = \lambda$, the average over M' and integral over $d\mu(u)$ was done and the integral over $d\mu(v)$ was parameterized via arbitrary constants μ_1 and μ_2 . Requiring that the resulting equation conserve probability necessitated a renormalization of the Jacobian from J to J^γ so that $P(\lambda)$ was redefined as

$$\bar{P}(\lambda) = \prod_{a \neq b}^N |\lambda_a - \lambda_b|^\gamma p(\lambda). \quad (9)$$

It also linked μ_1 and μ_2 together leaving their ratio $\gamma \equiv \mu_1/\mu_2$ as the only true degree of freedom, resulting in a one-parameter generalization of the DMPK equation:

$$\frac{\partial \bar{P}(\lambda)}{\partial t} = \frac{2\mu_2}{N+1} \sum_{a=1}^N \frac{\partial}{\partial \lambda_a} \lambda_a (1 + \lambda_a) \left[\frac{\partial}{\partial \lambda_a} - \sum_{b \neq a} \frac{\gamma}{\lambda_a - \lambda_b} \right] \bar{P}(\lambda). \quad (10)$$

The parameter μ_2 just renormalizes the mean free path, while the presence of γ evinces a disorder-dependent repulsion between the eigenvalues. It was demonstrated [18] that γ must be 1 in the weak disorder isotropic limit, and asymptotically approach 0 in the large disorder limit.

A more formal derivation of the GDMPK equation was given in [7]. Going back to equation (6), a model for the average over M' known to reproduce the metal–insulator transition was used [19], and it was found that the integral over $d\mu(u)$ could be done exactly. The resulting $p(\lambda'', v'')$ was Taylor expanded about $\lambda'' = \lambda$, but it was assumed that an expansion of v'' about v was unnecessary, especially in the insulating regime of interest because in that regime the eigenvectors settle into length-independent values. The required integrals over $d\mu(v)$ were formally done using a mean field approximation using the following definitions:

$$\begin{aligned} K_{ab} &\equiv \langle k_{ab} \rangle_v \equiv \sum_{\alpha} \langle |v_{a\alpha}|^2 |v_{b\alpha}|^2 \rangle_v \\ \gamma_{ab} &\equiv \frac{2K_{ab}}{K_{aa}}. \end{aligned} \quad (11)$$

Here the angular bracket denotes an ensemble average over all unitary matrices ν , with disorder strength, geometry L and L_z , as well as transmission eigenvalues λ held fixed:

$$\langle(\cdots)\rangle_\nu \equiv \int d\mu(\nu) p_{L_z}(\lambda, \nu)(\cdots). \quad (12)$$

Therefore K will in principle be a function of all three of these things. Parenthetically, K is typically a λ -independent quantity, though exceptions exist. This situation will be explored in more depth in section 3. In any event, requiring that the resulting equation obey probability conservation necessitated a renormalization of the Jacobian again so that $P(\lambda)$ was re-defined to be:

$$\hat{P}(\lambda) = \prod_{a \neq b}^N |\lambda_a - \lambda_b|^{\gamma_{ab}} p(\lambda). \quad (13)$$

One could then write the resulting GDMPK equation as

$$\frac{\partial \hat{P}(\lambda)}{\partial t} = \sum_{a=1}^N K_{aa} \frac{\partial}{\partial \lambda_a} \lambda_a (1 + \lambda_a) \left[\frac{\partial}{\partial \lambda_a} - \sum_{b \neq a} \frac{\gamma_{ab}}{\lambda_a - \lambda_b} \right] \hat{P}(\lambda). \quad (14)$$

As shown in [2], the matrix elements K_{ab} can be explicitly evaluated numerically and contain information not only about dimensionality but also the critical point in 3d. The quasi-1d DMPK equation is recovered when $\gamma_{ab} \rightarrow 1$. In the above derivation the eigenvector correlations were neglected under the assumption that they become small at very large disorder, restricting the validity of equation (14) to the deeply insulating limit. Within this limit, the conductance distributions obtained analytically from the solutions of equation (14) agree well with numerical results obtained from tight binding Anderson model [1, 2].

Despite the successes, there are a few major criticism of the derivation of equation (14). The first one is that it is not clear if and/or under what conditions the eigenvector correlations can be neglected. Second, it has heretofore been necessary to impose probability conservation as a separate constraint, which has required a ‘renormalization’ of $J(\lambda)$, and redefinition of $P(\lambda)$ seemingly at odds with equation (4). Last, there is the apparent conflict with the known constraint that in the limit $|\lambda_a - \lambda_b| \rightarrow 0$, the level repulsion should be characterized by the symmetry parameter $\beta = 1$ as determined by the exponent of the Jacobian and not by the matrix γ_{ab} , where $\gamma_{ab} \ll 1$ in the deeply insulating limit [23]. In this work we address all of these criticisms. To address the first and second criticisms we include *both* eigenvalue and eigenvector correlations systematically in a controlled way up to order $\delta L_z/l$. We will find that including the additional eigenvector correlations results ultimately in the same GDMPK in equation (14), but also restores the Jacobian to its original form in equation (3), resulting in an equation which *automatically* conserves probability. Thus the results for $\mathcal{P}(g)$ obtained earlier in [1, 2] turn out to be valid quite generally, beyond the deeply insulating limit. We address the last criticism by developing a self-consistent evolution equation for K_{ab} and argue that the equation implies $\gamma_{ab} \rightarrow 1$ as $|\lambda_a - \lambda_b| \rightarrow 0$, satisfying the symmetry requirement. In addition, we obtain an exact relationship between the eigenvectors and the Lyapunov exponents ν_n , defined by the relation $\langle \lambda_n \rangle \equiv \exp[(2L_z/l)\nu_n]$, leading to a sum rule for the Lyapunov exponents.

