Quantum transport in two dimensional ferromagnets

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Abstract

It has been long thought that electron transport in a ferromagnet does not show weak localization effect due to small phase coherent time. In addition, experiments on ferromagnetic films were not so popular because they oxidize quickly. But recent experiments show clear evidence of weak localization effects as well as a power law temperature dependence of the conductivity. It has been proposed that spin waves play an important role in these systems.

In this proposal I address the temperature and disorder dependence of longitudinal and anomalous Hall conductivity in ferromagnetic thin films, including the effects of spin waves. I also explore the possibility of temperature dependence of the anomalous Hall conductivity due to Berry phase.
I. MOTIVATION

Electron transport in ultrathin ferromagnetic films shows quite unusual quantum effects according to in-house experiment by Prof. Hebard’s group [1] [2] [3]. They observed signature of weak localization effect in ultrathin polycrystalline \(< 100\text{Å}\) Fe films over a wide range of disorder strength. They used a custom-built high vacuum system capable of doing in-situ transport measurements and found a logarithmic temperature dependence in both longitudinal and anomalous Hall conductivity for sheet resistances from 100 to 3000 \(\Omega\) at relatively low temperature \((T < 20\text{K})\). The logarithmic temperature dependence of the conductivity was also found by Bergmann and Ye in their experiment on amorphous, high resistance Fe films [4].

In Prof. Hebard group’s more recent experiments on Gd films, the signature of weak localization are seen in less disordered samples, whereas in more disordered samples \((R \gtrsim 10k\Omega)\) a power law behavior is observed [5]. Prof. Muttalib and Prof. P. Wölfle pointed out that at least a linear \(T\) behavior is possible by a large inelastic scattering off spin wave excitations [6]. The power law behavior, which was also seen earlier in the experiment by Ovadyahu and Imry [7] in indium oxide films and Davies and Pepper [8] in Si layers, has not been explained theoretically, except for a crude argument which states that in the hopping

![FIG. 1: \(\ln T\) dependence of \(R_{xx}\) and \(R_{xy}\) for Fe film with \(R_0 = 2733\Omega\). Inset: \(R_0 = 49\Omega\) [1]](image)

\[
\Delta^N(\sigma_{xy}) \\
\Delta^N(R_{xx}) \\
\Delta^N(R_{xy})
\]

\[
\text{Normalized Relative Change} \\
\text{T(K)}
\]

\[
-6.0 \\
-6.0 \\
-6.0
\]
FIG. 2: Power law behavior of conductivity in Si film. Two curves are \( \sigma = 2.27 \times 10^{-5} + 1.8610^{-5} \ln(T/3.2); \quad \sigma = 5.26 \times 10^{-5}(T/3.2)^{0.813} \) Experimental data better fit the second curve [8].

regime, one expects

\[
\sigma \propto D \propto \frac{\xi_{\text{loc}}^2}{\tau_{\text{in}}} \propto T^p,
\]

where \( D \) is the diffusion coefficient, \( \xi_{\text{loc}} \) is a localization length and \( \tau_{\text{in}} \) is an inelastic scattering time. However, a rigorous many-body calculation in this regime has not been done.

It is known that Berry phase effects cause transverse conductivity, and current theories claim that Berry phase contribution to the conductivity does not depend on temperature or disorder of the material, although it depends on the band structure of the material [9]. However, a recent study [10] shows that Berry phase may occur due to inhomogeneous magnetization, which is expected to depend on temperature. So we are motivated to derive the temperature dependence of the anomalous Hall conductivity due to Berry phase.

II. BACKGROUND FOR PROPOSED WORK

A. Quantum correction to longitudinal conductivity

Transport properties in metals at low temperatures show unusual quantum mechanical effects that cannot be explained by semiclassical transport theory. Through an extensive
theoretical and experimental research on quantum effects for the last five decades, now two mechanisms are considered important to understand transport properties in conductors at low temperature. One is Anderson localization and the other one is electron-electron interaction.

1. Weak localization correction and scaling theory

In the late 1950’s, through his study of electron wavefunction in random potential, Anderson showed that electron wavefunction localizes when disorder is large [11]. The strength of the localization depends on dimensionality; in 1D the localization correction is enormously important and can exceed the Drude result, whereas in 3D the localization occurs only when the disorder exceed a critical value. In 2D case, a small amount of disorder is still sufficient to localize electrons but the effect is not so large compared to 1D case, thus it is called “weak” localization.

