

Abstract

These lecture notes supplements my discussion class. We only have one session per week and normally we have to rush through one whole chapter in less than 50 minutes. Now I believe supplementary notes are necessary for you to further your understanding in physics on your own pace.

Each topic is independent. So you can just start with the section that interests you. Some topics are very basic, so I hope everyone will read them. Others are advanced, which often exceeds the textbook level. So if you feel the course is too easy and you're bored, try challenging problems I provide in these notes.

1 Range of Projectile Motion

1.1 Horizontal Range

Most of the basic physics textbooks talk about the horizontal range of the projectile motion. It is derived using the kinematics equations:

$$\begin{aligned} a_x &= 0 \\ v_x &= v_{0x} \\ \Delta x &= v_{0x}t \\ a_y &= -g \\ v_y &= v_{0y} - gt \\ \Delta y &= v_{0y}t - \frac{1}{2}gt^2 \end{aligned}$$

where

$$\begin{aligned} v_{0x} &= v_0 \cos \theta \\ v_{0y} &= v_0 \sin \theta \end{aligned}$$

Suppose a projectile is thrown from the ground level, then the range is the distance between the launch point and the landing point, where the projectile hits the ground. When the projectile comes back to the ground, the vertical displacement is zero, thus we have

$$0 = v_0 \sin \theta t - \frac{1}{2}gt^2$$

Solving for t , we have

$$t = 0, \frac{2v_0 \sin \theta}{g}$$

The first solution gives the time when the projectile is thrown and the second one is the time when it hits the ground. Plugging in the second solution into the displacement equation and using $2 \sin \theta \cos \theta = \sin(2\theta)$, we have

$$R = \Delta x(t = 2v_0 \sin \theta / g) = \frac{v_0^2}{g} \sin(2\theta)$$

Example

A baseball player can throw a ball at 30.0 m/s. What is the maximum horizontal range?

Solution

To maximize the range, s/he must throw a ball at an angle of 45° because at this angle $\sin 2\theta = 1$. The range is

$$R = \frac{v_0^2}{g} = \frac{30^2}{9.8} = 91.8 \text{ m}$$

1.2 Range on a Slope

Now what happens if you throw a ball on a slope? Do you still need to throw a ball at 45° above the horizontal to maximize the range? Or should you throw at 45° to the slope? Neither of them is correct. If you throw a ball at 45° on a slope of, say, 60° , then you're practically throwing the ball *toward* the slope so the range is actually zero. (We ignored the height of the launch point.) If you throw a ball at an angle of 45° to the slope of 45° , you're actually throwing a ball straight up, and the range is again zero.

Now we modify our theory for the horizontal range and derive the range of a projectile on a slope. Assuming a projectile is launched from the ground level, the range is defined as the distance between the launch point and the point where the projectile hits the ground. We take $+x$ direction horizontally and $+y$ vertically upward, so that we can still use our kinematic equations.

A slope of angle α is expressed as $\Delta y = \tan \alpha \Delta x$, and when the projectile hits the ground, the horizontal displacement and vertical displacement of the projectile must satisfy this relationship. Thus we have

$$v_0 \sin \theta t - \frac{1}{2} g t^2 = \tan \alpha (v_0 \cos \theta t)$$

Solutions for this equations are

$$t = 0, \frac{2v_0}{g} (\sin \theta - \tan \alpha \cos \theta)$$

The second solution gives the time at which the projectile hits the ground. Plugging this into the equation for Δx , we get

$$\Delta x = v_0 \cos \theta t = v_0 \cos \theta \left[\frac{2v_0}{g} (\sin \theta - \tan \alpha \cos \theta) \right]$$

We haven't reached the solution yet. We just got horizontal displacement, not the range on the slope. The range on slope is given by

$$R = \sqrt{\Delta x^2 + \Delta y^2} = \sqrt{\Delta x^2 + (\tan \alpha \Delta x)^2} = \frac{\Delta x}{\cos \alpha}$$

where we used the trig identity $1 + \tan^2 \alpha = \frac{1}{\cos^2 \alpha}$. Thus we get

$$\begin{aligned} R &= \frac{2v_0^2 \cos \theta}{g \cos \alpha} (\sin \theta - \tan \alpha \cos \theta) = \frac{2v_0^2 \cos \theta}{g \cos \alpha} \left(\frac{\sin \theta \cos \alpha - \sin \alpha \cos \theta}{\cos \alpha} \right) \\ &= \frac{2v_0^2 \cos \theta \sin(\theta - \alpha)}{g \cos^2 \alpha} = \frac{v_0^2}{g \cos^2 \alpha} [\sin(2\theta - \alpha) - \sin \alpha] \end{aligned}$$

where we used trig identity $\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$. You must note this reduces to the expression of horizontal range at $\alpha = 0$.