2. The generalized DMPK equation

We will start with equation (5). Using equation (1), and equating M'' with MM'^{-1} , it then follows that $u'' = u \cdot f(\lambda, \lambda', v v'^{\dagger})$. The invariance of the measure then allows one [7] to integrate over u , leading to an integral equation purely in terms of $p(\lambda, v)$,

$$p_{L_z+\delta L_z}(\lambda, v) = \int d\mu(u') d\mu(\lambda') d\mu(v') p_{\delta L_z}(\lambda', u', v') p_{L_z}(\lambda'', v''). \quad (15)$$

Next we write $\lambda'' = \lambda + \delta\lambda''$ and $v'' = v + \delta v''$ and Taylor expand $p(\lambda'', v'')$.

$$\begin{aligned} p_{L_z+\delta L_z}(\lambda, v) &= p_{L_z}(\lambda, v) + \sum_a \frac{\partial p_{L_z}(\lambda, v)}{\partial \lambda_a} \langle \delta \lambda''_a \rangle_{M'} + \frac{1}{2} \sum_{ab} \frac{\partial^2 p_{L_z}(\lambda, v)}{\partial \lambda_a \partial \lambda_b} \langle \delta \lambda''_a \delta \lambda''_b \rangle_{M'} \\ &+ \sum_{ab} \frac{\partial p_{L_z}(\lambda, v)}{\partial v_{ab}} \langle \delta v''_{ab} \rangle_{M'} + \sum_{a,bc} \frac{\partial^2 p_{L_z}(\lambda, v)}{\partial \lambda_a \partial v_{bc}} \langle \delta \lambda''_a \delta v''_{bc} \rangle_{M'} \\ &+ \frac{1}{2} \sum_{ab,cd} \frac{\partial^2 p_{L_z}(\lambda, v)}{\partial v_{ab} \partial v_{cd}} \langle \delta v''_{ab} \delta v''_{cd} \rangle_{M'} + \dots \end{aligned} \quad (16)$$

Here $\langle (\dots) \rangle_{M'}$ represents an average over M' given by

$$\langle (\dots) \rangle_{M'} \equiv \int d\mu(u') d\mu(\lambda') d\mu(v') p_{\delta L_z}(\lambda', u', v') (\dots). \quad (17)$$

Thereafter we integrate both sides over $d\mu(v)$ and use integration by parts to write:

$$\begin{aligned} p_{L_z+\delta L_z}(\lambda) &= p_{L_z}(\lambda) + \sum_a \int d\mu(v) \frac{\partial p_{L_z}(\lambda, v)}{\partial \lambda_a} \langle \delta \lambda''_a \rangle_{M'} \\ &+ \frac{1}{2} \sum_{ab} \int d\mu(v) \frac{\partial^2 p_{L_z}(\lambda, v)}{\partial \lambda_a \partial \lambda_b} \langle \delta \lambda''_a \delta \lambda''_b \rangle_{M'} - \sum_{ab} \int d\mu(v) p_{L_z}(\lambda, v) \left\langle \frac{\partial \delta v''_{ab}}{\partial v_{ab}} \right\rangle_{M'} \\ &- \sum_{a,bc} \int d\mu(v) \frac{\partial p_{L_z}(\lambda, v)}{\partial \lambda_a} \left\langle \frac{\partial \delta \lambda''_a \delta v''_{bc}}{\partial v_{bc}} \right\rangle_{M'} \\ &+ \frac{1}{2} \sum_{ab,cd} \int d\mu(v) p_{L_z}(\lambda, v) \left\langle \frac{\partial^2 \delta v''_{ab} \delta v''_{cd}}{\partial v_{ab} \partial v_{cd}} \right\rangle_{M'} + \dots \end{aligned} \quad (18)$$

Next we make two assumptions about $p_{L_z}(\lambda, v)$.

- (i) *Assumption 1.* The distribution $p_{L_z}(\lambda, v)$ depends only on the moduli $|v_{ab}|^2$ so that no phases are preferred. We will call this the random phase assumption.
- (ii) *Assumption 2.* The distribution $p_{L_z}(\lambda, v)$ and its derivatives with respect to λ are highly peaked about a particular set of moduli (for a given disorder strength, geometry, and set of eigenvalues, with different realizations of randomness).

As checked numerically in [2], $p_{L_z}(\lambda, v)$ is highly peaked and close to the average in the metallic ($l \gg a$ where a is the lattice spacing), critical ($l \sim a$), and strongly disordered regimes ($\xi \sim a$ where ξ is the localization length). In particular, the distributions $P(k_{11})$ and $P(k_{12})$ are size independent at the critical point (thanks to this, the critical point could be found from the size and disorder dependence of K_{11} and γ_{12} [2]). The finite width of the distribution was confirmed numerically also for the strongly disordered two-dimensional systems [21] where much larger samples could be considered.