Conceptual understanding of weak localization was first developed by Altshuler, Aronov, Larkin and Khmelnitskii, who pointed out that weak localization results from the constructive interference between electrons passing along time-reversal paths. For each path $P$ of an electron that comes back to its starting point, there is a corresponding time reversal path $\bar{P}$, and the total probability $P$ associated with the two paths is given by

$$P = |A_P + A_{\bar{P}}| = |A_P|^2 + |A_{\bar{P}}|^2 + 2\text{Re}(A_P^* A_{\bar{P}}).$$

(2.1)

Letting the probability amplitudes associated with the two paths be $A_P = \sqrt{p}e^{i\phi}$ and $A_{\bar{P}} = \sqrt{\bar{p}}e^{i\bar{\phi}}$, we have

$$P = p + \bar{p} + 2\sqrt{p\bar{p}}\cos(\phi - \bar{\phi}).$$

(2.2)

If the two paths are unrelated, the last term is zero, but since two paths are related through time-reversal symmetry, they interfere constructively. Therefore

$$P = 4p.$$  

(2.3)

So it doubles the return probability. The quantum mechanical coherence is considered to be stronger at lower temperature, thus the localization effect is more important at low temperature.
Diagramatically, this is expressed as maximally crossed diagrams, called cooperons. This type of diagrams were first examined by Langer and Neal in 1966 [12]. The crossed part is calculated as ladder with different direction of hole propagator. Calculating in the same way as diffusion propagator, the cooperon propagator is given by

\[ C(q) = \frac{1}{2\pi N(0)\tau^2 Dq^2 + |\nu_n|}. \]  

(2.4)

The weak localization correction to the conductivity is given by a pair bubble diagram with cooperon propagator:

\[ \delta\sigma^{\alpha\beta} = \frac{e^2T}{\nu_n} \sum_e v_k^\alpha v_{k+q}^\beta \frac{1}{(2\pi)^d} \int d^d k \frac{1}{(2\pi)^d} \int d^d q G^+(k) G^+(-k+q) G^-(k+q) G^-(k) C(q). \]

(2.5)

Integrating over the internal momentum and frequency, the weak localization correction in 2D is given by

\[ \delta\sigma = -\frac{\sigma_0}{2\pi k_F l} \ln \frac{\tau_0}{\tau} \propto \ln T \]

(2.6)

It has a logarithmic dependence on temperature. Whereas the weak localization corrections are not divergent for dimensions greater than 2, it becomes stronger in dimensions 2 and below. Thus 2D is called critical dimensionality.

Based on the result in 2D, Abrahams, Anderson, Licciardello, and Ramakrishnan proposed the scaling theory for localization [13]. They proposed that in any dimension, conductance \( G = 1/R \) can always be normalized to form a dimensionless parameter

\[ g(L) = \frac{G(L)}{\frac{e^2}{h}}, \]

(2.7)

which satisfy a one-parameter scaling equation

\[ \frac{d\ln g(L)}{d\ln L} = \beta(g). \]

(2.8)

When the conductance is large, the parameter is given by \( g(L) = \frac{ne^2}{m} L^{d-2} \). Therefore,

\[ \beta(g) = d - 2. \]

(2.9)

When the conductance is small, the parameter decays exponentially \( g(L) \propto e^{-L/L_c} \), thus

\[ \beta \propto -\ln g. \]

(2.10)

Abrahams et al connected these two asymptotic results. In dimensions below 2, \( \beta(g) \) is always negative, so the conductance is always scaled to zero and electrons are localized. In dimensions above 2, there is a critical conductance \( g_c \) and disorder-driven metal-insulator transition occurs at the critical conductance.
FIG. 3: Plot of $\beta(g)$ vs $\ln g(L)$ for $d > 2$, $d = 2$ and $d < 2$. $\ln g(L)$ is a normalized local conductance [13].

2. Interaction correction to the conductivity

Interaction correction to the longitudinal conductivity was first calculated by Altshuler and Aronov in 1979 [14]. The lowest order diagrams, which give dominant contribution to the corrections to the conductivity, are four diagrams that contain one interaction line and three diffusons.

Each diagram has a form of

$$\delta\rho_\alpha^\beta = \frac{e^2 T}{\Omega} \sum_\omega \int \frac{d^dk}{(2\pi)^d} \int \frac{d^d k'}{(2\pi)^d} \int \frac{d^dq}{(2\pi)^d} \Gamma(\omega, q) \Gamma(\omega - \Omega, q),$$

where $G^\pm$ are impurity averaged Green’s function and $\Gamma$ is a diffuson propagator, and $V$ is photon propagator.