Example

A baseball player throws a ball on a 20° slope toward the top of the slope. S/he can throw a ball at a speed of 30 m/s. What is the maximum range? (Ignore the height of the launch point.)

Solution

In our expression for the range, the second term is constant. Thus we need to maximize the first term. It is achieved when

$$\begin{aligned} \sin(2\theta - \alpha) &= 1 \\ 2\theta - \alpha &= 90^\circ \\ \theta &= \frac{90^\circ + \alpha}{2} = \frac{90 + 20}{2} = 55^\circ \end{aligned}$$

So s/he needs to throw a ball at an angle 55° to the horizontal (35° to the incline). The range with this projection angle is

$$R = \frac{v_0^2}{g \cos^2 \alpha} [1 - \sin \alpha] = \frac{30^2}{9.8 \cos^2 20^\circ} [1 - \sin 20^\circ] = 68.4 \text{ m}$$

The range on a slope is shorter than that on a level ground.

Problem

What is the range if the baseball player throws a ball toward the bottom of the slope (The inclination is -20°)?

2 Choice of Coordinate System

Before applying Newton's 2nd law to a given problem, you need to fix your coordinate system. We have one solid principle when we choose a coordinate system. *Take one of the axes along the direction of acceleration.* This is because with this coordinate system, the acceleration along the other axis is zero. This simplifies our algebra a lot.

Example

A block with mass m is sliding down a slope of angle θ . The coefficient of kinetic friction between the slope and the block is μ_k . Find the acceleration of the block.

Solution 1 We take $+x$ axis along and down the slope and $+y$ axis perpendicular to the surface, in this coordinate system, y component of the acceleration is zero. The y component of the equation of motion is

$$\begin{aligned} 0 &= N - mg \cos \theta \\ N &= mg \cos \theta \end{aligned}$$

The kinetic friction is given by $f_k = \mu_k N$. Applying the Newton's 2nd law in the x direction, we have

$$\begin{aligned} ma &= mg \sin \theta - f_k = mg \sin \theta - \mu_k(mg \cos \theta) \\ a &= g(\sin \theta - \mu_k \cos \theta) \end{aligned}$$

Solution 2 We take $+x$ direction horizontally and $+y$ axis vertically upward. In this coordinate system, x and y components of the equation of motion are, respectively,

$$\begin{aligned} ma \cos \theta &= N \sin \theta - \mu_k N \cos \theta \\ -ma \sin \theta &= N \cos \theta + \mu_k N \sin \theta - mg \end{aligned}$$

where $\mu_k N = f_k$. First we solve each equation for N to remove N from the simultaneous equation.

$$\begin{aligned} N &= \frac{ma \cos \theta}{\sin \theta - \mu_k \cos \theta} \\ N &= \frac{mg - ma \sin \theta}{\cos \theta + \mu_k \sin \theta} \end{aligned}$$

Then we equate them, and solve for a .

$$\begin{aligned} \frac{ma \cos \theta}{\sin \theta - \mu_k \cos \theta} &= \frac{mg - ma \sin \theta}{\cos \theta + \mu_k \sin \theta} \\ a \cos \theta (\cos \theta + \mu_k \sin \theta) &= (g - a \sin \theta)(\sin \theta - \mu_k \cos \theta) \\ a(\cos^2 \theta + \sin^2 \theta) &= g(\sin \theta - \mu_k \cos \theta) \\ a &= g(\sin \theta - \mu_k \cos \theta) \end{aligned}$$

where we used trigonometric identity $\cos^2 \theta + \sin^2 \theta = 1$. Now are you convinced? No? Let's see another example.