This being the case, we may take the bracket $\langle \dots \rangle_{M'}$ inside equation (16) to be approximately constant over the width of $p_{L_z}(\lambda, v)$, pull it outside the integral, and replace the integral with an ensemble average over v . So then equation (16) can be rewritten as

$$\frac{\partial p_{L_z}(\lambda)}{\partial L_z} = f(\lambda) p_{L_z}(\lambda) + \sum_a g_a(\lambda) \frac{\partial p_{L_z}(\lambda)}{\partial \lambda_a} + \sum_{ab} h_{ab}(\lambda) \frac{\partial^2 p_{L_z}(\lambda)}{\partial \lambda_a \partial \lambda_b} + \dots \quad (19)$$

where we define,

$$\begin{aligned}
f(\lambda) &= \frac{l}{\delta L_z} \left[\frac{1}{2} \sum_{ab,cd} \left\langle \frac{\partial^2 \delta v''_{ab} \delta v''_{cd}}{\partial v_{ab} \partial v_{cd}} \right\rangle_{M'v} - \sum_{ab} \left\langle \frac{\partial \delta v''_{ab}}{\partial v_{ab}} \right\rangle_{M'v} \right] \\
g_a(\lambda) &= \frac{l}{\delta L_z} \left[\langle \delta \lambda''_a \rangle_{M'v} - \sum_{bc} \left\langle \frac{\partial \delta \lambda''_a \delta v''_{bc}}{\partial v_{bc}} \right\rangle_{M'v} \right] \\
h_{ab}(\lambda) &= \frac{l}{2\delta L_z} \langle \delta \lambda''_a \delta \lambda''_b \rangle_{M'v}.
\end{aligned} \tag{20}$$

Here the average over v (in addition to the average over M') is defined as in equation (12), and equation (17). Note that in the limit $\delta L_z \rightarrow 0$, we only need to keep terms in the average that are at most linear in $\delta L_z/l$. In [7], only the terms corresponding to the changes in λ'' were assumed to be important, neglecting the changes in $\delta v''$. Since these also contain terms linear in $\delta L_z/l$, a systematic expansion needs to keep all five terms. All other terms in the series expansion beyond the second derivative (represented by (\dots) in equation (19)) will be higher order in $\delta L_z/l$ and therefore need not be considered. In order to evaluate the changes $\delta \lambda''$ and $\delta v''$, we would like a matrix whose eigenvectors and eigenvalues are given by v and λ respectively. Such a matrix, X , is given below [20]:

$$\begin{aligned}
X &= \frac{1}{4} [M^\dagger M + (M^\dagger M)^{-1} - 2I] \\
&= \begin{pmatrix} v^\dagger \lambda v & 0 \\ 0 & v^\dagger \lambda v \end{pmatrix}.
\end{aligned} \tag{21}$$

A little algebra, using equation (1), is sufficient to demonstrate the last line. From here we form X'' out of $M'' = M M'^{-1}$ to calculate the perturbative corrections to λ'' and v'' . After some manipulations one obtains

$$v''^\dagger \lambda'' v'' = v^\dagger \lambda v + v^\dagger Y v \tag{22}$$

where the matrix Y is given by

$$\begin{aligned}
Y &= -\lambda + v u' \lambda' u'^\dagger v^\dagger + v u' \sqrt{1 + \lambda'} v' v^\dagger \lambda v v'^\dagger \sqrt{1 + \lambda'} u'^\dagger v^\dagger \\
&\quad - v u' \sqrt{1 + \lambda'} v' v^\dagger \sqrt{\lambda(1 + \lambda)} v^* v'^T \sqrt{\lambda'} u'^\dagger v^\dagger \\
&\quad - v u' \sqrt{\lambda'} v'^* v^T \sqrt{\lambda(1 + \lambda)} v v'^\dagger \sqrt{1 + \lambda'} u'^\dagger v^\dagger \\
&\quad + v u' \sqrt{\lambda'} v'^* v^T \lambda v^* v'^T \sqrt{\lambda'} u'^\dagger v^\dagger.
\end{aligned} \tag{23}$$

Going back to equation (1), it is apparent that in the limit of vanishing thickness δL_z , we should have $M' \rightarrow 1$. This implies that $u' v' \rightarrow 1$. So we can say:

$$u' v' = 1 + \omega' + \dots \tag{24}$$

where ω' is of order δL_z ⁴. Note since $u' v'$ is unitary, this requires that ω' be anti-Hermitian. And now we will plug equation (24) into equation (23). Keeping only terms of order δL_z , we obtain:

$$\begin{aligned}
Y &= v v'^\dagger \lambda' v' v^\dagger + \frac{1}{2} v v'^\dagger \lambda' v' v^\dagger \lambda + \frac{1}{2} \lambda v v'^\dagger \lambda' v' v^\dagger \\
&\quad - \sqrt{\lambda(1 + \lambda)} v^* v'^T \sqrt{\lambda'} v' v^\dagger - v v'^\dagger \sqrt{\lambda'} v'^* v^T \sqrt{\lambda(1 + \lambda)} \\
&\quad + v v'^\dagger \sqrt{\lambda'} v'^* v^T \lambda v^* v'^T \sqrt{\lambda'} v' v^\dagger + v \omega' v^\dagger \lambda - \lambda v \omega' v^\dagger.
\end{aligned} \tag{25}$$