Integrating over $k$ and $k'$, because in 2D the constant term gives zero over the angular integral, thus you need to keep linear $q$ term. Thus,

$$\int \frac{d^2 k}{(2\pi)^2} (G_+^k) G_-^k \propto \tau^3 q.$$  \hspace{1cm} (2.12)

In addition, after summing over $\omega$ the diffuson part is proportional to $q^{-4}$. Therefore before the $q$ integral the integrand is proportional to $q^2/q^4 \propto q^{-2}$ As a result, interaction
correction in 2D has logarithmic temperature dependence,

$$\delta \sigma \propto \ln(T\tau).$$ \hspace{1cm} (2.13)

Since it has the same functional form as weak localization, it is difficult to distinguish between the two corrections. However, the first order diagrams of interaction correction gives zero for transverse conductivity, namely $\delta \sigma_{xy}^{\text{int}} = 0$. Thus the logarithmic dependence of the anomalous Hall conductivity in Fe films suggests the presence of weak localization corrections [1].

**B. Hall conductivity and Anomalous Hall effect**

For ferromagnetic metals, it is known that Hall effect can exist in the absence of external magnetic field, which is called Anomalous Hall effect (AHE). Anomalous Hall effect is different from normal Hall effect caused by Lorentz force on charge carriers. The transverse electric potential of AHE is proportional to the spontaneous magnetization. In ferromagnet in a magnetic field, the density of up spin and down spin in conduction band are different, and AHE arises because the spin interacts with orbital angular momentum during scattering events. Since AHE is proportional to the magnetization, the hall resistivity is expressed as $\rho_{xy} = R_H H + R_S M$ where $R_H$ and $R_S$ are the normal and the anomalous Hall coefficients respectively.

Experimentally, the normal and anomalous Hall part is separated by measuring the Hall resistivity as a function of the magnetic field. Since the magnetic saturation is reached at high magnetic field, Hall resistivity behave as a linear function with a slope of $R_H$. If you
extrapolate the function, the value at zero magnetic field corresponds to anomalous Hall resistivity.

Currently two mechanisms are known to be responsible for the anomalous Hall effect in ferromagnetic metals. One is the skew scattering proposed by Smit and the other is the side-jump effect proposed by Berger. A third mechanism, which is more important for semi-conductors, is due to the Berry phase.

1. Skew Scattering Mechanism

In 1955 [16], J. Smit proposed a model that explains anomalous Hall effect, which is called skew scattering. He explains that AHE arises because scattering amplitude of an electron wave packet is anisotropic due to the spin-orbit coupling; the amplitude depends on the direction of the spin and direction of scattered electrons. The Hamiltonian with disorder potential and spin-orbit interaction is expressed in second quantization as

$$ H = \sum_{k\sigma}(\epsilon_k - M_z\sigma)c_{k\sigma}^\dagger c_{k\sigma} + \sum_j V(k - k')e^{i(k - k')R_j} \left[ \delta_{\sigma\sigma'} - ig_\sigma \tau_{\sigma\sigma'} \cdot (k \times k') \right], $$

where $g_\sigma$ is a spin-orbit coupling constant, and $\tau_{\sigma\sigma'}$ is the Pauli matrix. The Hall conductivity is calculated by applying this potential to the lowest order diagram that contributes to the Hall conductivity, which is a pair bubble diagram with three scattering processes at the same impurity:

$$ \delta_{\sigma xy} = \frac{e^2 n_{\text{imp}} T}{\Omega} \sum_\omega \int \frac{d^2 k}{(2\pi)^2} \int \frac{d^2 k'}{(2\pi)^2} \int \frac{d^2 k''}{(2\pi)^2} \frac{\tau_{\sigma\sigma'}}{i} v_{k\sigma} v_{k'\sigma'} V_{k\sigma k'\sigma'} V_{k'\sigma' k''\sigma} V_{k''\sigma'' k\sigma}$$

$$ \times G_k(\omega)G_k(\omega - i\Omega)G_{k'}(\omega)G_{k'}(\omega - i\Omega)G_{k''}(i\omega)G_{k''}(i\omega - i\Omega), $$

where $V_{kk'k''\sigma} = V_0 [1 - ig_\sigma \tau_{\sigma\sigma'} \cdot (k \times k')]$. Integrating over internal momentum and frequency, the skew scattering contribution to the AH conductivity is given by

$$ \sigma_{xy}^{SS} = \frac{M_z}{\epsilon_F} V_0 N_0 \langle g_0 \rangle_\sigma \sigma_{xx}, $$

where $N_0$ is the spin averaged density of states at the Fermi level and $V_0$ is impurity potential. Thus the AH conductivity is proportional the longitudinal conductivity and magnetization. Assuming $\sigma_{xy} \ll \sigma_{xx}$, then longitudinal and Hall resistivity are simply given by $\rho_{xx} = 1/\rho_{xy}$ and $\rho_{xy} = \sigma_{xy}/\sigma_{xx}^2$. Thus AH resistivity is also proportional to the
longitudinal resistivity
\[ \rho_{xy} \propto \frac{\sigma_{xy}}{\sigma_{xx}^2} \propto \rho_{xx}. \] (2.17)

2. Side Jump Mechanism

According to Berger [17], transverse electric potential arises because the trajectory of scattered electrons by an impurity is shifted due to spin-orbit interaction. This corresponds to an additional term in the quantum mechanical velocity operator for the Hamiltonian with spin-orbit interaction,
\[ v = \frac{d}{dt} r = -i [r, H] = \frac{P}{m} + \frac{g}{4\pi n_{\sigma}} (\tau \times \nabla V). \] (2.18)
The second term depends on the spin orientation, and causes lateral shift of the scattered wavepacket. The side-jump contribution is calculated applying this velocity at vertices in the diagram that contains one scattering line which comes from a vertex,
\[ \sigma_{xy} = \frac{e^2 n_{\text{imp}} T}{\Omega} \sum_{\omega} \int \frac{d^2 k}{(2\pi)^2} \int \frac{d^2 k'}{(2\pi)^2} \times G_k(\omega)G_k(\omega - i\Omega)G_k'(\omega) \left( \frac{-i g_\sigma V_0^2}{\epsilon_{F\sigma}} \right) \left[ \tau_{\sigma\sigma'} \times \frac{k - k'}{2m} \right] v_y. \] (2.19)
Performing the integration over momentum and frequency, the side jump contribution to the Hall conductivity is given by
\[ \sigma_{xy}^{SJ} = \frac{e^2}{2\pi} \sum_{\sigma} \left[ \frac{g_\sigma}{1 + \frac{g_\sigma^2}{2}} \right]. \] (2.20)
Thus, AH conductivity in this model is independent of impurity concentration and depends only on the side jump displacement, and the effective mean path for the side-jump contribution is short, this contribution is more important in strongly disordered sample. The same property of this model was shown by Berger’s calculation using semi-classical model. Since anomalous Hall conductivity is independent of longitudinal conductivity, Hall resistivity is proportional to the square of longitudinal conductivity

\[ \rho_{xy}^{SJ} \propto \frac{\sigma_{xy}}{\sigma_{xx}^2} \propto \rho_{xx}^2. \] (2.21)

3. Berry Phase Mechanism

Berry phase mechanism arises when there is an anomalous velocity term. Originally it was pointed out by Karplus and Luttinger in 1954 [19] when they calculated the charge current of electrons moving in a Bravais lattice. They found a quantity \( \Omega \) that acts like magnetic field which causes a transverse velocity \( \mathbf{E} \times \Omega \). Recently this quantity was identified as Berry phase curvature [21] [20]. In Block representation, the Hamiltonian with a static potential \( V \) is given by

\[ H = V(\mathbf{R}) + \epsilon_n(\mathbf{k}). \] (2.22)

If the potential satisfies \( \nabla V = \mathbf{E} \), then the semiclassical equations of motion become

\[ \dot{\mathbf{k}} = e\mathbf{E} + e\mathbf{v} \times \mathbf{B} \] (2.23)

\[ \mathbf{v} = \nabla \epsilon_n - e \mathbf{E} \times \Omega, \] (2.24)

where \( \Omega = \nabla_k \times \mathbf{X} \) is an effective magnetic field due to Berry vector potential \( \mathbf{X} \):

\[ \mathbf{X} = \int_{\text{cell}} d^3\mathbf{r} u_{nk}^*(\mathbf{r}) \nabla_k u_{nk}(\mathbf{r}). \] (2.25)
FIG. 7: Anomalous Hall conductivity $\sigma_{xy}$ vs longitudinal conductivity $\sigma_{xx}$ in pure metal (Fe, Ni, Co, and Gd), oxides (SrRuO$_2$ and La$_{1-x}$Sr$_x$CoO$_3$), and calcogenide spinels (Cu$_{1-x}$Zn$_x$Cr$_2$Se$_4$) at low temperature [24].

Here $eE \times \Omega$ is called the Luttinger term.