⁴ We use a model of disorder characterized by a random potential $V(r) = \sum_i V_i \delta(r - r_i)$. Calculating the disorder averaged transfer matrix M to first order in L_z , and making simplifying approximations, we can reproduce the building blocks in equation (27). Under the same approximation, we find that $\langle \omega' \rangle_{M'}$ is proportional to δL_z

The perturbative changes due to M' are then given by

$$\begin{aligned}\delta\lambda_n'' &= Y_{nn} + \sum_{i \neq n} \frac{|Y_{in}|^2}{\lambda_n - \lambda_i} \\ \delta v_{mn}'' &= \sum_{i \neq n} \frac{Y_{in}}{\lambda_n - \lambda_i} v_{mi}^\dagger - \frac{1}{2} \sum_{i \neq n} \frac{|Y_{in}|^2}{(\lambda_i - \lambda_n)^2} v_{mn}^\dagger \\ &\quad + \sum_{i \neq n} \sum_{j \neq n} \frac{Y_{ij}}{\lambda_n - \lambda_i} \frac{Y_{jn}}{\lambda_n - \lambda_j} v_{mi}^\dagger - \sum_{i \neq n} \frac{Y_{mn}}{\lambda_n - \lambda_i} \frac{Y_{in}}{\lambda_n - \lambda_i} v_{mi}^\dagger.\end{aligned}\quad (26)$$

Note that the eigenvector corrections so written are normalized to second order. To make further progress, we follow [7] and use a very general model for the average over M' . Since $\lambda' \rightarrow 0$ as $\delta L_z \rightarrow 0$, we use $\lambda' \propto \delta L_z/l$ and the following ‘building blocks’:

$$\begin{aligned}\langle v'_{ab} \rangle_{M'} &= \langle v'^*_{ab} \rangle_{M'} = \langle v'_{ab} v'_{cd} \rangle_{M'} = \langle v'^*_{ab} v'^*_{cd} \rangle_{M'} = 0 \\ \sum_a \langle \sqrt{\lambda'_a} v'_{a\alpha} v'_{a\beta} \rangle_{M'} &= 0 \\ \sum_a \langle \lambda'_a v'^*_{a\alpha} v'_{a\beta} \rangle_{M'} &= \frac{\delta L_z}{l} \delta_{\alpha\beta} \\ \sum_{ab} \langle \sqrt{\lambda'_a \lambda'_b} v'^*_{a\alpha} v'^*_{a\beta} v'_{b\delta} v'_{b\gamma} \rangle_{M'} &= \frac{\delta L_z}{l} \delta_{\alpha\beta} \delta_{\alpha\gamma} \delta_{\beta\delta}. \\ \langle \omega'_{\alpha\beta} \rangle_{M'} &= -\frac{\delta L_z}{l} \delta_{\alpha\beta}.\end{aligned}\quad (27)$$

The first four are the building blocks used in the original DMPK equation and those suggested in [19] to describe the metal–insulator transition. The last is obtained via (see footnote no 4), though in fact, ω' makes no contribution below since it cancels out in all relevant terms due to its proportionality to the unit matrix. Now we can build a *systematic and controlled expansion* of p_{L_z} in powers of the small parameter $\delta L_z/l$. All of the correlations in equation (20) can be written in terms of the following two results:

$$\begin{aligned}\langle Y_{mn} \rangle_{M'} &= \frac{\delta L_z}{l} (1 + \lambda_n) \delta_{mn} + \frac{\delta L_z}{l} \sum_{\alpha\beta} \lambda_\alpha v_{m\beta} v_{n\beta}^* |v_{\alpha\beta}|^2 \\ \langle Y_{ij} Y_{mn} \rangle_{M'} &= \frac{\delta L_z}{l} \Lambda_{in} \sum_{\alpha} v_{i\alpha}^* v_{j\alpha}^* v_{m\alpha} v_{n\alpha} + \frac{\delta L_z}{l} \Lambda_{jm} \sum_{\alpha} v_{m\alpha}^* v_{n\alpha}^* v_{i\alpha} v_{j\alpha}\end{aligned}\quad (28)$$

where we make the definition

$$\Lambda_{ij} \equiv \sqrt{\lambda_i(1 + \lambda_i)\lambda_j(1 + \lambda_j)}.\quad (29)$$

As is also evident in equation (20), we need to take derivatives with respect to v_{ab} and we use the following two basic equations for this,

$$\frac{\partial v_{ab}}{\partial v_{ij}} = \delta_{ai} \delta_{bj}; \quad \frac{\partial v_{ab}^*}{\partial v_{ij}} = -v_{aj}^* v_{ib}^*.\quad (30)$$

The latter is obtained by implicitly differentiating the relationship $v^\dagger v = 1$. Finally, we need to then calculate averages over v . Using the random phase assumption, and after lengthy algebra we find that f , g , and h of equation (20) can all be written in terms of the K matrix,

$$\begin{aligned}f(\lambda) &= \sum_{k \neq m} \frac{1 + 2\lambda_m}{\lambda_m - \lambda_k} [K_{mm} - 2K_{mk}] + \sum_{k \neq m \neq n} \frac{\lambda_m(1 + \lambda_m)}{(\lambda_m - \lambda_k)(\lambda_m - \lambda_n)} [K_{mm} - 2K_{mk}] \\ g_a(\lambda) &= (1 + 2\lambda_a) K_{aa} + \sum_{k \neq a} \frac{2\lambda_a(1 + \lambda_a)}{\lambda_a - \lambda_k} [K_{aa} - K_{ak}] \\ h_{ab}(\lambda) &= \lambda_a(1 + \lambda_a) K_{aa} \delta_{ab}.\end{aligned}\quad (31)$$

Putting these results into equation (19) and using the definition of $P(\lambda)$ from equation (4) we find that we can write equation (19) in full generality as

$$\frac{\partial P(\lambda)}{\partial t} = \sum_{a=1}^N K_{aa} \frac{\partial}{\partial \lambda_a} \lambda_a (1 + \lambda_a) \left[\frac{\partial}{\partial \lambda_a} - \sum_{b \neq a} \frac{\gamma_{ab}}{\lambda_a - \lambda_b} \right] P(\lambda). \quad (32)$$

As one can see, this is the same GDMPK equation as used in [7]. But this time we do not need to assume anything special about the eigenvector correlations, and indeed by including them, the equation now automatically conserves probability. Finally, since this equation was derived under very general assumptions, we expect it to provide a description of the conductance distribution on both sides of and across the metal–insulator transition. The only remaining problem is that equation (32) implies that the level repulsion for neighboring levels is determined by the matrix γ_{ab} which is not obviously related to the symmetry parameter β . In the insulating regime, it is known that on average $\gamma_{ab} \ll 1$ which seems to contradict this symmetry requirement. In the next section we address this issue by writing down an evolution equation for K_{mn} .

3. Level repulsion between nearest neighbors

We begin developing an evolution equation for K_{mn} by expanding K_{mn} out to linear order in $\delta L_z/l$. We write

$$K_{mn} + \delta K_{mn} = \sum_a \langle |v_{ma} + \delta v_{ma}|^2 |v_{na} + \delta v_{na}|^2 \rangle_{M'v} \quad (33)$$

Then using the results of equation (26), equation (28), and the random phase assumption we find that we can write the evolution of K_{mn} with length L_z as

$$\frac{dK_{mn}}{dt} = \sum_{i \neq m} f_{im} (L_{ni}^{mi} - L_{nm}^{im}) + \sum_{i \neq n} f_{in} (L_{mi}^{ni} - L_{mn}^{in}) + 2 \sum_{i \neq m} f_{im} L_{mi}^{mi} \delta_{mn} - 2 f_{mn} L_{nm}^{nm} \delta_{m \neq n} \quad (34)$$

where f_{ij} is given by,

$$f_{ij} \equiv \frac{\lambda_i(1 + \lambda_i) + \lambda_j(1 + \lambda_j)}{(\lambda_i - \lambda_j)^2} \quad (35)$$

and L_{ik}^{jk} is given by,

$$\begin{aligned} L_{ik}^{jk} &\equiv \sum_{\alpha, \beta} \langle |v_{i\alpha}|^2 |v_{k\alpha}|^2 |v_{j\beta}|^2 |v_{k\beta}|^2 \rangle_v \\ &= \langle k_{ik} k_{jk} \rangle_v. \end{aligned} \quad (36)$$

Note that $f_{ij} \rightarrow 1$ in the limit of large L_z , independently of i and j since the eigenvalues λ_i increase exponentially with length L_z . It is important to note that this evolution equation for K conserves probability since if we start with a K whose columns sum to 1, as they must, then one can show that the evolution equation preserves this property. And it is interesting to observe that the evolution of K is determined solely by its second moment. To gain some insight we will split L into the sum of two terms—its mean field approximation and fluctuations about the mean:

$$L_{ik}^{jk} = \langle k_{ik} \rangle_v \langle k_{jk} \rangle_v + \Delta_{ik}^{jk} = K_{ik} K_{jk} + \Delta_{ik}^{jk}. \quad (37)$$

Δ can be roughly interpreted as the variance of k . In particular Δ_{mn}^{mm} is exactly the variance of k_{mn} . As an elementary application of equation (37), we show that equation (34) can describe a weakly disordered quasi-1d conductor. Regardless of geometry, we expect the fluctuations

Δ to be negligible for small disorder [2]. So using equation (37) with $\Delta = 0$, the evolution equation for K reduces to:

$$\begin{aligned} \frac{dK_{mn}}{dt} = & \sum_{i \neq m} f_{im}(K_{ni}K_{mi} - K_{nm}K_{im}) + \sum_{i \neq n} f_{in}(K_{mi}K_{ni} - K_{mn}K_{in}) \\ & + 2 \sum_{i \neq m} f_{im}K_{mi}^2\delta_{nn} - 2f_{mn}K_{nm}^2\delta_{m \neq n}. \end{aligned} \quad (38)$$

It is straightforward to show that the L_z independent solution to this equation is

$$K_{mn} = \frac{1 + \delta_{mn}}{N + 1} \quad (39)$$

as required by the quasi-1d DMPK equation. We now wish to argue that for any disorder strength $\gamma_{ij} \rightarrow 1$ as $\lambda_i \rightarrow \lambda_j$, as required by symmetry. To that end we set $m = n$ in equation (34), and combine it with equation (37) to obtain:

$$\frac{dK_{mm}}{dt} = K_{mm}^2 \sum_{i \neq m} f_{im} \left[\gamma_{mi}(1 - \gamma_{mi}) + \frac{2\Delta_{mi}^{mi} - 4\Delta_{mm}^{mi}}{K_{mm}^2} \right]. \quad (40)$$