The current density due to the Luttinger term is given by

$$J = e^2 E \times \sum_k \Omega(k) f^0_k,$$

(2.26)

where $f^0_k$ is the equilibrium distribution function. Solving this equation, the Berry phase contribution to the AH conductivity is given by

$$\sigma_{xy}^{BP} = e^2 n \langle \Omega \rangle,$$

(2.27)

where $\langle \Omega \rangle = n^{-1} \sum_k \Omega_z(k) f^0_k$ is the weighted average of the Berry curvature. The Berry phase mechanism does not depend on impurity scattering and mean free path, and its contribution to the AH conductivity is proportional to the square of the longitudinal conductivity, which is the same dependence as the side jump mechanism. Diagrammatic calculation can be performed by applying anomalous velocity term to the vertices, which is similar to the side jump calculation.

Since distinction between extrinsic contribution (skew scattering and side jump) and intrinsic contribution (Berry phase) is difficult, the interpretation for experimental results
has been a controversial issue. Recently Nagaosa et al suggested a unified theory that explains anomalous Hall effect comprehensively [22]. According to them, anomalous Hall effect is mostly due to skew scattering in the clean limit, however, extrinsic-to-intrinsic crossover occurs as disorder and damping rate $1/\tau$ increases.

III. ONGOING WORK

A. Calculation of dispersion relation for magnons

It has been argued that spin waves are important in thin ferromagnetic films. In particular, a linear temperature dependence of the conductivity in thin Gd films seen in recent experiment can be due to scattering from spin waves [1]. This requires calculation of spin susceptibility.

To calculate the correction to the longitudinal conductivity $\delta\sigma_{xx}$ in ferromagnetic films due to spin waves, we need the stiffness and damping rate of magnons. The transverse spin susceptibility in 2D is given by a pair polarization bubble diagram:

$$i\Pi^0 = \int \frac{d\epsilon}{2\pi} \int \frac{d^2k}{(2\pi)^2} G^0_\uparrow(k + Q, \epsilon + \Omega) G^0_\downarrow(k, \epsilon),$$

where $G^0_{\uparrow\downarrow}$ are Born approximation time-ordered Green's function with spin orientation

$$G^0_{\uparrow\downarrow}(k, \epsilon) = \frac{\theta(\xi_{k,\uparrow\downarrow})}{\epsilon - \xi_{k,\uparrow\downarrow} + \xi + \frac{1}{2\tau}} + \frac{\theta(-\xi_{k,\uparrow\downarrow})}{\epsilon - \xi_{k,\uparrow\downarrow} + \xi - \frac{1}{2\tau}}$$

$$\xi_{k,\uparrow\downarrow} = \frac{k^2}{2m} - \frac{k_F^2}{2m} \pm \frac{B}{2}.$$
Here \( B = \xi_{k,1} - \xi_{k,1} \) is Zeeman splitting. Integrating over the frequency, we get

\[
\Pi^0 = \int \frac{d^2 k}{(2\pi)^2} \left[ \frac{\theta(\xi_{k+Q,1})\theta(-\xi_{k,1})}{\Omega - \frac{1}{m}(kQ \cos \theta + \frac{Q^2}{2}) - B + \frac{i}{\tau}} - \frac{\theta(-\xi_{k+Q,1})\theta(\xi_{k,1})}{\Omega - \frac{1}{m}(kQ \cos \theta + \frac{Q^2}{2}) - B - \frac{i}{\tau}} \right]. \quad (3.4)
\]

We assume \( Q \ll B < \epsilon_F \), thus the only first term gives non-zero contribution due to the product of two theta functions. Then the real and imaginary parts of \( \Pi^0 \) are

\[
\text{Re}\Pi^0 = -\frac{1}{2\pi^2} \int_0^\infty kdk \int_0^\pi d\theta \theta(\xi_{k+Q,1})\theta(-\xi_{k,1}) \frac{B - \Omega + \frac{1}{m}(kQ \cos \theta + \frac{Q^2}{2})}{k^2Q^2/m^2 \cos \theta + \frac{2kQ}{m}(B - \Omega + \frac{Q^2}{2m}) \cos \theta + (B - \Omega + \frac{Q^2}{2m})^2 + \frac{1}{\tau^2}}
\]

\[
= -\frac{1}{2\pi^2} \int_0^\infty kdk \theta(\xi_{k+Q,1})\theta(-\xi_{k,1}) \frac{\tan^{-1} \left[ \frac{(\frac{kQ}{m}-(B-\Omega+\frac{Q^2}{2m})+\frac{i}{\tau}) \tan (\frac{\pi}{2})}{\sqrt{(\frac{kQ}{m})^2-(B-\Omega+\frac{Q^2}{2m})^2}} \right]}{\sqrt{(\frac{kQ}{m})^2-(B-\Omega+\frac{Q^2}{2m})^2}} |^0_0 + (C.C) \quad (3.5)
\]

\[
\text{Im}\Pi^0 = -\frac{1}{2\pi^2} \int_0^\infty kdk \int_0^\pi d\theta \theta(\xi_{k+Q,1})\theta(-\xi_{k,1}) \frac{\frac{k^2Q^2}{m^2} \cos \theta + \frac{2kQ}{m}(B - \Omega + \frac{Q^2}{2m}) \cos \theta + (B - \Omega + \frac{Q^2}{2m})^2 + \frac{1}{\tau^2}}{\frac{1}{\tau} \tan^{-1} \left[ \frac{(\frac{kQ}{m}-(B-\Omega+\frac{Q^2}{2m})+\frac{i}{\tau}) \tan (\frac{\pi}{2})}{\sqrt{(\frac{kQ}{m})^2-(B-\Omega+\frac{Q^2}{2m})^2}} \right]} |^0_0 + (C.C). \quad (3.6)
\]

The angular and radial integrals get restrictions. Solving \( \xi_{k+Q} > 0 \) for \( \cos \theta \) we have

\[
\cos \theta > \frac{k_F^2 - k^2 - mB}{2kQ} \equiv \lambda_0. \quad (3.7)
\]

Now we consider three cases: 1. \( 1 \geq \lambda_0 \geq 0 \), 2. \( 0 \geq \lambda_0 \geq -1 \) and 3. \( -1 \geq \lambda_0 \) Calculating the restriction for the radial integral for each case, with the restriction from the other theta function, we have

Case 1: \( \int_0^\infty kdk \int_0^\pi d\theta \theta(\xi_{k+Q,1})\theta(-\xi_{k,1}) \rightarrow \int \sqrt{\frac{k_F^2 - mB - Q^2}{k_F^2 - mB - Q}} kdk \int_0^{\theta_0=\cos^{-1} \lambda_0} d\theta \quad (3.8) \)

Case 2: \( \int_0^\infty kdk \int_0^\pi d\theta \theta(\xi_{k+Q,1})\theta(-\xi_{k,1}) \rightarrow \int \sqrt{\frac{k_F^2 - mB + Q}{k_F^2 - mB + Q}} kdk \int_0^{\theta_0=\cos^{-1} \lambda_0} d\theta \quad (3.9) \)

Case 3: \( \int_0^\infty kdk \int_0^\pi d\theta \theta(\xi_{k+Q,1})\theta(-\xi_{k,1}) \rightarrow \int \sqrt{\frac{k_F^2 + mB}{k_F^2 + mB + Q}} kdk \int_0^\pi d\theta. \quad (3.10) \)
Adding the contribution from all cases, we have

\[ Q \]

Thus case one and two can be combined. We introducing \( \tan(\frac{\theta}{2}) = \sqrt{\frac{1 - \lambda}{1 + \lambda}} \), and subtitute \( k = \sqrt{k^2_F - mB} \) and up to \( Q^2 \) term. Then the real part and imaginary part of \( \Pi^0 \) from case one and two are

\[
\text{Re} \Pi^0 = \frac{1}{\pi^2} \int_{\frac{Q}{k_F}}^{\sqrt{k^2_F - mB + Q}} \frac{\tan^{-1} \left[ \frac{-1 + \frac{kB}{B - \Omega + i\tau}}{1 + \frac{kB}{B - \Omega - i\tau}} \right]}{\frac{2\pi}{Q^2} \frac{\tau(B - \Omega)}{1 + \tau^2(B - \Omega)^2}} kdkR_e \]

(3.11)

\[
\text{Im} \Pi^0 = \frac{1}{\pi^2} \int_{\frac{Q}{k_F}}^{\sqrt{k^2_F - mB - Q}} \frac{\tan^{-1} \left[ \frac{-1 + \frac{kB}{B - \Omega + i\tau}}{1 + \frac{kB}{B - \Omega - i\tau}} \right]}{\frac{2\pi}{Q^2} \frac{\tau(B - \Omega)}{1 + \tau^2(B - \Omega)^2}} kdkR_e \]

(3.12)

For case 3, arctangent part gives \( \frac{\pi}{2} \), thus up to \( Q^2 \) term, we have

\[
\text{Re} \Pi^0 = \frac{mB\tau}{2\pi} \frac{\tau(B - \Omega)}{1 + \tau^2(B - \Omega)^2} + \frac{Q\sqrt{k^2_F - mB}\tau}{2\pi} \frac{\tau(B - \Omega)}{1 + \tau^2(B - \Omega)^2} + \frac{Q^2\tau}{4\pi} \frac{\tau(B - \Omega)}{1 + \tau^2(B - \Omega)^2} \]