If we neglect fluctuations again, we have:

$$\frac{dK_{mm}}{dt} = K_{mm}^2 \sum_{i \neq m} f_{im}\gamma_{mi}(1 - \gamma_{mi}). \quad (41)$$

Then if we let nearest neighbors λ_m and λ_n approach each other in equation (41), f_{mn} will become singular. But since dK_{mm}/dt must be bounded, it must be that γ_{mn} goes to 1. (Of course $\gamma_{mn} \rightarrow 0$ is also a possible solution, but this choice will contradict the case in the metallic limit where all neighboring λ_m and λ_n are close to each other and the corresponding $\gamma_{mn} = 1$. As we will show shortly, our numerical results verify that $\gamma_{mn} \rightarrow 1$ as nearest neighbors λ_m and λ_n approach each other even in the strong disorder limit.)

Now certainly we can neglect fluctuations for weak disorder, but what about strong disorder? Although fluctuations Δ are not generally negligible in the 3d strongly disordered regime—instead they are of order K [2]—we expect them to be less important when the eigenvalues assume metal-like configurations, namely $1 - \lambda_n/\lambda_{n+1} \ll 1/N$. This is because since k_{mn} takes the form of an inner product, in the insulating regime we can interpret it as a measure of the overlap between transmission eigenvector amplitudes through channels m and n , essentially a measure of the overlap of the paths through these channels. Since in the insulating regime different paths tend to have different transmission eigenvalues, if two eigenvalues λ_m and λ_n are close then it is likely because the paths are the same. Therefore as we average over realizations of disorder, while keeping eigenvalues close we can expect the overlap of the paths to remain high, and therefore the variance of the overlap, i.e. of k_{mn} , to be small. Thus even when λ_1 and λ_2 are far apart on average and consequently $\gamma_{12} \ll 1$ in the large disorder regime (as shown in numerical studies [2]), in the limit where their separation goes to zero, the repulsion would be determined by the usual symmetry parameter equal to $\beta = 1$ in the orthogonal case as considered here.

The argument above is not rigorous, but it is bolstered by numerical evidence as well. Numerically, it is difficult to access the small λ separation limit since for large disorder the lowest levels have a statistically negligible probability to become close. However there are some levels near the middle of the spectrum that do come closer for any given disorder strength and for sufficiently large system size [22] (see figure 1). Figure 1 illustrates the trend of how $\gamma_{n,n+1}$ changes when the separation of the levels becomes smaller. Here we have used the same anisotropic Anderson model as considered in [2]. We consider a cubic system of size

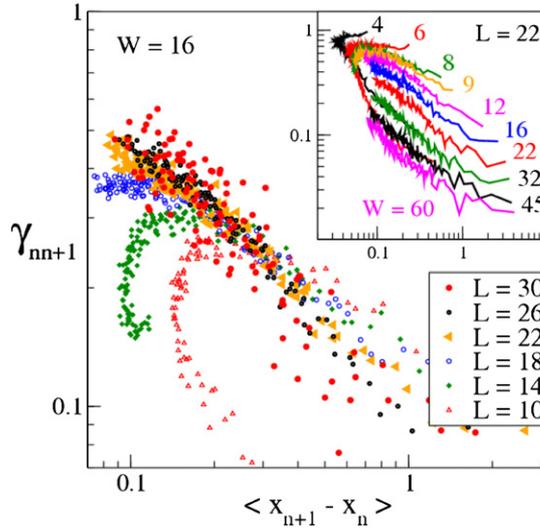


Figure 1. $\gamma_{n,n+1}$ as a function of the difference $\langle x_{n+1} - x_n \rangle$ for statistical ensembles of cubic samples of different sizes $10 \leq L \leq 30$ for a given disorder strength $W = 16$. (Only $N = 100$ samples are considered for $L = 30$.) For comparison, the critical disorder [2] $W_c \approx 9.4$. To avoid numerical inaccuracies, only channels with $\langle x_n - x_1 \rangle < 34$ were considered. The mean values of the logarithm of the conductance are -4.07 for $L = 10$ and -12.6 for $L = 30$. Typically, the level separation decreases with increasing n up to $n = N/2$, and then start to increase again; this leads to two different ‘branches’ (the upper one for $n < N/2$), as is visible for the two smallest sizes $L \leq 14$. For larger L this crossover occurs at larger n and smaller separation, the lower branch eventually becoming numerically inaccessible for sufficiently large L . Inset: The same ($\gamma_{n,n+1}$ versus the level separation) for a fixed length $L = 22$ but for different values of W .

$L \times L \times L$ but with anisotropic nearest neighbor hopping elements $t_z = 1$ and $t_x = t_y = 0.4$. The anisotropy guarantees that all channels are open, i.e. $N = L^2$. For a given disorder W and a fixed system size L , the matrix K and hence the parameters γ_{mn} as well as the eigenvalues λ_n are then evaluated using 10^4 statistical samples within orthogonal symmetry.