(3.13)

\[
\text{Im} \Pi^0 = (-\frac{mB\tau}{2\pi} + \frac{Q\sqrt{k^2_F - mB}\tau}{2\pi} + \frac{Q^2\tau}{4\pi}) \frac{1}{\tau^2(B - \Omega)^2 + 1} \]

(3.14)

Adding the contribution from all cases, we have

\[
\text{Re} \Pi^0 = -\frac{mB\tau}{2\pi} \frac{\tau(B - \Omega)}{1 + \tau^2(B - \Omega)^2} + \frac{(Q\tau)^2(k_F - mB)\tau^2(B - \Omega)^2 - 1}{\pi^2m} \frac{\tau(B - \Omega)}{1 + \tau^2(B - \Omega)^2} + \frac{Q^2\tau}{4\pi^2} \frac{\tau(B - \Omega)}{1 + \tau^2(B - \Omega)^2} \]

(3.15)

\[
\text{Im} \Pi^0 = \frac{mB\tau}{2\pi} \frac{1}{1 + \tau^2(B - \Omega)^2} + \frac{(Q\tau)^2(k_F - mB)}{\pi^2m} \frac{2\tau(B - \Omega)}{1 + \tau^2(B - \Omega)^2} + \frac{Q^2\tau}{4\pi^2} \frac{\tau(B - \Omega)}{1 + \tau^2(B - \Omega)^2} \]

(3.16)

The important thing to note is that the linear \( Q \) term in case 1 plus 2 exactly cancels that in case 3. The dispersion relation and damping constant of spin wave are determined by the equation

\[
1 = J\Pi^0(\Omega_Q - i\gamma_Q), \quad (3.17)
\]

14
where $\Pi^0_R$ is a retarded polarization, which is related to the time-ordered polarization as

$$\Pi^0_R = \text{Re}\Pi^0 + i\text{sgn}\omega\text{Im}\Pi^0. \quad (3.18)$$

As long as the damping is small, the real and imaginary part separate, and we have

$$1 = J\text{Re}\Pi^0_R = J\text{Re}\Pi^0 \quad (3.19)$$

$$\gamma_Q = \frac{\text{Im}\Pi^0_R}{\partial \text{Re}(\Pi^0_R)/\partial \Omega|_{\Omega_Q}} = \text{sgn}\Omega_Q\frac{\text{Im}\Pi^0}{\partial \text{Re}(\Pi^0)/\partial \Omega|_{\Omega_Q}}, \quad (3.20)$$

where $J$ is a coupling constant, which is negative because interaction is attractive. Assuming weak scattering $B\tau \gg 1$, then the dispersion and the damping rate are

$$\Omega_Q = B(1 - N_0|J|) + \frac{2Q^2(k_F^2 - mB)}{m^2B\pi} + \frac{Q^2|J|}{4\pi^2}(\pi - 2) \quad (3.21)$$

$$\gamma_Q = \frac{1}{\tau} \quad (3.22)$$

B. Discussion

Using this dispersion relation and damping rate, we may calculate the correction to the conductivity due to the diagrams that contains magnon propagator. However, the constant term should be zero as long as we consider short range interaction, which is guaranteed by the Goldstone theorem. According to the theorem, a collective excitation has a zero frequency mode, called the Goldstone mode, as long as the interaction is short ranged. It arises because the action of adding a magnon with an arbitrary long wavelength rotate the magnetization through an infinitesimal angle $\delta\theta$, which costs only infinitesimal energy.

Therefore, our non-zero gap implies that the spin conservation law is violated, and before proceeding to the conductivity calculation, we need to select proper set of diagrams and re-calculate the dispersion relation and damping rate. For this purpose we need to add vertex correction to the pair bubble and check if the constant part disappears. That will be my first project in the future.

IV. PROPOSED RESEARCH

A. Spin Wave

A rigorous study of spin wave in the presence of disorder based on standard many-body theory has not been carried out, even in weak disorder. Starting with Hamiltonian with
spin-spin interaction,
\[ H = \sum_{k\sigma} \epsilon_{k\sigma} a_{k\sigma}^\dagger a_{k\sigma} + J \sum_{k,k',q} a_{k+q\uparrow}^\dagger a_{k\uparrow} a_{k'-q\downarrow}^\dagger a_{k'\downarrow} + \sum_{j} \sum_{k,k'} e^{i(k-k'R_j)} a_{k\sigma}^\dagger a_{k'\sigma} \] (4.1)
\[ \epsilon_{k\sigma} = \epsilon_k - \sigma B. \] (4.2)

First I am going to calculate the spin susceptibility with proper vertex correction. This may require that I start with a conserving approximation that guarantee the Goldstone theorem. Then I will add disorder and interactions.