Since the eigenvalues $\lambda_n = \sinh^{-2}(x_n/2)$ increase exponentially with the size of the system, it turns out that many eigenvalues increase above our numerical accuracy. Therefore, we consider only channels where $\langle x_n - x_1 \rangle < 34$. This guarantees that $\lambda_n/\lambda_1 < 1.7 \times 10^{-15}$, although at the same time the restriction does not enable us to consider all γ_{mn} for a given N . For instance, in figure 1, $n \leq 120$. Nevertheless, figure 1 confirms our expectation that $\gamma_{n,n+1}$ increases towards unity when the difference $\langle x_{n+1} - x_n \rangle$ decreases.

In order to move beyond the small fluctuation limit, and more generally model K for a strongly disordered insulator, especially near the critical point, it is expected that Δ will become important for the lowest channels. Though little can be said at this point, one implication of equation (40) is that if K_{mm} reaches a limiting distribution in $L_z \gg L$ strongly disordered limit, the steady state value of γ_{mn} will depend on these fluctuations.

We emphasize that regardless of what the matrix L or K may look like, the validity of equation (32) is unaffected.

4. The K-matrix and the Lyapunov exponents

Finally, we develop an equation relating the Lyapunov exponents and the matrix K . The Lyapunov exponent ν is defined by

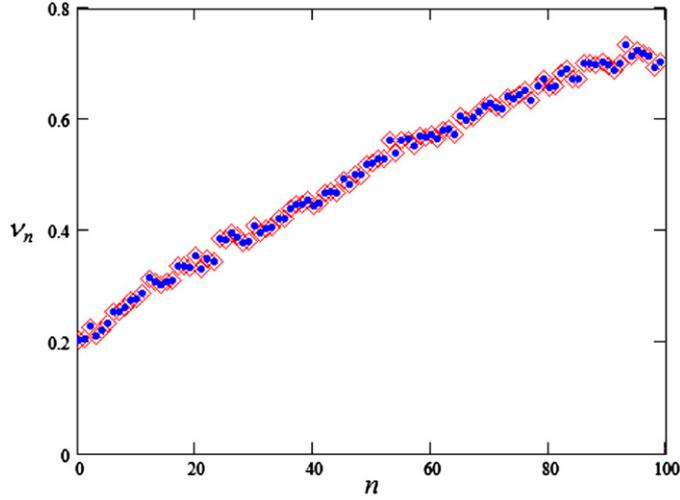


Figure 2. The Lyapunov exponents v_n plotted versus n . The red diamonds are from numerical simulations ($L = 10$, $W = 20$, and $\langle \ln g \rangle = -7.1$); the solid blue circles are from equation (48).

$$\lambda(L_z \rightarrow \infty) = e^{2\nu L_z}. \tag{42}$$

In order to develop an equation for ν we write

$$\lambda(L_z + \delta L_z) - \lambda(L_z) = e^{2\nu(L_z + \delta L_z)} - e^{2\nu L_z}. \tag{43}$$

Solving for ν we have

$$\nu = \frac{1}{2\delta L_z} \ln \left(1 + \frac{\delta \lambda}{\lambda} \right). \tag{44}$$

Expanding out to linear order in $\delta L_z/l$ and averaging over realizations of disorder we obtain

$$\nu_n = \frac{l}{2\delta L_z} \left[\left\langle \frac{\delta \lambda_n}{\lambda_n} \right\rangle_{M'\nu} - \frac{1}{2} \left\langle \left(\frac{\delta \lambda_n}{\lambda_n} \right)^2 \right\rangle_{M'\nu} \right]. \tag{45}$$

Note that this $\delta \lambda$ is the $\delta \lambda''$ of equation (26). So using equations (26) and (28), we find:

$$\langle (\delta \lambda_n'') \rangle_{M'} = \frac{\delta L_z}{l} (1 + 2\lambda_n) + \frac{\delta L_z}{l} \sum_{m, (i \neq n)} \frac{\lambda_i + \lambda_n + 2\lambda_i \lambda_n}{\lambda_n - \lambda_i} |v_{im}|^2 |v_{nm}|^2 \tag{46}$$

$$\langle (\delta \lambda_n'')^2 \rangle_{M'} = 2 \frac{\delta L_z}{l} \lambda_n (1 + \lambda_n) \sum_m |v_{nm}|^4. \tag{47}$$

Next we perform the disorder average over ν . Recalling equation (11) we find:

$$\nu_n = \frac{1}{2} K_{nn} + \sum_{m \neq n} \frac{1 + \lambda_n}{\lambda_n - \lambda_m} K_{nm} \tag{48}$$

for all disorder strengths. A similar equation was obtained in [19]. Equation (48) has been verified numerically in the insulating regime, where most doubt concerning its validity would reside. When plots of ν_n obtained via numerical simulations using tight binding Anderson model and from equation (48) are overlaid, as shown in figure 2, the points are indistinguishable.

Interestingly, if we sum this equation over n , we obtain the following relationship, independent of K and λ ,

$$\sum_{n=1}^N v_n = \frac{N}{2}. \quad (49)$$

This relation has been known to be true in the diffusive limit [24], but the present calculation implies it is true for large disorder as well. This has been confirmed numerically.