With these corrections the gap should be zero and I’m hoping to get temperature and disorder dependence for the damping constant. Then we can calculate the correction to the conductivity using the magnon propagator derived by the dispersion relationship and damping rate. If the damping rate has a non-trivial temperature dependence, then this might give a non-trivial temperature dependence for the conductivity.

Then I am going to work on strong disorder calculation. Strong disorder calculation is different from weak disorder calculation in the sense that we need to consider more diagrams that we could ignore for weak disorder.

Also, especially to analyze anomalous Hall conductivity, it is necessary to take anisotropic potential due to spin-orbit coupling into account. We treated impurity potential as isotropic, but this should be modified as

\[ \sum_{j} \sum_{k,k'} e^{i(k-k'R_j)} \rightarrow \sum_{k,k'\sigma\sigma'} e^{i(k-k'R_j)} \left[ \delta_{\sigma\sigma'} - ig_{SO} \hat{\tau}_{\sigma\sigma'} \cdot (\hat{k} \times \hat{k}') \right], \] (4.3)

where \( g_{SO} = \lambda_c^2 k_F^2 / (2\pi)^2 \) is a dimensionless spin-orbit coupling constant.

Another possible approach to strong disorder calculation is to start from localized state and consider hopping of the electrons. In this case the basis state is not the eigenfunction of momentum, thus we need to perform diagrammatic calculation in real space.

**B. Berry Phase**

Nagaosa et al’s comprehensive model explains the crossover between intrinsic and extrinsic contribution to anomalous Hall conductivity well. However, research on the possibility of a temperature dependence of the Berry phase has not been done yet.

Recent study on modification of the semiclassical equation of motion due to Berry phase shows an anomalous velocity caused by Berry curvature and inhomogeneous magnetization.
They start from Hamiltonian with inhomogeneous magnetic field as a perturbation

\[ H = \left( \frac{\mathbf{p}^2}{2m} + e\mathbf{A}(\mathbf{R}) + e\delta\mathbf{A}(\mathbf{R}) \right)^2 + V(\mathbf{R}), \]  

where \( \mathbf{A}(\mathbf{R}) \) and \( \delta\mathbf{A}(\mathbf{R}) \) are the homogeneous and inhomogeneous vector potential, respectively. Diagonalizing this Hamiltonian, they get Berry phase term in semiclassical equation of motions

\[ \dot{\mathbf{r}} = \frac{\partial E(\mathbf{k})}{\mathbf{k}} - \mathbf{k} \times \Theta(\mathbf{k}) \]  

\[ \dot{\mathbf{k}} = -e\mathbf{E} - e\dot{\mathbf{r}} \times \delta\mathbf{B}(\mathbf{r}) - M \frac{\partial \delta\mathbf{B}}{\partial \mathbf{r}}, \]

where \( \Theta(\mathbf{k}) \) is the Berry curvature. Since inhomogeneous magnetization involves domain-walls that fluctuate with temperature, we suspect anomalous Hall conductivity due to this anomalous velocity term has temperature dependence. Therefore I propose to calculate diagrammatically the correction to the conductivity due to Berry phase term due to inhomogeneous magnetization.

The Berry phase term can also be derived from Dirac Hamiltonian with applied electric field [25]

\[ \hat{H}(P^\alpha, R^\alpha) = \alpha_\beta (P^\alpha - \frac{e}{c}) A^\beta (R^\alpha)c + \beta mc^2, \]

where \( \alpha = (\alpha^0, \alpha) \) is Dirac matrix and \( A \) is the electromagnetic potential. Thus the contribution to the AH conductivity from this Hamiltonian is expected to be independent of the details of the band structure of materials, which require extensive numerical analysis. Thus Berry phase curvature that appears in the relativistic Hamiltonian can be the source of yet another contribution to the AHE. With this motivation, I propose to study weak relativistic limit of the Dirac Hamiltonian to get the anomalous velocity term in the velocity operator. I can then use the same diagrammatic technique that have been used for the side-jump contribution to the AHE.


[20] N.P. Ong and Wei-Li Lee, Geometry and the Anomalous Hall Effect in Ferromagnets