Note that these relations are independent of any model for the matrix K . A two-parameter model for K was chosen in [2] for large disorder, based entirely on numerical studies. The exact relationship should allow us to build a better model based on the known properties of the Lyapunov exponents.

5. Summary and Conclusion

In this work we developed a systematic and controlled derivation of the generalized DMPK equation by expanding the probability distribution $p_{L_c+\delta L_c}(\lambda)$ in a power series $\delta L_c/l$ and keeping all terms linear in $\delta L_c/l$ in the limit $\delta L_c/l \rightarrow 0$. This means including the eigenvector correlations that had been neglected before. The additional correlations do not change the generalized DMPK equation obtained earlier; instead, their inclusion automatically conserve probability by allowing the evolution of the Jacobian without having to redefine it with a disorder strength dependent exponent. In addition to providing a broader applicability of the generalized DMPK equation such as the conductance distributions obtained in [1], the derivation shows how the repulsion of closely spaced neighboring eigenvalues remain consistent with symmetry requirements, even though it can be very different when the eigenvalues are far apart. Moreover, we obtain an exact result relating the correlation matrix K and the eigenvalues λ with the Lyapunov exponents ν as well as a sum rule for the exponents independent of K or λ .

The challenge remains to construct a reasonable model for the phenomenological matrix K . The evolution of K involves quantities that cannot be factored into products of K 's in general, and at this point an analytic solution seems too complicated. On the other hand a crude phenomenological two-parameter model suggested by numerical studies and used in [1, 2] seems to work quite well in the strongly disordered regime, although further numerical studies are needed to check if/how the model might change when e.g. approaching the metal–insulator critical point. We expect the relationships with the Lyapunov exponents obtained here to provide useful insights.

Acknowledgments

We thank the anonymous referees for pointing out the necessity to go beyond the first term in equation (24). PM acknowledges financial support from the Slovak Research Development Agency under the contract no APVV-0108-11.

References

- [1] Douglas A and Muttalib K A 2010 *Phys. Rev. B* **82** 035121
Douglas A and Muttalib K A 2009 *Phys. Rev. B* **80** 161102
- [2] Muttalib K A, Markoš P and Wölfle P 2005 *Phys. Rev. B* **72** 125317
Markoš P, Muttalib K A, Wölfle P and Klauder J R 2004 *Europhys. Lett.* **68** 867

- [3] Markoš P 2003 Conductance statistics near the Anderson transition *Anderson Localization and its Ramifications (Lectures Notes in Physics vol 630)* ed T Brandes and S Kettelman (Berlin: Springer)
- [4] Soukoulis C M, Wang X, Li Q and Sigalas M M 1999 *Phys. Rev. Lett.* **82** 668
- [5] Slevin K and Ohtsuki T 1997 *Phys. Rev. Lett.* **78** 4083
- [6] Somoza J, Prior A M and Ortuno M 2009 *Eur. Phys. J. B* **70** 513
Somoza A M, Prior J and Ortuno M 2006 *Phys. Rev. B* **73** 184201
- [7] Muttalib K A and Gopar V A 2002 *Phys. Rev. B* **66** 11538
- [8] Dorokhov O N 1982 *JETP Lett.* **36** 318
Mello P A, Pereyra P and Kumar N 1988 *Ann. Phys., NY* **181** 290
Beenakker C W J 1997 *Rev. Mod. Phys.* **69** 731
- [9] Frahm K 1995 *Phys. Rev. Lett.* **74** 4706
Brouwer P A and Frahm K 1996 *Phys. Rev. B* **53** 1490
Rejaei B 1996 *Phys. Rev. B* **53** 13235
- [10] Altshuler B L, Kravtsov V E and Lerner I 1989 *Phys. Lett. A* **134** 488
- [11] Shapiro B 1990 *Phys. Rev. Lett.* **65** 1510
- [12] Mehta M L 1991 *Random Matrices* 2nd edn (Boston, MA: Academic)
- [13] Muttalib K A and Wölfle P 1999 *Phys. Rev. Lett.* **83** 3013
Wölfle P and Muttalib K A 1999 *Ann. Phys., Leipzig* **8** 753
- [14] Muttalib K A, Wölfle P, Garcia-Martin A and Gopar V A 2003 *Europhys. Lett.* **61** 95
- [15] Gopar V A, Muttalib K A and Wölfle P 2002 *Phys. Rev. B* **66** 174204
- [16] Wölfle K A, Muttalib P and Gopar V A 2003 *Ann. Phys.* **308** 156–200
- [17] Garcia-Martin A and Saenz J J 2001 *Phys. Rev. Lett.* **87** 116603
- [18] Muttalib K A and Klauder J R 1999 *Phys. Rev. Lett.* **82** 4272
- [19] Chalker J T and Bernhardt M 1993 *Phys. Rev. Lett.* **70** 982
- [20] Muttalib K A, Pichard J-L and Stone A D 1987 *Phys. Rev. Lett.* **59** 2475
- [21] Markoš P 2010 *Phys. Rev. B* **82** 094203
- [22] Pichard J-L and André G 1986 *Europhys. Lett.* **2** 477
- [23] Markos P and Schweitzer L 2010 *Phys. Rev. B* **81** 113412
- [24] Pichard J-L 1991 *Quantum Coherence in Mesoscopic Systems (NATO ASI vol 254)* ed B Kramer (New York: Plenum) p 361